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CONTENTS OF VOLUME XIX

Memoirs

Miscellanea

List of Plates, etc.

Errata et Corrigenda.

[illegible] the value is correct in the original manuscript—217N has been printed for [illegible] numerator of the third expression in the coefficient of $-\frac{4M\bar{\nu}_6\bar{\nu}_2}{M-1}$ in the value of [illegible] Mr Church's paper in Vol. XVIII. The correct value is given in the first line on [illegible]. The owners of Vol. XVIII are requested to make this correction in the text.

[illegible] rather puzzling slips were overlooked in the proof of B. G. E. Hooke's paper on the [illegible] Street Crania. They occur in the "Remarks" in the Sheets of Measurements facing [illegible] of Vol. XVIII.

Sheet I.	Skull No. 33: For "Pars-occipital" read "Par-occipital."
" "	Skull No. 44: Read "Sylvian groove" for "spheno-parietal sulcus."
Sheet II.	Skull No 48: "Accessory cusp on 1st molar," this is a *fifth* cusp.
" "	Skull No. 50: The tiny ossicles are at junction of anterior (not ante) and middle thirds, i.e. in region of metopic fontanelle.
" "	Skull No. 55: For "super-mastoid" read "supra-mastoid."
" "	Skull No. 62: For "spheno-parietal sulcus" read "Sylvian groove."
" "	Skull No. 67: For "petro-mastoid process" read "retro-mastoid process" (on inner side of mastoidal groove).
" "	Skull No. 70: For "pterygo-spinus" read "pterygo-spinous."
" "	Skull No. 75: Read again "supra-mastoid" for "super-mastoid."
Sheet III.	Skull No. 197: For "lacrimalis" read "lacrymalis."
" "	Skull No. 198: The "transverse bridge on foramen" alluded to here will be found elucidated in the Remarks on Skull No. 187.
Sheet IV.	Skull No. 211: "Posterior interparietal bone," cancel this and replace by "ossicle of λ."
Sheet V.	Skull No. 100: "petro-mastoid" should be "retro-mastoid."

BIOMETRIKA

ON THE APPROXIMATE QUADRATURE OF CERTAIN SKEW CURVES, WITH AN ACCOUNT OF THE RESEARCHES OF THOMAS BAYES.

BY JOHN WISHART, M.A., B.SC.

CONTENTS.

Introduction. The Incomplete Beta Function is defined by the relation

$$I_x(p, q) = \frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)} \int_0^x x^{p-1}(1-x)^{q-1}\,dx \qquad (1),$$

and thus represents the area, up to the point x, of the skew curve:

$$y = y_0 x^{p-1}(1-x)^{q-1},$$

when the total area is taken as unity. The problem of the quadrature of this curve, if not one of the classical problems of post-Newtonian mathematics, can at any rate lay claim to have been worked at for a very long time. The earliest discussion is contained in the celebrated memoir of Thomas Bayes: *An Essay towards solving a Problem in the Doctrine of Chances**. Bayes, in this and a second paper, both of which were communicated posthumously to the Royal Society by Dr Richard Price, was alive to the necessity of finding approximate methods of evaluating the integral (1), when p, or q, or both, became large. Within recent years the problem has re-doubled its importance, for upon its successful solution depends the use of a very considerable part of Pearson's analysis of skew frequency. There can be no doubt that a large number of the distributions met with in statistical experience are capable of good representation by one or other of the familiar Type I curves, or of the particular cases of this Type met with in Types II, VI

* *Phil. Trans.* Vol. LIII. (1763), p. 370 et seq.

and VII. The practical statistician, then, needs a quick and accurate evaluation of (1). This want will in the near future in part be met by the issue of the *Tables of the Incomplete Beta Function,* at present being computed. But no matter how far the range of these Tables be extended, the labour of the calculation and the cost of printing must limit them, and beyond this point we have to meet the needs of the occasional user who finds on fitting his curve that the constants fall beyond the range of the Tables. It is the object of the later sections of this memoir to supply approximate formulae for the Incomplete Beta Function, when p and q are large. It will, however, be found instructive to begin by studying Bayes' work, in order to see, first, what it was that he really proved, and secondly, to obtain a clue which, when followed up, will lead to the kind of expansion we are in search of.

Section A. The proof given by Bayes has been expounded so often that nothing more than a brief summary will be given here. His design, at first, was to find out a method by which we might judge concerning the probability that an event has of happening, under given circumstances, upon the supposition that we know nothing concerning it but that, under the same circumstances, it has happened a certain number of times and failed a certain other number of trials. To develop his method he makes use of the idea of balls being placed at random on a square table. A first ball is put down, and is supposed equally likely to be anywhere on the table. This is the assumption of the "equal distribution of ignorance." Expressed in more general terms, "to estimate the chance that the probability for the happening of an event, perfectly unknown, should lie between any two named degrees of probability, antecedently to any experiments made about it," we must "suppose the chance the same that it should lie between any two equidifferent degrees*." The position of the first ball is then held to determine the success or failure of a subsequent series of n trials.

For, let $ABCD$ represent the table (Fig. 1), and suppose the first ball to rest at E (distant x from the nearest point of AD). Let $AB = 1$.

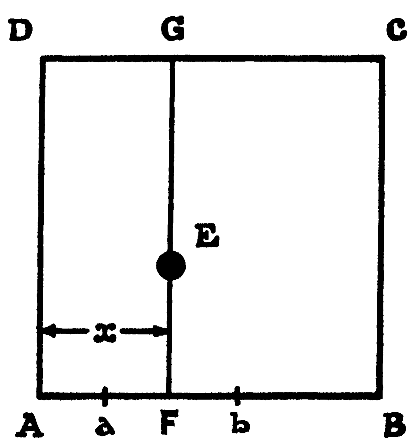

Fig. 1.

* *Phil. Trans.* Vol. LIII. (1763), pp. 370—371.

Then the chance that the first ball should lie between x and $x + \delta x$ (on Bayes' assumption) $= \delta x$.

A second ball is now put down n times: a success is counted if it should be between FG and AD, a failure if outside. Let there be l successes and m failures $(l + m = n)$. The chance of this particular result is

$$\frac{n!}{l!\,m!}\,x^l(1-x)^m,$$

since, when the first ball has been placed at x, the chance of the event is x ($AB = 1$). Combining this with the previous probability, and taking the total probability for all possible positions of the first ball between a and b, say, we find that the probability of the first ball being between a and b, and of a subsequent series of l successes and m failures out of n trials, is given by

$$\frac{n!}{l!\,m!}\int_a^b x^l(1-x)^m\,dx.$$

Now we assume that the first ball must rest somewhere between A and B. Hence *before* the first ball is put down, the probability that the event will happen l times and fail m times in n trials is

$$\frac{n!}{l!\,m!}\int_0^1 x^l(1-x)^m\,dx.$$

This is a corollary to the previous result, got by putting $a = 0$, $b = 1$.

The theory is now to be applied to an event concerning which we know nothing except that it has happened l times and failed m times in n trials. Let P be the probability that the chance of success, before any trials were made, lay between a and b. So P is the chance, to be determined *à posteriori*, that the first ball put down should lie between any two points a and b. Then by a simple application of the theorem of compound probability we have

$$\frac{n!}{l!\,m!}\int_a^b x^l(1-x)^m\,dx = P \times \frac{n!}{l!\,m!}\int_0^1 x^l(1-x)^m\,dx,$$

whence

$$P = \frac{\int_a^b x^l(1-x)^m\,dx}{\int_0^1 x^l(1-x)^m\,dx} \qquad \text{.............................(2).}$$

Such is the theorem reached by Bayes. It is important historically as being one of the first results reached in the theory of inverse probability. A simpler illustration than that of balls being placed on a table might easily be furnished. Obviously the width of the table (AD in Fig. 1) is immaterial to the result. There is, in fact, no need for the introduction of a second dimension at all. The nature of Bayes' experiment tends to obscure the issue, which is a very simple one. Consider a teetotum whose sides are painted in a definite, but unknown, proportion of black and white. It is spun n times and the upper face in each case observed when it has come to rest. What is the probability after noting l blacks and m whites that the proportion of black to white should be between a and b,

where a and b are at our choice? This is the same problem, and the assumption in solving it is that every possible proportion of black to white has the same chance of occurring in the beginning. Or again consider the immunity of a person to a certain disease. All grades of immunity are possible, varying from one person to another. Some people will catch the disease, others will escape, thus defining a point X (see Fig. 2), of immunity $\xi\,(0<\xi<1)$, representing

Curve of Frequency for ξ

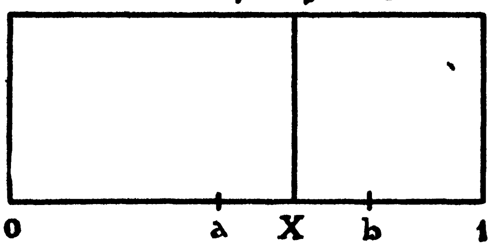

Fig 2.

the critical immunity below which all will succumb. We have no knowledge of the position of X, but we observe n cases and note that l succumb while m escape. Our result (2) then enables us to determine the probability that X lies between a and b, say, on the assumption that all values for this critical immunity are equally likely to occur, i.e. the distribution of frequency is a rectangle.

P being now determined by equation (2) its value is retained and used for the prognostication of events in a further series of trials. It is evident, however,

(a) that his assumption of the "equal distribution of ignorance" is one that is difficult to justify,

(b) that his theorem as given by (2) is *not* the theorem which is now called after him.

The first of these statements has been discussed in the long series of papers that have since been written on the subject, for a bibliography of which the reader is referred to K. Pearson, *Biometrika*, Vol. XIII. (1920), pp. 1—3, J. M. Keynes, *A Treatise on Probability*, pp. 370—383, and E. S. Pearson, *Biometrika*, Vol. XVII. (1925), pp. 388—389 (footnotes in all cases) It is not altogether clear, however, that the statement (b) above is generally accepted. Bayes' Theorem, as now understood, is this: an event has been observed to happen l times and fail m times in n trials; the chance of r successes and s failures in a further series of t trials ($t=r+s$) is then given by

$$C_{(p,\,q)\,(r,\,s)}=\frac{(r+s)\,!}{r\,!\,s\,!}\,\frac{\int_0^1 z^{l+r}(1-z)^{m+s}F(z)\,dz}{\int_0^1 z^l(1-z)^m F(z)\,dz}\quad\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

where, if ξ represents the limiting value of the variate x, thus determining the success or failure of the event (see above), $F(z)$ is the ratio of two functions $\phi(\xi)$ and $f(\xi)$, and $y=\phi(\xi)$ is the frequency curve of the *à priori* possible values of ξ,

while $Nf(x)$ is the frequency curve of values of x in the population N which is under investigation*. Even if we assume that $\phi(\xi)/f(\xi)$ is constant, thus giving the result of Condorcet and Laplace, we are stating a very different theorem from that given by Bayes himself. His result (see Equation (2)) depends essentially on the evaluation of an Incomplete Beta Function, and the immediate interest of his paper to us is in the examination of the methods by which he attempted to solve this problem. Although his methods appear crude to us now, the next section will show that he made at least one interesting contribution to the common stock of our knowledge.

Section B. Bayes solved his problem successfully for small l and m by expanding and integrating term by term. He also gave the well-known formula, obtained by integration by parts, which is quoted by H. E. Soper in *Tracts for Computers*, No. VII. p. 5, Equation (5). But he was not so successful in his methods of approximating to the area required when p and q are large. His general method will be indicated below, although it is of little more than historical interest now.

Let ANB represent the curve $y = x^l(1-x)^m$, where $AB = 1$, and let MN be the maximum ordinate, so that $AM = \frac{l}{n} = a$, and $BM = 1 - \frac{l}{n} = \frac{m}{n} = b$. Let $CM = ME = z$.

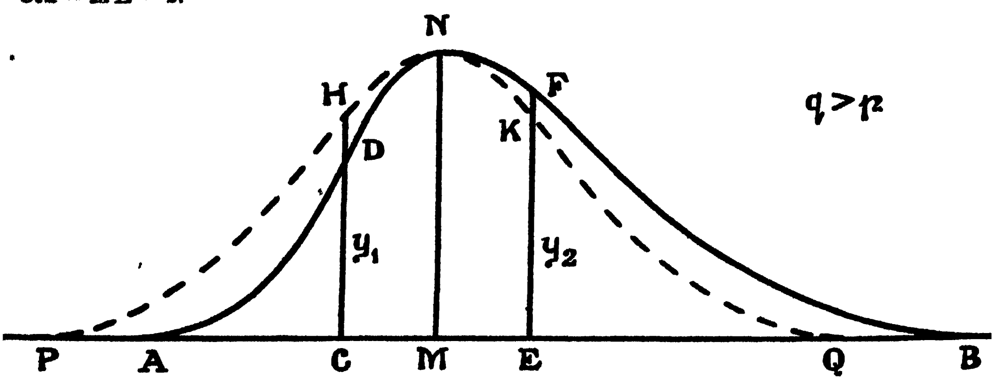

Fig. 3.

The problem then is to determine the two areas under the curve adjacent to the mode, and bounded by the equidistant ordinates CD and EF.

1. From Soper's Equation (5), referred to above, on putting x equal, first to a, and then to b (and interchanging p and q), we have

$$\int_a^1 x^l(1-x)^m\,dx - \int_0^a x^l(1-x)^m\,dx$$
$$= \frac{a^l b^m}{n} \times \left[\left\{\frac{m}{m+1} + \frac{l\,.\,m^2}{(m+1)(m+2)\,l} + \frac{l\,(l-1)\,m^3}{(m+1)(m+2)(m+3)\,l^2} + \dots\right\}\right.$$
$$\left. - \left\{\frac{l}{l+1} + \frac{m\,.\,l^2}{(l+1)(l+2)\,m} + \frac{m\,(m-1)\,l^3}{(l+1)(l+2)(l+3)\,m^2} + \dots\right\}\right].$$

* For a fuller discussion, see K. Pearson, *Biometrika*, Vol. XVI. (1924), pp. 190—198.

Let u_r be any term in the first series and u_r' the corresponding term in the second series. Then it can be shown that

$$\frac{u_{r+2}}{u_r'} < 1 \text{ provided } m > l.$$

Hence

$$\int_a^1 x^l (1-x)^m \, dx - \int_0^a x^l (1-x)^m \, dx < \frac{a^l b^m}{n} \left\{ \frac{m}{m+1} + \frac{m^2}{(m+1)(m+2)} \right\}$$
$$< \frac{2a^l b^m}{n} \frac{m}{m+2},$$
$$< \frac{2a^l b^m}{n} \quad \text{..............................(4).}$$

Thus Bayes reaches a very crude upper limit to the difference of areas on either side of the mode. It consists in taking only the first two terms of the reduction by parts formula (and even approximating here), and neglecting

(*a*) the difference between the terms u_{r+2} and u_r' for values of r from 1 to $l-1$,

(*b*) the remaining terms of the second series for values of r from l to $m+1$.

2. The argument then proceeds as follows:

$$y_1 = (a-z)^l (b+z)^m,$$
$$y_2 = (a+z)^l (b-z)^m. \quad \text{(See Fig. 3.)}$$

So if $m > l$, $y_2 > y_1$, and $\frac{y_2}{y_1}$ increases as z increases: for

$$\log \frac{y_2}{y_1} = \frac{2nz^3}{3} \cdot \frac{(b^2-a^2)}{a^2 b^2} + \dots,$$

which is positive if $m > l$.

3.
$$\frac{\int_a^1 x^l (1-x)^m \, dx}{\int_0^a x^l (1-x)^m \, dx} > \frac{\int_a^{a+z} x^l (1-x)^m \, dx}{\int_{a-z}^a x^l (1-x)^m \, dx}.$$

4. Suppose the symmetrical curve PNQ to be drawn whose equation is

$$y = a^l b^m \left(1 - \frac{n^2 z^2}{lm}\right)^{\frac{n}{2}}$$

(shown by dotted line), so that its maximum ordinate coincides with that of the skew curve. It is easy to show, by differentiating, that

$$\left(1 - \frac{nz}{m}\right)^m \left(1 + \frac{nz}{l}\right)^l > \left(1 - \frac{n^2 z^2}{lm}\right)^{\frac{n}{2}} > \left(1 - \frac{nz}{l}\right)^l \left(1 + \frac{nz}{m}\right)^m, \text{ if } m > l.$$

Hence

$$\int_0^z \left(1 + \frac{nz}{l}\right)^l \left(1 - \frac{nz}{m}\right)^m dz > \int_0^z \left(1 - \frac{n^2 z^2}{lm}\right)^{\frac{n}{2}} dz > \int_0^{-z} \left(1 + \frac{nz}{l}\right)^l \left(1 - \frac{nz}{m}\right)^m dz.$$

Now let
$$u = \sqrt{\frac{n^3}{lm}}\, z,$$
then
$$\int_0^z \left(1 - \frac{n^2 z^2}{lm}\right)^{\frac{n}{2}} dz = \sqrt{\frac{lm}{n^3}} \left(u - \frac{1}{6} u^3 + \frac{n-2}{8n} \cdot \frac{u^5}{5} - \frac{(n-2)(n-4)}{48n^2} \cdot \frac{u^7}{7} + \dots\right) \dots (5).$$
Therefore
$$\frac{\int_0^z \left(1 - \frac{n^2 z^2}{lm}\right)^{\frac{n}{2}} dz}{\int_0^{\frac{\sqrt{lm}}{n}} \left(1 - \frac{n^2 z^2}{lm}\right)^{\frac{n}{2}} dz} = \frac{n+1}{2^n} \sqrt{\frac{2}{m}} \cdot \frac{n\,!}{\frac{n}{2}\,!\,\frac{n}{2}\,!} \left(u - \tfrac{1}{6} u^3 + \dots\right) \quad \dots\dots\dots (6).$$

5. We have
$$\int_a^1 x^l (1-x)^m \, dx - \int_0^a x^l (1-x)^m \, dx < \frac{2a^l b^m}{n} \text{ from Article 1,}$$
and
$$\int_a^1 x^l (1-x)^m \, dx + \int_0^a x^l (1-x)^m \, dx = \frac{l\,!\, m\,!}{(n+1)\, n\,!},$$
therefore
$$2 \int_a^1 x^l (1-x)^m \, dx < \frac{l\,!\, m\,!}{(n+1)\, n\,!} + \frac{2a^l b^m}{n},$$
and
$$2 \int_0^a x^l (1-x)^m \, dx > \frac{l\,!\, m\,!}{(n+1)\, n\,!} - \frac{2a^l b^m}{n}.$$

Combining, we have
$$\frac{\int_a^1 x^l (1-x)^m \, dx}{\int_0^a x^l (1-x)^m \, dx} < \frac{1 + \frac{2n\,!}{l\,!\, m\,!} a^l b^m \left(1 + \frac{1}{n}\right)}{1 - \frac{2n\,!}{l\,!\, m\,!} a^l b^m \left(1 + \frac{1}{n}\right)}.$$

So from Art. 3
$$\frac{\int_a^{a+z} x^l (1-x)^m \, dx}{\int_{a-z}^a x^l (1-x)^m \, dx} < \frac{1+R}{1-R},$$
where
$$R = \frac{2n\,!}{l\,!\, m\,!} a^l b^m \left(1 + \frac{1}{n}\right),$$
therefore
$$\frac{\int_a^{a+z} x^l (1-x)^m \, dx}{\text{area from } a \text{ to } a+z \text{ of symmetrical curve}} < \frac{1+R}{1-R};$$
likewise
$$\frac{\int_a^{a-z} x^l (1-x)^m \, dx}{\text{area from } a \text{ to } a+z \text{ of symmetrical curve}} > \frac{1-R}{1+R}.$$

6. This leads directly to Rule II of Bayes; for let the ratio of the area from a to $a+z$ of the symmetrical curve to the whole area of the skew curve be called Σ, then

$$\Sigma \frac{1+R}{1-R} > \frac{\int_a^{a+z} x^l (1-x)^m \, dx}{\int_0^1 x^l (1-x)^m \, dx} > \Sigma \quad \text{.....................(7)},$$

and

$$\Sigma > \frac{\int_a^{a-z} x^l (1-x)^m \, dx}{\int_0^1 x^l (1-x)^m \, dx} > \Sigma \frac{1-R}{1+R} \quad \text{.....................(8)}.$$

Combining these results we have

$$\frac{2\Sigma}{1-R} > \frac{\int_{a-z}^{a+z} x^l (1-x)^m \, dx}{\int_0^1 x^l (1-x)^m \, dx} > \frac{2\Sigma}{1+R} \quad \text{.....................(9)}.$$

Bayes' Rule II is comprised in the three results of this Article. The statement must, of course, be reversed if $l > m$.

7. The full value of Σ is, from Art. 4,

$$\Sigma = \frac{n+1}{n} \sqrt{\frac{lm}{n}}\, a^l b^m \frac{n!}{l!\,m!} \left(u - \frac{1}{6} u^3 + \frac{n-2}{2^2\, 2!\, n} \cdot \frac{u^5}{5} - \frac{(n-2)(n-4)}{2^3 \cdot 3!\, n^2} \cdot \frac{u^7}{7} + \dots \right) \text{...(10)}.$$

Bayes shows how to approximate to the first part of this result by using Stirling's theorem, and gets

$$\frac{n!}{l!\,m!} a^l b^m = \sqrt{\frac{n}{2\pi l m}}\, e^{\frac{1}{12}\left(\frac{1}{n} - \frac{1}{l} - \frac{1}{m}\right) - \frac{1}{360}\left(\frac{1}{n^3} - \frac{1}{l^3} - \frac{1}{m^3}\right) + \dots} \quad \text{......(11)}.$$

The series in (10) may be used when u is less than 1, or even when $u < \sqrt{6}$. But for large values of u Bayes has shown how to obtain a series better suited to computation, and here he has made an important contribution to our subject. The following is a development of his idea in modern dress. It is to be noted, in passing, that where Bayes expanded $\int_0^u \left(1 - \frac{2u^2}{n}\right)^{\frac{n}{2}} du$, which takes in the limit when $n \to \infty$ the value $\int_0^u e^{-u^2} du$, we are treating of the analogous form $\int_0^u \left(1 - \frac{u^2}{n}\right)^{\frac{n}{2}} du$, giving as the limit $\int_0^u e^{-\frac{u^2}{2}} du$. This is in line with the traditions of the biometrical school.

But $$\int_0^u \left(1 - \frac{u^2}{n}\right)^{\frac{n}{2}} du = \int_0^{\sqrt{n}} \left(1 - \frac{u^2}{n}\right)^{\frac{n}{2}} du - \int_u^{\sqrt{n}} \left(1 - \frac{u^2}{n}\right)^{\frac{n}{2}} du$$

$$= \frac{2^{n-1} n \sqrt{n}}{n+1} \frac{\{\Gamma(\frac{1}{2}n)\}^2}{\Gamma(n)} - \int_u^{\sqrt{n}} \left(1 - \frac{u^2}{n}\right)^{\frac{n}{2}} du.$$

Now $\int_u^{\sqrt{n}}\left(1-\frac{u^2}{n}\right)^{\frac{n}{2}}du=-\frac{n}{n+2}\int_u^{\sqrt{n}}\frac{1}{u}\,d\left(1-\frac{u^2}{n}\right)^{\frac{n}{2}+1}$

$$=-\frac{n}{n+2}\left[\frac{1}{u}\left(1-\frac{u^2}{n}\right)^{\frac{n}{2}+1}\right]_u^{\sqrt{n}}-\frac{n}{n+2}\int_u^{\sqrt{n}}\frac{1}{u^2}\left(1-\frac{u^2}{n}\right)^{\frac{n}{2}+1}du$$

$$=\frac{n}{n+2}\cdot\frac{1}{u}\left(1-\frac{u^2}{n}\right)^{\frac{n}{2}+1}+\frac{n}{n+2}\cdot\frac{n}{n+4}\int_u^{\sqrt{n}}\frac{1}{u^3}\,d\left(1-\frac{u^2}{n}\right)^{\frac{n}{2}+2}$$

$$=\frac{n}{n+2}\cdot\frac{1}{u}\cdot\left(1-\frac{u^2}{n}\right)^{\frac{n}{2}+1}-\frac{n^2}{(n+2)(n+4)}\cdot\frac{1}{u^3}\left(1-\frac{u^2}{n}\right)^{\frac{n}{2}+2}$$

$$+\frac{n^3}{(n+2)(n+4)(n+6)}\cdot\frac{1.3}{u^5}\left(1-\frac{u^2}{n}\right)^{\frac{n}{2}+3}+\dots$$

$$+\frac{(-1)^{s+1}n^s}{(n+2)(n+4)\dots(n+2s)}\cdot\frac{1.3.5\dots 2s-3}{u^{2s-1}}\left(1-\frac{u^2}{n}\right)^{\frac{n}{2}+s}$$

$$+\frac{(-1)^s n^s.1.3.5\dots 2s-1}{(n+2)(n+4)\dots(n+2s)}\int_u^{\sqrt{n}}\frac{1}{u^{2s}}\left(1-\frac{u^2}{n}\right)^{\frac{n}{2}+s}du.$$

Hence on integrating by parts we have

$$(2s-1)\int_u^{\sqrt{n}}\frac{1}{u^{2s}}\left(1-\frac{u^2}{n}\right)^{\frac{n}{2}+s}du=\frac{1}{u^{2s-1}}\left(1-\frac{u^2}{n}\right)^{\frac{n}{2}+s}$$
$$\cdot\frac{n+2s}{n}\int_u^{\sqrt{n}}\frac{1}{u^{2s-2}}\left(1-\frac{u^2}{n}\right)^{\frac{n}{2}+s-1}du,$$

so if we denote the remainder after s terms by $R_s(u)$ and the sth term of the series itself by $U_s(u)$ we have $|R_s(u)|<|U_s(u)|$ always, so that the error involved in taking the sum of s terms of the series as the value of the integral is always less than the sth term itself.

But we know that $\qquad \Gamma(n)=\frac{2^{n-1}}{\sqrt{\pi}}\Gamma(\tfrac{1}{2}n)\,\Gamma\{\tfrac{1}{2}(n+1)\}.$

Hence we have

$$\int_0^u\left(1-\frac{u^2}{n}\right)^{\frac{n}{2}}du=\frac{\sqrt{n\pi}}{2}\cdot\frac{n}{n+1}\cdot\frac{\Gamma(\frac{1}{2}n)}{\Gamma\{\frac{1}{2}(n+1)\}}-\frac{n}{n+2}\cdot\frac{1}{u}\left(1-\frac{u^2}{n}\right)^{\frac{n}{2}}$$
$$\times\left[1-\frac{n}{n+4}\cdot\frac{1}{u^2}\left(1-\frac{u^2}{n}\right)+\frac{n^2}{(n+4)(n+6)}\cdot\frac{1.3}{u^4}\left(1-\frac{u^2}{n}\right)^2-\dots\right]\quad\dots(12).$$

This is the series whose first few terms are given by Bayes. It is an asymptotic expansion representing the area of the "tail" of the symmetrical curve, and useful when u is greater than unity. The error has been shown to be always less than the last term retained, and so taking the series to s terms and then to $s+1$ terms will give two fairly close limits between which the value of the integral must lie.

Bayes now combines the Stirling approximation (11) and the asymptotic expansion (12) and so gives an approximate value for Σ. This, taken with Rule II, leads directly to Rule III, which we will give in Bayes' own words:

"If nothing is known of an event but that it has happened l times and failed m times in $l+m$ or n trials, and from hence I judge that the probability of its happening in a single trial lies between $\frac{l}{n}+z$ and $\frac{l}{n}-z$, my chance to be right is greater than

$$\frac{\sqrt{2\pi lm}\,h}{\sqrt{2\pi lm}+2h\,(n^{\cdot 5}+n^{-\cdot 5})}\times\left[H-\frac{1}{\sqrt{\pi}}\cdot\frac{n+1}{n+2}\cdot\frac{\left(1-\frac{2k^2z^2}{n}\right)^{\frac{n}{2}+1}}{kz}\right],$$

and less than

$$\frac{\sqrt{2\pi lm}\,h}{\sqrt{2\pi lm}-2h\,(n^{\cdot 5}+n^{-\cdot 5})}\times\left[H-\frac{1}{\sqrt{\pi}}\cdot\frac{n+1}{n+2}\cdot\frac{\left(1-\frac{2k^2z^2}{n}\right)^{\frac{n}{2}+1}}{kz}\right.$$

$$\left.+\frac{1}{\sqrt{\pi}}\cdot\frac{n}{n+2}\cdot\frac{n+1}{n+4}\cdot\frac{\left(1-\frac{2k^2z^2}{n}\right)^{\frac{n}{2}+2}}{2k^3z^3}\right];"$$

where Bayes' $\qquad h=e^{\frac{1}{12}\left(\frac{1}{n}-\frac{1}{l}-\frac{1}{m}\right)-\frac{1}{360}\left(\frac{1}{n^3}-\frac{1}{l^3}-\frac{1}{m^3}\right)+\cdots},$

and $\qquad H=e^{\frac{2^2-1}{12n}-\frac{2^4-1}{360n^3}+\cdots},$

that is, H is the value of h^{-1} that results from putting $l=m=\frac{n}{2}$ and $k=\sqrt{\frac{n^3}{2lm}}$, so that $\sqrt{2}kz$ corresponds to our u.

The above result is obtained by substituting for R (Equation (9)) by the use of (11), and by noting that the real sum of the series in (10) lies between the sum of 2 and 3 terms of the expansion (12). The form given here will be found in Timerding's edition of Bayes' essays*. Bayes' own result is slightly different, owing to the use of K to signify $\frac{\pi}{2}$. But one correction is here made. I have written $\sqrt{2\pi lm}+2h\,(n^{\cdot 5}+n^{-\cdot 5})$, where Bayes had the erroneous value

$$\sqrt{2\pi lm}+h\,(n^{\cdot 5}+n^{-\cdot 5}).$$

This error does not appear to have been detected by Timerding.

It will be noticed that Bayes does not give many terms of the series in his result. He is wise here, for his limits to the value of the required probability are already so wide that it is of little use proceeding to a closer approximation to the area of the auxiliary symmetrical curve.

* *Versuch zur Lösung eines Problems der Wahrscheinlichkeitsrechnung*, von Thomas Bayes. Oswald's series of *Klassiker*, Leipzig, 1908.

Let us take an example. Suppose $l=56, m=44$, so that $n=100$; further let $z=\cdot 144$; therefore $kz=2\cdot 0512904$. We find

$$h = \cdot 9974546,$$
$$H = 1\cdot 00250{,}3086.$$

Substituting in Bayes' Rule III we have this result; if an event has happened 56 times and failed 44 times in 100 trials, the chance that the probability of its happening in a single trial will lie between ·416 and ·704 is greater than ·85795, and less than 1·18989. Now the required chance cannot be greater than unity, but we see that values greater than this are not excluded from Bayes' results. We suspect that this is why Price, in one of the examples he worked, only gave the *lower* limit to the answer. (See p. 416 of the first Essay.) In the next volume of the *Philosophical Transactions* (Vol. LIV.), Price does, in fact, endeavour to improve on the above limits, and after a great deal of involved reasoning, reaches the conclusion that the chance required must be less than 2Σ, and greater than

$$\Sigma\left(1+\frac{1-2\left(1+\frac{1}{n}\right)\frac{n!}{l!\,m!}a^l b^m}{1+\delta\left(1+\frac{1}{n}\right)\frac{n!}{l!\,m!}a^l b^m}\right),$$

where he takes $\delta=1$ for small l and m, $\delta=\frac{1}{2}$ if either l or m be greater than 10, and $\delta=\frac{1}{10}$ in an example where $l=100$, $m=1000$. He might equally well have taken $\delta=0$, provided l and m were reasonably large, and the integration were not carried too far from the mode. This would mean that the chance must lie between 2Σ and

$$2\Sigma\left\{1-\left(1+\frac{1}{n}\right)\frac{n!}{l!\,m!}a^l b^m\right\}.$$

As it is, Price's result determines the limits of our chance to be ·90250 and ·99975. But since his day we have considerably improved our methods of quadrature for this particular skew curve, and a comparison of the above results with others reached by more modern methods will not be without interest. Bayes tried to minimise his error by determining the area between two ordinates equidistant from the mode. When this is done a good result can be obtained by replacing the skew curve by a symmetrical one, although it may be a poor fit on one side or the other. A positive error on one side of the mode is partially neutralised by a negative error on the other, and as long as we are not integrating over too great a distance from the mode, the net error is small. But unfortunately this will not do. In actual practice we require the areas, or the frequencies, on either side of the mode separately and not together. The following table shows the numerical results obtained for our example by approximating (1) with a normal curve using Sheppard's Tables, (2) with a symmetrical (Type II) curve, by the method given in *Biometrika*, Vol. XVII. p. 470, (3) with a short series of Incomplete Normal Moment Functions, after the method described by H. E. Soper in *Tracts for Computers*, No. VII. p. 44 and (4) with a formula of

Numerical Integration (Weddle—24 ordinates). The area is given first for one side of the mode only and then for both sides.

Method	Chance of probability being between	
	·56 and ·704	·416 and ·704
Normal curve	·49838	·99675
Type II (2 terms only)	·49855	·99710
Moment functions (2 terms only)	·492405	·997106
Weddle's Rule (correct to number of figures shown)	·4925697	·9971041

Leaving Bayes and returning to our equation (12), we have the following asymptotic expansion for the symmetrical probability integral (Pearson Types II and VII):

$$I_x(\tfrac{1}{2}n+1, \tfrac{1}{2}n+1) = 1 - \frac{\Gamma\{\frac{1}{2}(n+1)\}}{\sqrt{n\pi}\,\Gamma(\frac{1}{2}n)} \cdot \frac{n+1}{n+2} \cdot \frac{1}{u}\left(1-\frac{u^2}{n}\right)^{\frac{n}{2}+1}$$
$$\times\left[1 - \frac{n}{n+4}\cdot\frac{1}{u^2}\left(1-\frac{u^2}{n}\right) + \frac{n^2}{(n+4)(n+6)}\cdot\frac{1.3}{u^4}\left(1-\frac{u^2}{n}\right)^2 - \frac{n^3}{(n+4)(n+6)(n+8)}\,\frac{1.3.5}{u^6}\left(1-\frac{u^2}{n}\right)^3 + \dots\right] \quad \text{......(13)},$$

where $u = 2\sqrt{n}\,(x-\frac{1}{2})$.

If n tends towards infinity $\left(1-\frac{u^2}{n}\right)^{\frac{n}{2}}$ becomes, in the limit, $e^{-\frac{1}{2}u^2}$, and the above expression then becomes the well-known asymptotic expansion for the Error Function, which is

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{u} e^{-\frac{1}{2}t^2}\,dt = 1 - \frac{1}{\sqrt{2\pi}}\,\frac{e^{-\frac{1}{2}u^2}}{u} \times \left[1 - \frac{1}{u^2} + \frac{1.3}{u^4} - \frac{1.3.5}{u^6} + \dots\right] \text{...(14)}.$$

Cf. Whittaker and Robinson: *Calculus of Observations*, p. 181. This expression is the $\frac{1}{2}(1+\alpha_u)$ of Sheppard's Tables*.

The formula (13) is due, as we have seen, to Bayes, and is very useful so far as it goes. For example, let us take $n = 100$, $x = \cdot 65$. This is the example worked out by other methods in *Biometrika*, Vol. XVII. pp. 71—72 and also on p. 472.

We have $$u = 2\sqrt{n}\,(x-\tfrac{1}{2}) = 3.$$

Then

$$I_{\cdot 65}(51, 51) = 1 - \frac{\Gamma(50{\cdot}5)}{10\sqrt{\pi}\,\Gamma(50)}\cdot\frac{101}{102}\cdot\frac{1}{3}\cdot(\cdot 91)^{51}$$
$$\left\{1 - \frac{100}{104}\cdot\frac{1}{9}(\cdot 91) + \frac{100^2}{104.106}\cdot\frac{3}{81}(\cdot 91)^2 + \dots\right\}.$$

* *Tables for Statisticians*, pp. 2—8.

The series, using 5 terms, amounts to ·919, and using 6 terms to ·926. After the 6th term the terms of the series begin to increase.

So using 5 terms $\qquad I_{.65}(51, 51) = 1 - \cdot 001\,0704 \times \cdot 919$

$\qquad\qquad\qquad\qquad = \cdot 999\,0163;$

using 6 terms $\qquad I_{.65}(51, 51) = \cdot 999\,0088.$

These two results give fairly close limits between which the required area must lie. Actually the answer, correct to seven places of decimals, is ·999 0127, while the arithmetic mean of the two results we have given is ·999 0126. But this may be regarded as fortuitous, and we are probably not justified in retaining more than six places of decimals, owing to the size of the terms neglected.

This example shows the limit of application of the formula under discussion, i.e. just at the point where modal expansions begin to fail, our new expansion comes in, and thereafter onwards to the "tail" (13) becomes increasingly accurate. We proceed to show now how (13) can be improved, and when this is done we shall have a complete solution to the problem of determining the probability integrals for symmetrical frequency curves over the entire range, when n is large.

In practice the expansion (14) for the Error Function is seldom used by the statistician, owing to the slowness with which the successive terms decrease. A much better formula is due to Schlömilch*, and is as follows:

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{u} e^{-\frac{1}{2}t^2}\,dt = 1 - \frac{1}{\sqrt{2\pi}}\frac{e^{-\frac{1}{2}u^2}}{u}$$
$$\times\left[1 - \frac{1}{u^2+2} + \frac{1}{(u^2+2)(u^2+4)} - \frac{5}{(u^2+2)(u^2+4)(u^2+6)}\right.$$
$$\left. + \frac{9}{(u^2+2)(u^2+4)(u^2+6)(u^2+8)} - \dots\right] \quad\dots\dots(15).$$

For suitable values of u the terms of this series decrease very rapidly, and two or three are very often enough to enable us to determine the integral with great accuracy. The numerators of the next two terms after those given are 129 and 57.

To obtain a corresponding formula in inverse factorials for the Symmetrical Beta Integral, we proceed as follows:

$$\int_u^{\sqrt{n}}\left(1-\frac{v^2}{n}\right)^{\frac{n}{2}} dv = \tfrac{1}{2}\int_z^n\left(1-\frac{w}{n}\right)^{\frac{n}{2}}\frac{dw}{\sqrt{w}}, \text{ on putting } v^2 = w,$$
$$= \tfrac{1}{2}\int_0^{n-z}\left(1-\frac{z+t}{n}\right)^{\frac{n}{2}}\frac{dt}{\sqrt{z+t}}, \text{ by } w = z+t,$$
$$= \tfrac{1}{2}\left(1-\frac{z}{n}\right)^{\frac{n}{2}}\int_0^{n-z}\left(1-\frac{t}{n-z}\right)^{\frac{n}{2}}\frac{dt}{\sqrt{z+t}} \quad\dots\dots\dots(16).$$

* *Zeitschrift für Mathematik und Physik*, Bd. IV. (1859), p. 401. The form is easily seen to be equivalent to that given above on putting $x^2 = \frac{1}{2}u^2$.

Now just as Schlömilch expanded $\dfrac{1}{x+t}$ in a series of inverse factorials, disposing satisfactorily of the remainder*, so we can expand $\dfrac{1}{\sqrt{z+t}}$ in the same way. We have, in fact,

$$\frac{1}{\sqrt{z+t}}=\frac{1}{\sqrt{z}}\left[1-\frac{\frac{1}{2}t}{z+2}+\frac{\frac{3}{8}t^2-t}{(z+2)(z+4)}-\frac{\frac{5}{16}t^3-\frac{9}{4}t^2+4t}{(z+2)(z+4)(z+6)}\right.$$
$$\left.+\frac{\frac{35}{128}t^4-\frac{15}{4}t^3+\frac{33}{2}t^2-24t}{(z+2)(z+4)(z+6)(z+8)}-\dots\right].$$

Substituting in (16) and remembering that $z=u^2$, we have

$$\int_u^{\sqrt{n}}\left(1-\frac{v^2}{n}\right)^{\frac{n}{2}}dv=\frac{1}{2u}\left(1-\frac{u^2}{n}\right)^{\frac{1}{2}n}\int_0^{n-z}\left(1-\frac{t}{n-z}\right)^{\frac{n}{2}}$$
$$\times\left(1-\frac{\frac{1}{2}t}{z+2}+\frac{\frac{3}{8}t^2-t}{(z+2)(z+4)}-\dots\right)dt.$$

Now let $\dfrac{t}{n-z}=y$,

$$\int_u^{\sqrt{n}}\left(1-\frac{v^2}{n}\right)^{\frac{n}{2}}dv=\frac{n}{2u}\left(1-\frac{u^2}{n}\right)^{\frac{1}{2}n+1}\times\left[\int_0^1(1-y)^{\frac{n}{2}}dy-\frac{n\left(1-\frac{u^2}{n}\right)}{2(u^2+2)}\int_0^1 y(1-y)^{\frac{n}{2}}dy\right.$$
$$\left.+\left\{\frac{3}{8}\cdot\frac{n^2\left(1-\frac{u^2}{n}\right)^2}{(u^2+2)(u^2+4)}\int_0^1 y^2(1-y)^{\frac{n}{2}}dy-\frac{n\left(1-\frac{u^2}{n}\right)}{(u^2+2)(u^2+4)}\int_0^1 y(1-y)^{\frac{n}{2}}dy\right\}-\dots\right]$$
$$=\frac{n}{n+2}\cdot\frac{1}{u}\left(1-\frac{u^2}{n}\right)^{\frac{1}{2}n+1}\times\left[1-\frac{n}{n+4}\cdot\frac{1-\frac{u^2}{n}}{u^2+2}\right.$$
$$+\frac{n}{n+4}\cdot\frac{1-\frac{u^2}{n}}{(u^2+2)(u^2+4)}\left\{\frac{1.3n}{n+6}\left(1-\frac{u^2}{n}\right)-2\right\}$$
$$-\frac{n}{n+4}\cdot\frac{1-\frac{u^2}{n}}{(u^2+2)(u^2+4)(u^2+6)}\left\{\frac{1.3.5n^2}{(n+6)(n+8)}\left(1-\frac{u^2}{n}\right)^2-\frac{1.3.6n}{n+6}\left(1-\frac{u^2}{n}\right)+8\right\}$$
$$+\frac{n}{n+4}\cdot\frac{1-\frac{u^2}{n}}{(u^2+2)(u^2+4)(u^2+6)(u^2+8)}\left\{\frac{1.3.5.7n^3}{(n+6)(n+8)(n+10)}\left(1-\frac{u^2}{n}\right)^3\right.$$
$$\left.\left.-\frac{1.3.5.12n^2}{(n+6)(n+8)}\left(1-\frac{u^2}{n}\right)^2+\frac{1.3.44n}{n+6}\left(1-\frac{u^2}{n}\right)-48\right\}-\dots\right]\dots\dots(17).$$

Since all else remains the same, we have only to substitute the series in square brackets in (17) for that in (13) to reach a new formula for our Integral. If we let $n\to\infty$ the above result reduces simply to the form given by Schlömilch for the Error Function (Equation (15) above).

* *Loc. cit.* pp. 394 and 395.

The general term of the series may be written down without much difficulty For the $(p+1)$th term we have

$$\frac{(-1)^p}{(u^2+2)(u^2+4)\dots(u^2+2p)}\left\{\frac{1.3.5\dots(2p-1)\,n^p\left(1-\frac{u^2}{n}\right)^p}{(n+4)(n+6)\dots\{n+2(p+1)\}}\right.$$

$$-\frac{1.3\dots(2p-3)\,p\,(p-1)\,n^{p-1}\left(1-\frac{u^2}{n}\right)^{p-1}}{(n+4)(n+6)\dots(n+2p)}$$

$$+\frac{1.3\dots(2p-5)\,p\,(p-1)(p-2)(3p-1)\,n^{p-2}\left(1-\frac{u^2}{n}\right)^{p-2}}{6(n+4)(n+6)\dots\{n+2(p-1)\}}$$

$$-\frac{1.3\dots(2p-7)\,p^2(p-1)^2(p-2)(p-3)\,n^{p-3}\left(1-\frac{u^2}{n}\right)^{p-3}}{6(n+4)(n+6)\dots\{n+2(p-2)\}}+\dots$$

$$\left.+(-1)^{p-1}\frac{2.4.6\dots(2p-2)\,n\left(1-\frac{u^2}{n}\right)}{n+4}\right\}.$$

The terms of the series decrease very rapidly, and it appears to be the best expansion we can get for the case of u being greater than unity. For the example already worked (page 13), we have for the first few terms of the series

$$1, \quad -\cdot07955, \quad +\cdot00352, \quad -\cdot00139,$$

giving for $I_{\cdot 65}(51, 51)$ the following approximations:

$$\left.\begin{array}{l}\left.\begin{array}{ll}(1) & \cdot999015\\ (2) & \cdot999011\end{array}\right\}\text{average }\cdot999013\\ \begin{array}{ll}(3) & \cdot9990125\end{array}\end{array}\right\}\text{average }\cdot9990128,$$

the last of which is only 2 out in the seventh decimal place.

From the point of view, however, of the computer, the series (17) is unwieldy. It is possible to modify it so that the coefficients become simple functions of n, which can be tabulated, even although we are no longer able to give the general term. The result is

$$I_x(\tfrac{1}{2}n+1, \tfrac{1}{2}n+1)=1-\frac{\Gamma\{\frac{1}{2}(n+1)\}}{\sqrt{n\pi}\,\Gamma(\frac{1}{2}n)}\cdot\frac{1}{u}\left(1-\frac{u^2}{n}\right)^{\frac{n}{2}+1}$$

$$\times\left[1-\frac{a_1}{u^2+2}+\frac{a_2}{(u^2+2)(u^2+4)}-\frac{a_3}{(u^2+2)(u^2+4)(u^2+6)}+\dots\right]\dots(18),$$

where $u=2\sqrt{n}\,(x-\tfrac{1}{2})$, and

$$a_1=\frac{n}{n-1},$$

$$a_2=1+\frac{10}{n}+\frac{37}{n^2}+\frac{118}{n^3}+\dots,$$

$$a_3=5+\frac{71}{n}+\frac{644}{n^2}+\frac{4238}{n^3}+\dots,$$

$$a_4 = 9 + \frac{540}{n} + \frac{9078}{n^2} + \frac{105432}{n^3} + \ldots,$$

$$a_5 = 129 + \frac{4509}{n} + \frac{122859}{n^2} + \frac{2249259}{n^3} + \ldots,$$

$$a_6 = 57 + \ldots.$$

It is to be noted that for $n \to \infty$ the a's become the Schlömilch numbers. His general coefficient a_p' is given by

$$a_p' = 1.3.5\ldots(2p-1) + 2C_1\,1.3.5\ldots(2p-3)$$
$$+ 2^2.\,C_2\,1.3.5\ldots(2p-5) + \ldots + 2^{p-1}\,C_{p-1} \quad\ldots\ldots(19),$$

where C_s is the coefficient of t^{p-s} in the expansion of $t(t-1)(t-2)\ldots(t-p+1)$. The numbers determined by the terms of (19), for different values of p, give the numerical coefficients of the terms in curled brackets of (17), but each is in this case modified by the introduction of a function of n and $\left(1 - \frac{u^2}{n}\right)$.

The Gamma Functions may be removed altogether from equation (18). Since

$$\frac{\Gamma\{\frac{1}{2}(n+1)\}}{\sqrt{n\pi}\,\Gamma(\frac{1}{2}n)} = \frac{1}{\sqrt{2\pi}} \times \left(1 - \frac{1}{4n} + \frac{1}{32n^2} + \frac{5}{128n^3} - \ldots\right)$$

(see *Biometrika*, Vol. XVII. p. 69), we have as our final form:

$$I_x(\tfrac{1}{2}n+1, \tfrac{1}{2}n+1) = 1 - \frac{1}{u}\left(1 - \frac{u^2}{n}\right)^{\frac{n}{2}+1}$$
$$\times \left[A_0 - \frac{A_1}{u^2+2} + \frac{A_2}{(u^2+2)(u^2+4)} - \frac{A_3}{(u^2+2)(u^2+4)(u^2+6)} + \ldots\right] \ldots(20),$$

where
$$A_0 = \frac{1}{\sqrt{2\pi}}\left(1 - \frac{1}{4n} + \frac{1}{32n^2} + \frac{5}{128n^3}\right),$$

$$A_1 = \frac{1}{\sqrt{2\pi}}\left(1 + \frac{3}{4n} + \frac{25}{32n^2} + \frac{105}{128n^3}\right),$$

$$A_2 = \frac{1}{\sqrt{2\pi}}\left(1 + \frac{39}{4n} + \frac{1105}{32n^2} + \frac{13965}{128n^3}\right),$$

$$A_3 = \frac{1}{\sqrt{2\pi}}\left(5 + \frac{279}{4n} + \frac{20045}{32n^2} + \frac{522165}{128n^3}\right),$$

$$A_4 = \frac{1}{\sqrt{2\pi}}\left(9 + \frac{2151}{4n} + \frac{286185}{32n^2} + \frac{13207005}{128n^3}\right),$$

$$A_5 = \frac{1}{\sqrt{2\pi}}\left(129 + \frac{17907}{4n} + \frac{3895545}{32n^2} + \frac{283992345}{128n^3}\right).$$

Generally, let
$$A_s = {}_1A_s + \frac{{}_2A_s}{n} + \frac{{}_3A_s}{n^2} + \ldots.$$

The coefficients ${}_1A_s$, ${}_2A_s$, ${}_3A_s$ have been tabulated (see Table I for $r = 0$). The reciprocals of powers of n up to the third have been calculated for values of n from 50 to 101 by 1, and from 101 to 200 at intervals of 2, and appear in Table II.

With the aid of these tables the computer should be able to calculate a number of terms of the expansion (20) in a short space of time. One point here is worth noting. The contributions from the successive terms seem, on our experience, to come out roughly of the same order in pairs. That is, the terms in A_2 and A_3 are not very different, then there is a drop and the terms in A_4 and A_5 are again not very different, and so on. This is due to the peculiar values of the Schlömilch coefficients. So if the computer is not satisfied with the value of the series as given by the terms up to A_3, he should go on and take in the next *two* terms, involving A_4 and A_5. It should not be necessary to go beyond this point.

Applied to the same example as before:

$$I_{\cdot 65}(51, 51) = 1 - \tfrac{1}{3}(\cdot 91)^{51} \times \begin{cases} \ 0\cdot 39795 \\ \ \cdot 03654 \\ + \ \cdot 00307 \\ - \ \cdot 00107 \\ + \ \cdot 00017 \\ - \ \cdot 00010. \end{cases}$$

The next term is not significant in the fifth place.

Using three terms only beyond the unity term gives

$$I_{\cdot 65}(51, 51) = \cdot 9990129,$$

or, using all five, $\quad = \cdot 9990127,$

which is correct to the number of figures shown.

My example is not by any means a favourable one. It was chosen in the first place to illustrate the modal expansions of my earlier paper, so the u, in practice, is more likely to be greater than the value in this example than less. Again, since the *Tables of the Incomplete Beta Function* will ultimately extend to $p = q = 50$, the value of n chosen (100) is not unduly large. Whole numbers were taken purely for ease in checking.

Section C. The formula (20), taken in conjunction with that given in *Biometrika*, Vol. XVII. p. 470, Equation (4), provides a complete solution to the problem of the determination of the probability integrals of symmetrical frequency curves, when n is large. I shall now endeavour to generalise these results by giving a similar solution for the much larger class of skew curves of Type I. We have seen how Bayes attempted to do this by expressing his area in terms of the corresponding area of the symmetrical curve, diminishing his risk of error by going from an ordinate on one side of the mode to one at the same distance on the other side of the mode. Even so his approximation is a poor one. The problem is not easy, for complexities are introduced as soon as we pass from the symmetrical to the asymmetrical curve. Formulae of the reduction by parts type are numerous, and are applicable to various cases, but in general a very large number of terms are needed and the process is long. H. E. Soper, in *Tracts for Computers*, No. VII. has discussed these, and has also drawn attention to two methods which are of particular interest when the indices are large. These are (a) the method of

expansion in terms of Incomplete Normal Moment Functions, and (*b*) the method of expansion in a series of Incomplete Gamma Functions. We shall have something to say about each of these methods later, but the present section will be occupied with an attempt at generalising the asymptotic series for the Symmetrical Function, which was developed in the previous section.

From our earlier work we have, if $n = l + m$:

$$I_x(l+1, m+1) = \frac{\Gamma(n+2)}{\Gamma(l+1)\Gamma(m+1)} \left(\frac{l}{n}\right)^l \left(\frac{m}{n}\right)^m \int_{-\frac{l}{n}}^{z} \left(1 + \frac{nz}{l}\right)^l \left(1 - \frac{nz}{m}\right)^m dz,$$

z being the distance from the mode of the ordinate at x. We then put $u = \sqrt{\frac{n^3}{lm}}\, z$ and approximated to the part outside the Integral by Stirling's theorem (see Equation (11)).

On expansion we have

$$e^{\frac{1}{12}\left(\frac{1}{n} - \frac{1}{l} - \frac{1}{m}\right) - \frac{1}{360}\left(\frac{1}{n^3} - \frac{1}{l^3} - \frac{1}{m^3}\right) + \dots} = 1 - \frac{1+r}{12n} + \frac{(1+r)^2}{288n^2} + \dots,$$

where
$$r = \frac{l}{m} + \frac{m}{l}.$$

For ordinary purposes sufficient accuracy is obtained by neglecting the terms beyond that in $\frac{1}{n^2}$, which leaves us with an expression which is very easy to compute. Only when r is very large will the next term be of any significance. Actually the term in $\frac{1}{n^3}$ is

$$+\frac{139r^3 + 417r^2 - 15r - 725}{51840n^3}.$$

Suppose $r = 10$, $n = 100$, then this contribution to the series is only $\cdot 0000035$.

Our formula now becomes

$$I_x(l+1, m+1) = \frac{1}{\sqrt{2\pi}} \left(1 + \frac{1}{n}\right) \left\{1 - \frac{1+r}{12n} + \frac{(1+r)^2}{288n^2}\right\}$$
$$\times \int_{-\sqrt{\frac{nl}{m}}}^{u} \left(1 + \sqrt{\frac{m}{nl}}\, u\right)^l \left(1 - \sqrt{\frac{l}{nm}}\, u\right)^m du \quad \text{.........(21)},$$

where
$$u = \sqrt{\frac{n^3}{lm}} \left(x - \frac{l}{n}\right) = \sqrt{(r+2)}\, n \left(x - \frac{l}{n}\right).$$

We shall now take the integral part into consideration. Since

$$\frac{d}{du}\left\{\left(1 + \sqrt{\frac{m}{nl}}\, u\right)^{l+1} \left(1 - \sqrt{\frac{l}{nm}}\, u\right)^{m+1}\right\}$$
$$= -\frac{n+2}{n}\left(u + \frac{n\alpha}{n+2}\right)\left(1 + \sqrt{\frac{m}{nl}}\, u\right)^l \left(1 - \sqrt{\frac{l}{nm}}\, u\right)^m,$$

where $\alpha = \dfrac{l-m}{\sqrt{lmn}}$, we have

$$\int_u^{\sqrt{\frac{nm}{l}}} \left(1+\sqrt{\frac{m}{nl}}\,u\right)^l \left(1-\sqrt{\frac{l}{nm}}\,u\right)^m du$$

$$= -\frac{n}{n+2}\int_u^{\sqrt{\frac{nm}{l}}} \frac{1}{u+\dfrac{n\alpha}{n+2}}\, d\left\{\left(1+\sqrt{\frac{m}{nl}}\,u\right)^{l+1}\left(1-\sqrt{\frac{l}{nm}}\,u\right)^{m+1}\right\}$$

$$= \frac{n}{n+2}\,\frac{\left(1+\sqrt{\dfrac{m}{nl}}\,u\right)^{l+1}\left(1-\sqrt{\dfrac{l}{nm}}\,u\right)^{m+1}}{u+\dfrac{n\alpha}{n+2}}$$

$$-\frac{n}{n+2}\int_u^{\sqrt{\frac{nm}{l}}} \frac{\left(1+\sqrt{\dfrac{m}{nl}}\,u\right)^{l+1}\left(1-\sqrt{\dfrac{l}{mn}}\,u\right)^{m+1}}{\left(u+\dfrac{n\alpha}{n+2}\right)^2}\,du.$$

Proceeding as before, we have for the first few terms of the series

$$\frac{n}{n+2}\,\frac{\left(1+\sqrt{\dfrac{m}{nl}}\,u\right)^{l+1}\left(1-\sqrt{\dfrac{l}{nm}}\,u\right)^{m+1}}{u+\dfrac{n\alpha}{n+2}} \times \left[1-\frac{n}{n+4}\cdot\frac{1-\dfrac{u^2}{n}-\alpha u}{\left(u+\dfrac{n\alpha}{n+2}\right)\left(u+\dfrac{2n\alpha}{n+4}\right)}\right.$$

$$\left.+\frac{1\,.\,3\,.\,n^2}{(n+4)(n+6)}\cdot\frac{\left\{u+\dfrac{n(5n+12)\alpha}{3(n+2)(n+4)}\right\}\left(1-\dfrac{u^2}{n}-\alpha u\right)^2}{\left(u+\dfrac{n\alpha}{n+2}\right)^2\left(u+\dfrac{2n\alpha}{n+4}\right)^2\left(u+\dfrac{3n\alpha}{n+6}\right)}-\dots\right] \quad \text{......(22).}$$

On putting $l=m$, so that $\alpha=0$, this series reduces to that already given for the symmetrical integral in (13). We have therefore effected a generalisation of the earlier solution, but no one who contemplates it will feel very happy about the result. More terms have been worked out, but it does not seem worth while to give them here for two reasons, (a) there seems to be no simple law of formation of successive terms, and the actual terms will be very difficult to compute, (b) the formula (13) for the symmetrical integral was abandoned owing to the slowness with which the terms decreased, and the present formula also suffers from the same defect, and will have to be replaced by another. But first let us note that if Bayes had found this expansion and had used, say, only the first two terms, he would have reached a far better result than he did. For he would have abandoned the "middle" curve as an approximation to what he was seeking and would be performing a quadrature of the actual skew curve. To illustrate with the same example as we took earlier, we have

$$l=56,\quad m=44,\quad \alpha=\cdot 0241747,\quad u=2{\cdot}9009627.$$

$$I_{\cdot 704}(57,\,45)=1-\frac{1}{\sqrt{2\pi}}\cdot\frac{n+1}{n+2}\cdot\left(1-\frac{1+r}{12n}\right)\frac{\left(1+\sqrt{\frac{m}{nl}}\,u\right)^{l+1}\left(1-\sqrt{\frac{l}{nm}}\,u\right)^{m+1}}{u+\frac{n\alpha}{n+2}}$$

$$\times\left[1-\frac{n}{n+4}\cdot\frac{1-\frac{u^2}{n}-\alpha u}{\left(u+\frac{n\alpha}{n+2}\right)\left(u+\frac{2n\alpha}{n+4}\right)}\right],$$

approximately.

With these values substituted and $r=2{\cdot}058442$, we have

$$I_{\cdot 704}(57,\,45)=1-{\cdot}00101={\cdot}99899.$$

The actual result can be found by an application of the reduction by parts formula and is $1-{\cdot}0010301={\cdot}9989699$, correct to 7 places.

Thus, for a case where the curve is moderately skew, a very good result can be obtained by retaining only one term beyond the unity term in (22). Bayes would have inferred from this answer that the chance for the probability of the event being greater than ·704 was ·00101, and he would have been right within 2 per cent. of error. Similarly he could, if desired, find the chance for the probability being less than ·416, or some other figure. This he would accomplish by exchanging l and m in the above formula, so that α would become negative, while u, r and n would remain the same.

But a better result still is to be obtained from an expansion in inverse factorials, analogous to that obtained from the symmetrical function. Reference to equations (13) and (17) will show that, at any rate as far as the first term of the series is concerned, the improved formula was obtained by replacing $\frac{1}{u^2}$ by the first term of its expansion in inverse factorials. In fact

$$\frac{1}{u^2}=\frac{1}{u^2+2}+\frac{2}{(u^2+2)(u^2+4)}+\frac{8}{(u^2+2)(u^2+4)(u^2+6)}+\dots,$$

which can be obtained by evaluating the convergents, up to any desired point, of the continued fraction

$$\frac{1}{u^2}=\frac{1}{u^2+r-}\;\frac{r(u^2+r)}{u^2+2r}\;-\;\frac{r(u^2+2r)}{u^2+3r}\;-\text{etc.},$$

for the particular case of $r=2$.

A first approximation to the result we are seeking can therefore be written down without further trouble. We have

$$I_x(l+1,\,m+1)=1-\frac{1}{\sqrt{2\pi}}\cdot\frac{n+1}{n+2}\cdot\left\{1-\frac{1+r}{12n}+\frac{(1+r)^2}{288n^2}\right\}\frac{1}{u+\frac{n\alpha}{n+2}}$$

$$\times\left(1+\sqrt{\frac{m}{nl}}\,u\right)^{l+1}\left(1-\sqrt{\frac{l}{nm}}\,u\right)^{m+1}$$

$$\times\left[1-\frac{n}{n+4}\cdot\frac{1-\dfrac{u^2}{n}-\alpha u}{\left(1+\dfrac{n\alpha}{(n+2)u}\right)\left(1+\dfrac{2n\alpha}{(n+4)u}\right)(u^2+r)}\right]\quad\ldots\ldots(23).$$

In this formula
$$n=l+m,\quad r=\frac{l}{m}+\frac{m}{l},$$
$$\alpha=\frac{l-m}{\sqrt{lmn}}=\pm\sqrt{\frac{r-2}{n}},$$
$$u=\sqrt{(r+2)\,n}\left(x-\frac{l}{n}\right).$$

Applied to the case of $l=56$, $m=44$, $x=\cdot 704$, we have

$$I_{\cdot 704}(57,45)=1-\cdot 0011146\,(1-\cdot 07579)$$
$$=1-\cdot 0010301=\cdot 9989699,$$

or we have the surprising result that an answer, correct to 7 decimal places, can be obtained by considering only one term of the series beyond the unity term.

For a second example consider $l=64$, $m=36$. If we take $x=\cdot 784$ we have $\alpha=\cdot 05833$, $r=2\cdot 34028$, $u=3$.

Then $I_{\cdot 784}(65,37)=1-\cdot 000402=\cdot 999598$, the correct result (by a reduction by parts formula) being $\cdot 999599$.

More terms in (23) can be worked out, but as the resulting expression is rather complicated in character we prefer to modify the result, and present a series better suited to rapid computation. This can be done in more than one way. The simplest method, although it is not absolutely rigorous mathematically, is as follows, since:

$$\frac{d}{du}\left\{\left(1+\sqrt{\frac{m}{nl}}\,u\right)^{l}\left(1-\sqrt{\frac{l}{nm}}\,u\right)^{m}\right\}=-u\left(1+\sqrt{\frac{m}{nl}}\,u\right)^{l-1}\left(1-\sqrt{\frac{l}{nm}}\,u\right)^{m-1}$$
$$=-\frac{u}{1-\dfrac{u^2}{n}-\alpha u}\left(1+\sqrt{\frac{m}{nl}}\,u\right)^{l}\left(1-\sqrt{\frac{l}{nm}}\,u\right)^{m},$$

we have

$$\int_u^{\sqrt{\frac{nm}{l}}}\left(1+\sqrt{\frac{m}{nl}}\,u\right)^{l}\left(1-\sqrt{\frac{l}{nm}}\,u\right)^{m}du$$
$$=-\int_u^{\sqrt{\frac{nm}{l}}}\frac{1-\dfrac{u^2}{n}-\alpha u}{u}\,d\left\{\left(1+\sqrt{\frac{m}{nl}}\,u\right)^{l}\left(1-\sqrt{\frac{l}{nm}}\,u\right)^{m}\right\}$$
$$=\frac{\left(1+\sqrt{\dfrac{m}{nl}}\,u\right)^{l+1}\left(1-\sqrt{\dfrac{l}{nm}}\,u\right)^{m+1}}{u}-\int_u^{\sqrt{\frac{nm}{l}}}\frac{1+\dfrac{u^2}{n}}{u^2}\left(1+\sqrt{\frac{m}{nl}}\,u\right)^{l}\left(1-\sqrt{\frac{l}{nm}}\,u\right)^{m}du$$

$$= \frac{\left(1+\sqrt{\frac{m}{nl}}u\right)^{l+1}\left(1-\sqrt{\frac{l}{nm}}u\right)^{m+1}}{u}$$

$$+\int_u^{\sqrt{\frac{nm}{l}}} \frac{\left(1+\frac{u^2}{n}\right)\left(1-\frac{u^2}{n}-au\right)}{u^3}\, d\left\{\left(1+\sqrt{\frac{m}{nl}}u\right)^{l}\left(1-\sqrt{\frac{l}{nm}}u\right)^{m}\right\}$$

$$= \frac{\left(1+\sqrt{\frac{m}{nl}}u\right)^{l+1}\left(1-\sqrt{\frac{l}{nm}}u\right)^{m+1}}{u}\times\left(1-\frac{1+\frac{u^2}{n}}{u^2}\right),$$

$$+\int_u^{\sqrt{\frac{nm}{l}}} \frac{3-2au+\frac{u^4}{n^2}}{u^4}\left(1+\sqrt{\frac{m}{nl}}u\right)^{l}\left(1-\sqrt{\frac{l}{nm}}u\right)^{m} du.$$

Proceeding in this way we have, on making use of equation (21), the following result:

$$I_x(l+1, m+1) = 1 - \frac{1}{\sqrt{2\pi}}\left(1+\frac{1}{n}\right)\left\{1-\frac{1+r}{12n}+\frac{(1+r)^2}{288n^2}\right\}$$

$$\times\frac{\left(1+\sqrt{\frac{m}{nl}}u\right)^{l+1}\left(1-\sqrt{\frac{l}{nm}}u\right)^{m+1}}{u}$$

$$\times\left[1-\frac{1+\frac{u^2}{n}}{u^2}+\frac{3-2au+\frac{u^4}{n^2}}{u^4}-\frac{15-20au+6a^2u^2-\frac{9u^2}{n}+\frac{4au^3}{n}+\frac{u^4}{n^2}+\frac{u^6}{n^3}}{u^6}\right.$$

$$+\frac{105-210au+130a^2u^2-120\frac{u^2}{n}-24a^3u^3+132\frac{au^3}{n}+30\frac{u^4}{n^2}-30\frac{a^2u^4}{n}-10\frac{au^5}{n^2}+\frac{u}{n^4}}{u^8}$$

$$\left. - \ldots\ldots\right] \qquad\ldots\ldots\ldots\ldots(24).$$

The terms of this series increase enormously in complexity as we go on. For example the term in $\frac{1}{u^{10}}$ contains 16 terms in the numerator, while that in $\frac{1}{u^{12}}$ contains 21 terms. These have been worked out, but are not given here. Examination of (24) will show that the series is composed of an odd and an even part. We multiply out by $1+\frac{1}{n}$, and substitute for $\frac{1}{u^2}$ the series in inverse factorials which begins with $\frac{1}{u^2+r}$, and for $\frac{1}{u^4}$ a similar series beginning with $\frac{1}{(u^2+r)(u^2+2r)}$, and so on. On collecting terms and replacing a by $\sqrt{\frac{r-2}{n}}$, we reach finally the following result:

$$I_x(l+1, m+1) = 1 - \left(1 + \sqrt{\frac{m}{nl}}\,u\right)^{l+1}\left(1 - \sqrt{\frac{l}{nm}}\,u\right)^{m+1}$$
$$\times\left[\frac{1}{u}\left\{B_0 - \frac{B_1}{u^2+r} + \frac{B_2}{(u^2+r)(u^2+2r)} - \frac{B_3}{(u^2+r)(u^2+2r)(u^2+3r)} + \ldots\right\}\right.$$
$$\left. - \left\{\frac{C_1}{(u^2+r)(u^2+2r)} - \frac{C_2}{(u^2+r)(u^2+2r)(u^2+3r)} + \ldots\right\}\right] \quad \ldots\ldots(25),$$

where u and r have their old values, and

$$B_0 = \frac{1}{\sqrt{2\pi}}\left(1 - \frac{1+r}{12n} + \frac{(1+r)^2}{288n^2} + \ldots\right),$$

$$B_1 = \frac{1}{\sqrt{2\pi}}\left(1 + \frac{11-r}{12n} + \frac{265 - 22r + r^2}{288n^2} + \ldots\right),$$

$$B_2 = \frac{1}{\sqrt{2\pi}}\left(3 - r + \frac{285 - 86r + r^2}{12n} + \frac{31395 - 11059r + 169r^2 - r^3}{288n^2} + \ldots\right),$$

$$B_3 = \frac{1}{\sqrt{2\pi}}\left(15 - 9r + 2r^2 + \frac{4725 - 2430r + 247r^2 - 2r^3}{12n}\right.$$
$$\left. + \frac{1295175 - 696675r + 70607r^2 - 485r^3 + 2r^4}{288n^2} + \ldots\right),$$

$$B_4 = \frac{1}{\sqrt{2\pi}}\left(105 - 90r + 33r^2 - 6r^3 + \frac{77175 - 57015r + 12585r^2 - 891r^3 + 6r^4}{12n}\right.$$
$$\left. + \frac{41435625 - 30302328r + 6122214r^2 - 346494r^3 + 1749r^4 - 6r^5}{288n^2} + \ldots\right),$$

$$B_5 = \frac{1}{\sqrt{2\pi}}\left(945 - 1050r + 525r^2 - 150r^3 + 24r^4\right.$$
$$\left. + \frac{1340955 - 1301895r + 452025r^2 - 69375r^3 + 4014r^4 - 24r^5}{12n} + \ldots\right);$$

$$C_1 = \sqrt{\frac{2(r-2)}{n\pi}}\left(1 + \frac{35-r}{12n} + \frac{1945 - 70r + r^2}{288n^2} + \ldots\right),$$

$$C_2 = \sqrt{\frac{2(r-2)}{n\pi}}\left(10 - 3r + \frac{1190 - 259r + 3r^2}{12n}\right.$$
$$\left. + \frac{184810 - 42487r + 508r^2 - 3r^3}{288n^2} + \ldots\right),$$

$$C_3 = \sqrt{\frac{2(r-2)}{n\pi}}\left(105 - 60r + 11r^2 + \frac{28875 - 12789r + 1309r^2 - 11r^3}{12n}\right.$$
$$\left. + \frac{9152385 - 3949698r + 356180r^2 - 2558r^3 + 11r^4}{288n^2} + \ldots\right),$$

$$C_4 = \sqrt{\frac{2(r-2)}{n\pi}}\left(1260 - 1050r + 350r^2 - 50r^3\right.$$
$$\left. + \frac{660660 - 448602r + 98140r^2 - 7140r^3 + 50r^4}{12n} + \ldots\right).$$

If we write
$$B_s = {}_1B_s + \frac{{}_2B_s}{n} + \frac{{}_3B_s}{n^2},$$
and
$$C_s \doteq \frac{{}_1C_s}{\sqrt{n}} + \frac{{}_2C_s}{n^{1\cdot5}} + \frac{{}_3C_s}{n^{2\cdot5}},$$
we have ${}_1B_0 = {}_1B_1 = 0{\cdot}39894228$, while the other twenty-six coefficients have been tabled for values of r from 2 to 10, going by ·05 to 2·2, then by ·1 to 2·6, and afterwards by intervals of ·2 (Table I). These, with the table of reciprocals of powers of n (Table II), will enable the computer with the minimum of labour to evaluate his integral subject to an error which may be estimated as less than the value of the last term he retains. Table II ought to prove useful for other purposes. A table exists in *Zehnstellige Logarithmen*, Bd. I. by Peters und Stein, Tafel 3, giving the values of $\frac{1}{n^s}$ for values of n up to 100 and for integral powers, but the writer does not know of any existing table giving $\frac{1}{\sqrt{n}}$ or higher fractional powers.

The advantage claimed for Equation (25) is that it should prove equally useful when the curve is nearly symmetrical, and when it is very skew. For although the coefficients become large with r, so do the denominators, since they are of the form $(u^2+r)(u^2+2r)\ldots$. Again the formula may be applied equally well at either end of the range. In Equation (22) α is positive when $l > m$, negative when $l < m$; in the latter case the successive factors of the denominators are continually decreasing, the reduction being considerable if the skewness is large. In (25), however, there is no such change. r is independent of the sign of $l-m$; to apply the formula to the case of $m > l$, the only changes will be in the part outside, and in the sign in front of the series of C's, which will become positive, owing to the change in sign of $\alpha = \pm\sqrt{\frac{r-2}{n}}$.

The only case in which it may not be profitable to apply (25) is where the ratio of the indices is so great that values of r outside the range given in Table I occur. The terms of the series, besides being difficult to compute, would then decrease very slowly, and an appreciable error might still remain, even if a large number of terms were included. But in such cases the form of the curve at the start approximates closely to that of a Type III curve, and a solution can be obtained in a series of Incomplete Gamma Functions, and is quickly and accurately reached so long as we do not go too far from the start of the curve. This case will be dealt with in detail later.

As was pointed out in the case of the symmetrical integral, the best approximations are obtained by going in jumps of two terms at a time; i.e. we may (*a*) stop at B_1, ignoring the C series (although we shall not get such a good result in general as we would with (23)), or (*b*) include terms up to B_3 and C_2, or (*c*) go up to B_5 and C_4. It would appear to be sufficient, for seven figure accuracy in the integral, to calculate the terms of the series to five places.

Numerical examples.

Example 1. Suppose: $l = 56$, $m = 44$, $x = {\cdot}704$.

Then $$r = 2{\cdot}058442, \quad u = 2{\cdot}90096, \quad u^2 = 8{\cdot}41558.$$

We have

$$I_{\cdot 704}(57, 45) = 1 - (1{\cdot}2571429)^{57}({\cdot}6727273)^{45}$$

$$\times\left[\frac{1}{2{\cdot}90096}\times\{{\cdot}39793 - {\cdot}03838 + {\cdot}00316 - {\cdot}00117\} - \{{\cdot}00015 - {\cdot}00004\}\right]$$

$$= 1 - (1{\cdot}2571429)^{57}({\cdot}6727273)^{45}({\cdot}12452).$$

Logs.

$5{\cdot}6649249$
$\bar{8}{\cdot}2527573$
$\bar{1}{\cdot}0952391$
$\bar{3}{\cdot}0129213$

therefore $$I_{\cdot 704}(57, 45) = 1 - {\cdot}0010302$$
$$= {\cdot}9989698,$$

which is only wrong by 1 in the seventh place. If we take in the next two terms our answer is found to be correct to within 2 in the eighth place. This shows that we are not introducing a large error by ignoring the terms after the fourth. In fact the value of the fifth term in the series of B's is ·00016 while the difference of the fifth and sixth terms is only ·00005.

Example 2. Let $l = 20$, $m = 80$. Take $z = {\cdot}12$ ($\sigma = {\cdot}04$), then $n = 100$, $r = 4{\cdot}25$, $u = 3$. We find

$$I_{\cdot 32}(21, 81) = 1 - (1{\cdot}6)^{21}({\cdot}85)^{81}$$
$$\times\left[\tfrac{1}{3}\{{\cdot}39720 - {\cdot}03028 - {\cdot}00225 - {\cdot}00092 - {\cdot}00045 - {\cdot}00022\} + \{{\cdot}00053 + {\cdot}00006 + {\cdot}00004 + {\cdot}00002\}\right].$$

Logs.

$4{\cdot}2865196$
$\bar{6}{\cdot}2829330$
$\bar{1}{\cdot}0852192$
$\bar{3}{\cdot}6546718$

Going up to B_3 and C_3 the value of the series is ·12184, which becomes ·12168 on taking in the next 2 terms. So

$$I_{\cdot 32}(21, 81) = 1 - {\cdot}004515,$$
$$= {\cdot}995485.$$

Actually we find by the use of a reduction by parts formula, of which 15 terms were computed,

$$I_{\cdot 32}(21, 81) = {\cdot}995489.$$

Example 3. Case of one index many times the other. Let $l = 1$, $m = 80$. Take $x = {\cdot}1$. Then

$$r = 80{\cdot}0125; \quad n = 81, \quad u = 7{\cdot}14796.$$

We get

$$I_{\cdot 1}(2, 81) = 1 - (8{\cdot}1037)^2({\cdot}91120375)^{81}$$

$$\left[\frac{1}{7{\cdot}14796}\{{\cdot}36788 - {\cdot}00283 - {\cdot}00112 - {\cdot}00062\} + \{{\cdot}00003 + {\cdot}00002\}\right].$$

The series part for the number of terms taken amounts to $\cdot 05088$, giving

$$I_{\cdot 1}(2, 81) = 1 - \cdot 0017897 = \cdot 998{,}2103.$$

Actually we have: $I_{\cdot 1}(2, 81) = 1 - 81{\cdot}82 \int_0^{0\cdot 9} x^{80}(1-x)\,dx$

$$= 1 - (9{\cdot}1)(\cdot 9)^{81} = \cdot 998{,}2107.$$

The first approximation thus gives an answer correct to four in the seventh place.

Examples 2 and 3 show that when one index is many times the other the convergence of the series is slower owing (i) to the high value of r, (ii) to the fact that the successive terms become of like sign. Hence the remainder, which we are neglecting, is of greater importance. A more correct result in the case of Example 2 could in fact be got by assuming the remainder to be equal in size to the last term retained, and correcting the two series by this amount. This in Example 2 would lead to a value $\cdot 12162$ for the series part, giving $I_{\cdot 32}(21, 81) = \cdot 995487$, thus doubling the accuracy. The justification for introducing this additive term lies in the fact that when the series is only slowly convergent, the remainder will be not very different in value from the last term retained.

Example 3 shows that our formula is applicable even when there is a considerable amount of skewness. We have in this case $\beta_1 = 1{\cdot}8$, $\beta_2 = 5{\cdot}6$, while the skewness, which is approximately equal to $1/\sqrt{l+1}$ when l is small compared with m, is $0{\cdot}7$. This encourages us in the hope that any curve of Type I which is likely to turn up in practical statistics can be successfully integrated by the method indicated.

Our Examples can be worked out equally well on the other side of the mode. The only changes are

(1) the part outside the series will alter owing to the interchange of l and m;

(2) the second series will have to be subtracted from the first.

To illustrate by one instance, suppose in Example 2 we let $z = -\cdot 12$.

Then we have without trouble

$$I_{\cdot 08}(21, 81) = (1{\cdot}15)^{81}(\cdot 4)^{21} \times \cdot 12038$$
$$= \cdot 0000437,$$

the correct result being $\cdot 0000436$. Even this discrepancy of 1 in the last place is removed by correcting for the remainder, as was done before.

These examples show that, if he takes the trouble, the computer can evaluate the Integral with very great accuracy even for the case where one index is many times greater than the other. On the other hand, the average statistician, who is probably content with a much smaller degree of accuracy than we have been striving after, ought to be able to get out a result in an exceedingly short space of time by using the following shortened form of (25):

$$I_x(l+1, m+1) = 1 - \frac{1}{u}\left(1 + \sqrt{\frac{m}{nl}}\,u\right)^{l+1}\left(1 - \sqrt{\frac{l}{nm}}\,u\right)^{m+1}\left(B_0 - \frac{B_1}{u^2 + r}\right).$$

The C terms are neglected, as being of the order of the rejected B terms. For Example 1 this would give a result ·998975, so that the error of the tail is less than one-half of one per cent. A similar procedure in the case of Example 2 would yield $1 - {\cdot}004538$, or ·995462, the error this time being 0·6 of one per cent., or little more than in the other example. Greater refinement can, of course, be obtained by the use of (23), but the above is simpler, and the coefficients of B_0 and B_1 are tabled. Actually, equation (23), applied to Example 2, gives ·995473, and, on the other side of the mode, ·0000440.

Section D. Other Methods.

(a) *Expansion in terms of Incomplete Gamma Functions.*

If we suppose one index large in comparison with the square of the other, we have the case dealt with by Soper (*Tracts for Computers*, No. VII. p. 40). The mode lies close up to one end of the curve, and the bulk of the area is distributed over a comparatively short portion of the range of x. The form of the curve at this end is then very like that of Type III, and its integral may be expressed as a series of Incomplete Gamma Functions. Such a series, up to a point, was worked out by Soper (p. 41), and the following, which is given without proof, is merely an extension of his result to include more terms.

Let l and m represent the indices as before, and suppose the integration to be carried up to a point distant x from the start of the curve. ($m > l$.) Then

$$I_x(l+1, m+1) = \frac{\Gamma(n+2)}{m^{l+1}\,\Gamma(m+1)}$$

$$\times \left[I(u_0, l) - M_1\frac{\Gamma(l+3)}{\Gamma(l+1)}I(u_2, l+2) - M_2\frac{\Gamma(l+4)}{\Gamma(l+1)}I(u_3, l+3)\right.$$

$$\left. + M_3\frac{\Gamma(l+5)}{\Gamma(l+1)}I(u_4, l+4) + M_4\frac{\Gamma(l+6)}{\Gamma(l+1)}I(u_5, l+5) - \dots\right] \quad \dots(26),$$

where
$$I(u, p) = \frac{1}{\Gamma(p+1)}\int_0^{u\sqrt{p+1}} v^p e^{-v}\,dv,$$

$$u_s = \frac{mx}{l+s+1},$$

and
$$M_1 = \frac{1}{2m}, \quad M_2 = \frac{1}{3m^2}, \quad M_3 = \frac{1}{8m^2}\left(1 - \frac{2}{m}\right),$$

$$M_4 = \frac{1}{6m^3}\left(1 - \frac{6}{5m}\right), \quad M_5 = \frac{1}{48m^3}\left(1 - \frac{26}{3m} + \frac{8}{m^2}\right),$$

$$M_6 = \frac{1}{24m^4}\left(1 - \frac{22}{5m} + \frac{24}{7m^2}\right), \quad M_7 = \frac{1}{384m^4}\left(1 - \frac{68}{3m} + \frac{348}{5m^2} - \frac{48}{m^3}\right),$$

$$M_8 = \frac{1}{144m^5}\left(1 - \frac{202}{45m} + \frac{892}{35m^2} - \frac{16}{m^3}\right), \quad M_9 = \frac{1}{3840m^5}\left(1 - \frac{140}{3m} + \frac{580}{3m^2} - \frac{4080}{7m^3} + \frac{384}{m^4}\right).$$

The terms are exact so far as they have been worked out, but obviously the computer may stop short at any point he desires, according to the accuracy sought for. The successive approximations to the value of this series should be taken up

to the odd M's, i.e. if a result correct to the required number of decimal places is not reached by going up to M_5, then *two* more terms should be computed, not one. The signs of the terms are equal in pairs, i.e. M_5 and M_6 are negative, M_7 and M_8 positive, and so on. The multiplier outside may perhaps be more conveniently expressed as follows provided l is an integer:

$$\frac{\Gamma(n+2)}{m^{l+1}\,\Gamma(m+1)}=\left(1+\frac{1}{m}\right)\left(1+\frac{2}{m}\right)\cdots\left(1+\frac{l+1}{m}\right).$$

The successive Incomplete Gamma Functions are obtained by interpolation from the *Tables of the Incomplete Gamma Function**. In one respect, the interpolation is awkward, for since the function is tabled in terms of its standard deviation as unit, the u with which we enter the tables changes for every function, necessitating a fresh computation of the coefficients of the Interpolation formula. But if Everett's formula be used in conjunction with Thompson's *Table of Coefficients*†, the work is reduced to a minimum.

If the integration be required only for the "tail" of the curve at its steep side a very few terms of the series will suffice. It is when we wish to integrate through the mode to a point on the far side that we may find the method tedious. But in no case should it be necessary to go beyond M_9.

To illustrate with the example partially worked out by Soper, let $l = 9{\cdot}2$, $m = 500$. Now the mode is at $x = {\cdot}018$.

(*a*) Let $x = {\cdot}01$; then $mx = 5$, and we have

$I_{\cdot01}(10{\cdot}2, 501)$ 1st approximation	(using 2 Γ-functions)	$= {\cdot}029\ 7376$
2nd "	(" 4 ")	$= {\cdot}029\ 7399$ (1)
Correct result (by parts)		$= {\cdot}029\ 7399$ (2)

(*b*) Let $x = {\cdot}03$; then $mx = 15$. For $I_{\cdot03}(10{\cdot}2, 501)$, we have

1st approximation	(using 2 Γ-functions)	$= {\cdot}930\ 1649$
2nd "	(" 4 ")	$= {\cdot}935\ 0065$
3rd "	(" 6 ")	$= {\cdot}934\ 9063$
4th "	(" 8 ")	$= {\cdot}934\ 9028$
5th "	(" 10 ")	$= {\cdot}934\ 9032$
Correct result (see Soper, p. 42)		${\cdot}934\ 9032$

A large number of terms have to be taken in the second case to get seven figure accuracy, the time occupied being roughly equivalent to that required for the application of a reduction by parts process. We have therefore nearly reached the limit of usefulness of our formula, and if we required integrals to points beyond $x = {\cdot}03$, we would have recourse to the asymptotic expansion of the previous section. Actually for $x = {\cdot}03$, the u of that expansion in equal to 2, approximately.

* Published by H.M. Stationery Office.

† *Tracts for Computers*, No. v.

(*b*) *Modal Expansions.*

When the indices are roughly of the same order and large, the area is concentrated over a short range near the mode, and the best method of evaluating the Incomplete Beta Function under these circumstances is to use the method of expansion in terms of Incomplete Normal Moment Functions. This was first developed by Prof. Pearson, and is illustrated by Soper (see pp. 43—46 of his Tract). An extended formula for the particular case of equal indices was given in *Biometrika*, Vol. XVII. p. 69. The generalisation of this to the case of unequal indices is given below in a form suited to the computer. An indication of the proof is given by Soper (p. 44).

$$I_x(l+1, m+1) = k_0 \times [m_0(u) - k_3 m_3(u) - k_4 m_4(u) - k_5 m_5(u) + k_6 m_6(u) + k_7 m_7(u) + k_8 m_8(u) - k_9 m_9(u) - k_{10} m_{10}(u) + k_{12} m_{12}(u)] \quad \text{........(27)},$$

where
$$m_s(u) = \frac{1}{\sqrt{2\pi}} \int_0^u t^s e^{-\frac{1}{2}t^2} dt \Big/ (s-1)(s-3) \ldots 2 \text{ or } 1,$$

$$r = \frac{l}{m} + \frac{m}{l}, \quad u = \sqrt{(r+2)n}\left(x - \frac{l}{n}\right) \text{ as in Section C,}$$

and
$$k_0 = \left(1 + \frac{1}{n}\right)\left(1 - \frac{1+r}{12n} + \frac{(1+r)^2}{288n^2}\right) = \sqrt{2\pi}\left(1 + \frac{1}{n}\right) B_0 \text{ (Table I)},$$

$$k_3 = \pm \frac{2}{3}\sqrt{\frac{r-2}{n}},$$

$$k_4 = \frac{3}{4} \cdot \frac{r-1}{n},$$

$$k_5 = \pm \frac{8}{5} \cdot \frac{r}{n}\sqrt{\frac{r-2}{n}},$$

$$k_6 = \frac{5}{6}\left\{\frac{r-2}{n} \quad \frac{3(r^2-r-1)}{n^2}\right.$$

$$k_7 = \pm \frac{4(r-1)}{n}\sqrt{\frac{r-2}{n}},$$

$$k_8 = \frac{7}{32} \cdot \frac{47r^2 - 94r + 15}{n^2}$$

$$k_9 = \pm \frac{64}{27} \cdot \frac{r-2}{n}\sqrt{\frac{r-2}{n}},$$

$$k_{10} = \frac{105}{8} \cdot \frac{(r-1)(r-2)}{n^2},$$

$$k_{12} = \frac{385}{72} \cdot \frac{(r-2)^2}{n^2}.$$

In this result we have not gone beyond the terms in $\frac{1}{n^2}$ for two reasons: (*a*) accurate results are obtainable by the use of formula (27) as it stands, over a wide range of l and m, (*b*) the expansion is asymptotic in character, and it is found

in general that when the terms in $\frac{1}{n^3}$ are computed, which takes us up to $m_{18}(u)$, the minimum term has been passed, and thereafter the terms increase rapidly, and a true sum is not obtained. The Incomplete Normal Moment Functions, up to the tenth, are tabled in *Tables for Statisticians*, pp. 2—8 and 22, 23, while the twelfth, computed specially for this investigation, appeared in *Biometrika*, Vol. XVII. p. 78.

We note in passing that when the indices l and m are equal, so that $r = 2$, our formula (27) reduces to

$$I_x(\tfrac{1}{2}n+1, \tfrac{1}{2}n+1) = \left(1 + \frac{3}{4n} - \frac{7}{32n^2}\right) m_0(x) - \left(\frac{3}{4n} + \frac{9}{16n^2}\right) m_4(x) - \frac{5}{2n^2} m_6(x) + \frac{105}{32n^2} m_8(x),$$

which is one of the formulae given for the Symmetric Function in my earlier paper*.

A word is necessary as to the alternative signs placed before the odd coefficients. These are to be taken as positive when $x > \frac{l}{n}$, i.e. when we are integrating on the right hand side of the mode. If we wish to integrate on the other side of the mode $\left(x < \frac{l}{n}\right)$ we make use of the fundamental formula

$$I_{x'}(p, q) = I_{-x'}(q, p),$$

x' being the distance from the mode. This is illustrated by the diagram (Fig. 4) which also shows the general form taken by the curve when the indices are large.

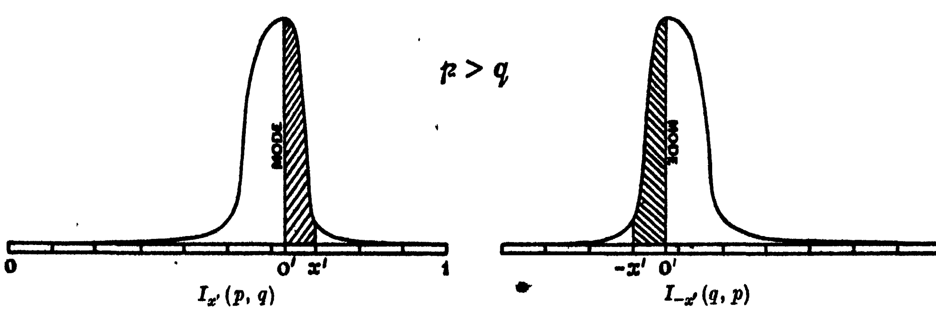

Fig. 4.

So integrating on the left hand side of the mode is equivalent to interchanging the indices, or, what is the same thing, changing the sign of

$$\alpha = \frac{l-m}{\sqrt{lmn}} = \sqrt{\frac{r-2}{n}} \text{ (see p. 21).}$$

Alternatively we might regard the change as being due to the odd Incomplete Normal Moment areas becoming negative with u, while the even ones remain positive. No other change in (27) is necessary so long as we are integrating to the same distance (x') on both sides of the mode.

* *Biometrika*, Vol. XVII. p. 70, Equation (8).

Some help in the computation of the k's may be had from Table I, by noting that $\sqrt{r-2} = 1{\cdot}2533{,}1414\,.\,{}_1C_1$, and also from Table II.

A number of illustrative examples have been worked out over a large range of l and m, and for distances from the mode ranging up to about three times the standard deviation. As in Section C, the results are found to be best when n is large and when the ratio of l to m is not great. The examples are presented in the form of a table which is given below. The method of checking was that of computing twenty-four ordinates of the curve and applying Weddle's Rule.

l	m	σ	x_1'	$I_{x_1'}(pq)$	Weddle	x_2'	$I_{x_2'}(pq)$	Weddle
56	44	·049	−·144	·504536	·504534	·144	·492567	·492570*
64	36	·047	·072	·4305316	·4305305	·144	·484152	·484161*
324	576	·016	·048	·5035287	·5035288	−·048	·4682821	·4682822
100	50	·038	·048	·3924982	·3924982	·12	·484415	·484419*
250	50	·022	·036	·4412788	·4412782	·06	·471971	·471975*
450	50	·014	·0216	·4335826	·4335820	·042	·468282	·468282
950	50	·007	·0096	·4083375	·4083370	·0192	·464919	·464923*
1024	36	·006	—	—	—	·0168	·458057	·458055
300	100	·022	·0336	·4347589	·4347588	·0672	·4843083	·4843090
500	100	·015	·024	·4356928	·4356927	·048	·4803804	·4803809
900	100	·010	·0144	·4279270	·4279272	·0288	·4772828	·4772835
500	250	·017	·024	·4169320	·4169320	·048	·4911916	·4911916
1000	250	·011	·0168	·4278626	·4278626	·0336	·4878834	·4878834
1000	500	·012	·012	·3376381	·3376381	·036	·4939278	·4939278

This table of examples was designed as a framework, in order to show the suitability of our expansion in Incomplete Normal Moment Functions for different values of l and m. It was calculated before the asymptotic formulae for the "tail" were developed, and as it stands, shows several things very clearly. To begin with it is very accurate for short distances round the mode, say up to about $1{\cdot}5\sigma$, no matter what the indices are. Secondly, for indices not very different, and for n fairly large, the formula applies quite well up to about 3σ from the mode. The examples marked with an asterisk show where the results are not so good. We may take it that when n is small, and when the ratio of the indices is great, the range of applicability of the formula (27) is narrowed down. But these are precisely those cases for which the expansions (25) and (26), of this and the last sections, have been found to yield excellent results. The results given above, for $l = 56$, $m = 44$, and for $l = 64$, $m = 36$, should be compared with those given on p. 21 for the same x. There should be no doubt in anyone's mind which is the correct formula to use under these circumstances.

Conclusion. An attempt has been made to present a variety of formulae all with the same object in view, namely, to enable the computer to evaluate an Incomplete Beta Function approximately when one index lies, or both lie outside the projected range of the Tables. When the Tables are published two alternative methods may be pursued: (a) the required Integral may be thrown back on the Tables by lowering

p or q (or both) by units till we come within the bounds of the Table. This is a feasible method where an index lies just a little way outside, but it becomes laborious if the reduction is for more than two or three units; (*b*) an approximate formula may be employed, and here there are special cases, each with a formula best suited to itself. Small areas round the mode, when p and q are not very different and large, are best worked out by a modal expansion such as (27). If one index is very small compared with the other, these modal areas are best determined by an expansion in terms of Incomplete Gamma Functions, such as (26). But when we get to points where our u, or $\sqrt{(r+2)}\, n \left(x - \frac{l}{n}\right)$, is of the order of 2 or more, then, *whatever* be the ratio of p to q, the asymptotic expansion for the "tail" (Equations (23) and (25)) provides easily the best solution. The actual number of terms which should be used depends on the particular problem, and on the zeal of the computer. We have already pointed out that, at any rate in certain cases, there is not much to choose between the full series given in (25) and the shortened form in (23). When the indices are actually equal, the simplified formula (18) applies. Thus all the cases that are likely to occur in practice are allowed for, and the transition from one to another of the formulae put forward is made as easy as possible.

I wish to record my indebtedness to Professor Pearson, who first drew my attention, in his lectures, to the exact nature of the problem whose solution was attempted by Bayes, and whose kindly criticism throughout the progress of this memoir has been much valued. My thanks are also due to Mr J. O. Irwin for aid in checking the laborious arithmetic involved in the preparation of the table of Examples on page 31.

TABLE I.

Table of the Coefficients of the Expansion (25) *on page* 23.

r	${}_2B_0$	${}_3B_0$	${}_2B_1$	${}_3B_1$	${}_1B_2$	${}_2B_2$	${}_3B_2$
2·00	−·0997356	+·012467	+·299207	+·31167	+·398942	+3·88969	+13·7760
2·05	−·1013978	+·012886	+·297544	+·31043	+·378995^{+}	+3·75347	+13·0566
2·10	−·1030601	+·013312	+·295882	+·30919	+·359048	+3·61741	+12·3383
2·15	−·1047223	+·013745^{-}	+·294220	+·30796	+·339101	+3·48152	+11·6212
2·20	−·1063846	+·014185^{-}	+·292558	+·30674	+·319154	+3·34580	+10·9051
2·3	−·1097091	+·015085^{+}	+·289233	+·30432	+·279260	+3·07485^{-}	+9·4765^{-}
2·4	−·1130336	+·016013	+·285909	+·30192	+·239365^{+}	+2·80456	+8·0523
2·5	−·1163582	+·016969	+·282584	+·29955^{+}	+·199471	+2·53495^{-}	+6·6326
2·6	−·1196827	+·017952	+·279260	+·29721	+·159577	+2·26599	+5·2174
2·8	−·1263317	+·020003	+·272611	+·29261	+·079788	+1·73008	+2·4003
3·0	−·1329808	+·022163	+·265962	+·28812	+·000000	+1·19683	−0·3989
3·2	−·1396298	+·024435^{+}	+·259312	+·28375^{-}	−·079788	+0·66623	−3·1805^{-}
3·4	−·1462788	+·026818	+·252663	+·27948	−·159577	+0·13830	−5·9443
3·6	−·1529279	+·029311	+·246014	+·27533	−·239365^{+}	−0·38697	−8·6906
3·8	−·1595769	+·031915^{+}	+·239365^{+}	+·27128	−·319154	−0·90959	−11·4193
4·0	−·1662260	+·034630	+·232716	+·26735^{-}	−·398942	−1·42954	−14·1306
4·2	−·1728750^{-}	+·037456	+·226067	+·26352	−·478731	−1·94684	−16·8245^{-}
4·4	−·1795240	+·040393	+·219418	+·25981	−·558519	−2·46147	−19·5010
4·6	−·1861731	+·043440	+·212769	+·25621	−·638308	−2·97345^{-}	−22·1603
4·8	−·1928221	+·046599	+·206120	+·25272	−·718096	−3·48277	−24·8023
5·0	−·1994711	+·049868	+·199471	+·24934	−·797885^{-}	−3·98942	−27·4273
5·2	−·2061202	+·053248	+·192822	+·24607	−·877673	−4·49342	−30·0352
5·4	−·2127692	+·056738	+·186173	+·24291	−·957461	−4·99476	−32·6260
5·6	−·2194183	+·060340	+·179524	+·23986	−1·037250^{-}	−5·49344	−35·2000
5·8	−·2260673	+·064052	+·172875^{-}	+·23693	−1·117038	−5·98945^{+}	−37·7571
6·0	−·2327163	+·067876	+·166226	+·23410	−1·196827	−6·48281	−40·2973
6·2	−·2393654	+·071810	+·159577	+·23139	−1·276615^{+}	−6·97351	−42·8209
6·4	−·2460144	+·075854	+·152928	+·22878	−1·356404	−7·46155^{+}	−45·3277
6·6	−·2526634	+·080010	+·146279	+·22629	−1·436192	−7·94693	−47·8180
6·8	−·2593125^{-}	+·084277	+·139630	+·22391	−1·515981	−8·42965^{+}	−50·2918
7·0	−·2659615^{+}	+·088654	+·132981	+·22163	−1·595769	−8·90971	−52·7490
7·2	−·2726106	+·093142	+·126332	+·21947	−1·675558	−9·38711	−55·1899
7·4	−·2792596	+·097741	+·119683	+·21742	−1·755346	−9·86185^{+}	−57·6144
7·6	−·2859086	+·102451	+·113034	+·21548	−1·835134	−10·33393	−60·0227
7·8	−·2925577	+·107271	+·106385^{-}	+·21366	−1·914923	−10·80336	−62·4148
8·0	−·2992067	+·112203	+·099736	+·21194	−1·994711	−11·27012	−64·7907
8·2	−·3058557	+·117245^{-}	+·093087	+·21033	−2·074500^{-}	−11·73422	−67·1506
8·4	−·3125048	+·122398	+·086437	+·20884	−2·154288	−12·19567	−69·4945^{-}
8·6	−·3191538	+·127662	+·079788	+·20745^{-}	−2·234077	−12·65445^{-}	−71·8224
8·8	−·3258029	+·133036	+·073139	+·20618	−2·313865^{+}	−13·11057	−74·1344
9·0	−·3324519	+·138522	+·066490	+·20501	−2·393654	−13·56404	−76·4307
9·2	−·3391009	+·144118	+·059841	+·20396	−2·473442	−14·01484	−78·7112
9·4	−·3457500^{-}	+·149825^{-}	+·053192	+·20302	−2·553231	−14·46299	−80·9761
9·6	−·3523990	+·155643	+·046543	+·20219	−2·633019	−14·90847	−83·2253
9·8	−·3590481	+·161572	+·039894	+·20147	−2·712808	−15·35130	−85·4590
10·0	−·3656971	+·167611	+·033245^{+}	+·20086	−2·792596	−15·79147	−87·6773

${}_1B_0 = {}_1B_1 = 0·3989\,4228.$

TABLE I (*continued*).

Table of the Coefficients of the Expansion (25) *on page* 23.

r	${}_1B_3$	${}_2B_3$	${}_3B_3$	${}_1B_4$	${}_2B_4$	${}_3B_4$	${}_1B_5$
2·00	+ 1·99471	+ 27·8262	+ 249·90	+ 3·5905-	+ 214·53	+ 3568	+ 51·46
2·05	+ 1·97676	+ 25·4089	+ 221·04	+ 2·9888	+ 186·59	+ 2895+	+ 52·03
2·10	+ 1·96280	+ 23·0305+	+ 192·66	+ 2·3793	+ 159·86	+ 2250+	+ 53·00
2·15	+ 1·95282	+ 20·6912	+ 164·74	+ 1·7602	+ 134·31	+ 1633	+ 54·41
2·20	+ 1·94684	+ 18·3907	+ 137·29	+ 1·1298	+ 109·93	+ 1043	+ 56·26
2·3	+ 1·94684	+ 13·9063	+ 83·79	− 0·1723	+ 64·67	− 56	+ 61·36
2·4	+ 1·96280	+ 9·5770	+ 32·16	− 1·5415+	+ 23·62	− 1051	+ 68·48
2·5	+ 1·99471	+ 5·4023	− 17·62	− 2·9921	− 13·09	− 1943	+ 77·79
2·6	+ 2·04258	+ 1·3819	− 65·53	− 4·5384	− 45·73	− 2736	+ 89·50-
2·8	+ 2·18620	− 6·1977	− 155·81	− 7·9757	− 99·44	− 4034	+ 121·03
3·0	+ 2·39365+	− 13·1651	− 238·70	− 11·9683	− 138·83	− 4968	+ 165·16
3·2	+ 2·66493	− 19·5234	− 314·24	− 16·6311	− 165·22	− 5559	+ 224·36
3·4	+ 3·00005-	− 25·2759	− 382·46	− 22·0791	− 179·91	− 5828	+ 301·44
3·6	+ 3·39899	− 30·4257	− 443·38	− 28·4270	− 184·18	− 5797	+ 399·61
3·8	+ 3·86176	− 34·9761	− 497·04	− 35·7899	− 179·34	− 5487	+ 522·43
4·0	+ 4·38837	− 38·9301	− 543·48	− 44·2826	− 166·66	− 4921	+ 673·81
4·2	+ 4·97880	− 42·2911	− 582·71	− 54·0200	− 147·41	− 4118	+ 858·07
4·4	+ 5·63306	− 45·0621	− 614·77	− 65·1170	− 122·86	− 3101	+ 1079·87
4·6	+ 6·35116	− 47·2465-	− 639·69	− 77·6884	− 94·26	− 1890	+ 1344·23
4·8	+ 7·13309	− 48·8473	− 657·50+	− 91·8493	− 62·87	− 507	+ 1656·57
5·0	+ 7·97885-	− 49·8678	− 668·23	− 107·7144	− 29·92	+ 1028	+ 2022·64
5·2	+ 8·88843	− 50·3111	− 671·91	− 125·3987	+ 3·35-	+ 2693	+ 2448·58
5·4	+ 9·86185+	− 50·1806	− 668·56	− 145·0171	+ 35·71	+ 4469	+ 2940·90
5·6	+10·89910	− 49·4792	− 658·22	− 166·6845-	+ 65·94	+ 6334	+ 3506·46
5·8	+12·00018	− 48·2103	− 640·92	− 190·5157	+ 92·84	+ 8267	+ 4152·51
6·0	+13·16510	− 46·3770	− 616·69	− 216·6257	+ 115·19	+10249	+ 4886·64
6·2	+14·39384	− 43·9826	− 585·55-	− 245·1293	+ 131·82	+12258	+ 5716·85-
6·4	+15·68641	− 41·0301	− 547·53	− 276·1415-	+ 141·51	+14275-	+ 6651·45+
6·6	+17·04281	− 37·5229	− 502·67	− 309·7771	+ 143·10	+16279	+ 7699·17
6·8	+18·46305-	− 33·4641	− 450·99	− 346·1510	+ 135·42	+18250-	+ 8869·09
7·0	+19·94711	− 28·8568	− 392·51	− 385·3782	+ 117·29	+20167	+10170·63
7·2	+21·49501	− 23·7044	− 327·28	− 427·5736	+ 87·56	+22017	+11613·03
7·4	+23·10674	− 18·0099	− 255·32	− 472·8519	+ 45·08	+23763	+13208·25+
7·6	+24·78229	− 11·7765+	− 176·65+	− 521·3282	− 11·29	+25401	+14965·05+
7·8	+26·52168	− 5·0075+	− 91·31	− 573·1173	− 82·69	+26907	+16894·94
8·0	+28·32490	+ 2·2939	+ 0·69	− 628·3341	− 170·25-	+28260	+19009·20
8·2	+30·19195+	+ 10·1246	+ 99·30	− 687·0935+	− 275·08	+29441	+21319·48
8·4	+32·12283	+ 18·4814	+ 204·50+	− 749·5104	− 398·29	+30430	+23837·81
8·6	+34·11754	+ 27·3611	+ 316·27	− 815·6997	− 540·99	+31209	+26576·56
8·8	+36·17609	+ 36·7604	+ 434·57	− 885·7763	− 704·28	+31757	+29548·49
9·0	+38·29846	+ 46·6762	+ 559·38	− 959·8551	− 889·24	+32055-	+32766·73
9·2	+40·48466	+ 57·1054	+ 690·68	−1038·0510	−1096·96	+32084	+36244·75-
9·4	+42·73470	+ 68·0447	+ 828·42	−1120·4789	−1328·50+	+31826	+39996·42
9·6	+45·04856	+ 79·4908	+ 972·60	−1207·2536	−1584·94	+31260	+44035·96
9·8	+47·42626	+ 91·4408	+1123·17	−1298·4901	−1867·34	+30368	+48377·96
10·0	+49·86779	+103·8912	+1280·11	−1394·3033	−2176·73	+29131	+53037·38

TABLE I (*continued*).

Table of the Coefficients of the Expansion (25) *on page* 23.

r	$_2B_5$	$_1C_1$	$_2C_1$	$_3C_1$	$_1C_2$	$_2C_2$	$_3C_2$
2·00	+ 1786	+ ·000000	+ ·00000	+ 0·0000	+ 0·00000	+ 0·0000	+ 0·00
2·05	+ 1465⁻	+ ·178412	+ ·48989	+ 1·1186	+ 0·68689	+ 9·9860	+ 61·84
2·10	+ 1164	+ ·252313	+ ·69176	+ 1·5791	+ 0·93356	+ 13·8631	+ 85·68
2·15	+ 883	+ ·309019	+ ·84594	+ 1·9305⁻	+ 1·09702	+ 16·6617	+ 102·77
2·20	+ 621	+ ·356825⁻	+ ·97532	+ 2·2250⁺	+ 1·21320	+ 18·8737	+ 116·18
2·3	+ 150⁻	+ ·437019	+1·19088	+ 2·7151	+ 1·35476	+ 22·2214	+ 136·17
2·4	− 256	+ ·504626	+1·37090	+ 3·1238	+ 1·41295⁺	+ 24·6291	+ 150·21
2·5	− 604	+ ·564190	+1·52801	+ 3·4797	+ 1·41047	+ 26·3876	+ 160·09
2·6	− 899	+ ·618039	+1·66870	+ 3·7979	+ 1·35969	+ 27·6510	+ 166·80
2·8	− 1356	+ ·713650⁻	+1·91496	+ 4·3533	+ 1·14184	+ 29·0408	+ 172·87
3·0	− 1673	+ ·797885⁻	+2·12769	+ 4·8316	+ 0·79788	+ 29·2558	+ 171·32
3·2	− 1889	+ ·874039	+2·31620	+ 5·2541	+ 0·34962	+ 28·5461	+ 163·75⁻
3·4	− 2038	+ ·944070	+2·48605⁺	+ 5·6334	− 0·18881	+ 27·0696	+ 151·14
3·6	− 2151	+1·009253	+2·64088	+ 5·9782	− 0·80740	+ 24·9353	+ 134·22
3·8	− 2254	+1·070474	+2·78323	+ 6·2943	− 1·49866	+ 22·2230	+ 113·48
4·0	− 2368	+1·128379	+2·91498	+ 6·5862	− 2·25676	+ 18·9944	+ 89·32
4·2	− 2509	+1·183454	+3·03753	+ 6·8568	− 3·07698	+ 15·2981	+ 62·06
4·4	− 2690	+1·236077	+3·15200	+ 7·1089	− 3·95545⁻	+ 11·1741	+ 31·96
4·6	− 2918	+1·286550⁺	+3·25926	+ 7·3448	− 4·88889	+ 6·6558	− 0·77
4·8	− 3197	+1·335116	+3·36004	+ 7·5658	− 5·87451	+ 1·7713	− 35·95⁺
5·0	− 3525⁻	+1·381977	+3·45494	+ 7·7736	− 6·90988	− 3·4549	− 73·42
5·2	− 3896	+1·427299	+3·54446	+ 7·9693	− 7·99288	− 9·0015⁺	− 113·03
5·4	− 4301	+1·471226	+3·62903	+ 8·1538	− 9·12160	− 14·8496	− 154·68
5·6	− 4725⁻	+1·513880	+3·70900	+ 8·3282	−10·29438	− 20·9824	− 198·24
5·8	− 5149	+1·555363	+3·78472	+ 8·4932	−11·50969	− 27·3848	− 243·63
6·0	− 5552	+1·595769	+3·85644	+ 8·6493	−12·76615⁺	− 34·0431	− 290·74
6·2	− 5905⁻	+1·635177	+3·92442	+ 8·7973	−14·06252	− 40·9448	− 339·51
6·4	− 6177	+1·673657	+3·98888	+ 8·9375⁺	−15·39764	− 48·0786	− 389·85⁺
6·6	− 6333	+1·711272	+4·05001	+ 9·0707	−16·77046	− 55·4338	− 441·71
6·8	− 6332	+1·748077	+4·10798	+ 9·1971	−18·18001	− 63·0007	− 495·01
7·0	− 6133	+1·784124	+4·16296	+ 9·3171	−19·62537	− 70·7703	− 549·71
7·2	− 5685⁺	+1·819457	+4·21507	+ 9·4312	−21·10570	− 78·7340	− 605·74
7·4	− 4938	+1·854116	+4·26447	+ 9·5394	−22·62022	− 86·8839	− 663·05⁻
7·6	− 3836	+1·888139	+4·31125⁺	+ 9·6423	−24·16819	− 95·2126	− 721·59
7·8	− 2319	+1·921560	+4·35554	+ 9·7402	−25·74891	−103·7130	− 781·34
8·0	− 322	+1·954410	+4·39742	+ 9·8331	−27·36174	−112·3786	− 842·22
8·2	+ 2221	+1·986717	+4·43700	+ 9·9214	−29·00606	−121·2029	− 904·23
8·4	+ 5383	+2·018506	+4·47435⁺	+10·0053	−30·68129	−130·1802	− 967·30
8·6	+ 9240	+2·049803	+4·50957	+10·0851	−32·38688	−139·3046	−1031·42
8·8	+13870	+2·080628	+4·54271	+10·1607	−34·12231	−148·5707	−1096·52
9·0	+19356	+2·111004	+4·57384	+10·2325⁺	−35·88707	−157·9735⁻	−1162·61
9·2	+25787	+2·140949	+4·60304	+10·3006	−37·68070	−167·5078	−1229·62
9·4	+33252	+2.170481	+4·63036	+10·3653	−39·50275⁻	−177·1691	−1297·56
9·6	+41846	+2·199616	+4·65585⁺	+10·4265⁺	−41·35278	−186·9527	−1366·38
9·8	+51667	+2·228370	+4·67958	+10·4845⁻	−43·23038	−196·8542	−1436·04
10·0	+62817	+2·256758	+4·70158	+10·5394	−45·13517	−206·8695⁺	−1506·55⁻

TABLE I (*continued*).

Table of the Coefficients of the Expansion (25) *on page* 23.

r	${}_1C_3$	${}_2C_3$	${}_3C_3$	${}_1C_4$	
2·00	+ 0·0000	+ 0·00	+ 0	+ 0·00	+ 0
2·05	+ 5·0361	+ 119·89	+ 1568	+ 26·34	+ 1380
2·10	+ 6·9411	+ 161·67	+ 2107	+ 34·18	+ 1813
2·15	+ 8·2964	+ 188·51	+ 2448	+ 38·15+	+ 2058
2·20	+ 9·3631	+ 206·89	+ 2676	+ 39·82	+ 2197
2·3	+ 11·0085+	+ 227·65+	+ 2916	+ 38·52	+ 2278
2·4	+ 12·2927	+ 234·20	+ 2961	+ 32·70	+ 2198
2·5	+ 13·3995+	+ 230·94	+ 2869	+ 23·27	+ 2018
2·6	+ 14·4374	+ 220·39	+ 2675−	+ 10·63	+ 1778
2·8	+ 16·5852	+ 183·58	+ 2057	− 23·98	+ 1209
3·0	+ 19·1492	+ 132·45−	+ 1221	− 71·81	+ 624
3·2	+ 22·4104	+ 72·39	+ 237	− 134·95+	+ 100
3·4	+ 26·5850+	+ 7·25−	− 846	− 216·38	− 318
3·6	+ 31·8520	− 60·05−	− 1990	− 319·73	− 600
3·8	+ 38·3658	− 127·10	− 3164	− 449·17	− 736
4·0	+ 46·2635+	− 191·92	− 4343	− 609·32	− 721
4·2	+ 55·6697	− 252·77	− 5504	− 805·22	− 563
4·4	+ 66·6987	− 308·13	− 6628	− 1042·26	− 275+
4·6	+ 79·4573	− 356·66	− 7699	− 1326·18	+ 122
4·8	+ 94·0456	− 397·13	− 8701	− 1663·02	+ 604
5·0	+ 110·5581	− 428·41	− 9618	− 2059·15	+ 1140
5·2	+ 129·0849	− 449·50−	− 10440	− 2521·18	+ 1695+
5·4	+ 149·7120	− 459·42	− 11153	− 3056·03	+ 2231
5·6	+ 172·5217	− 457·31	− 11746	− 3670·86	+ 2704
5·8	+ 197·5934	− 442·33	− 12209	− 4373·06	+ 3069
6·0	+ 225·0034	− 413·70	− 12531	− 5170·29	+ 3278
6·2	+ 254·8260	− 370·70	− 12704	− 6070·43	+ 3277
6·4	+ 287·1326	− 312·63	− 12719	− 7081·58	+ 3013
6·6	+ 321·9929	− 238·84	− 12567	− 8212·05+	+ 2427
6·8	+ 359·4747	− 148·70	− 12240	− 9470·38	+ 1460
7·0	+ 399·6438	− 41·63	− 11731	− 10865·32	+ 50−
7·2	+ 442·5647	+ 82·94	− 11032	12405·78	− 1868
7·4	+ 488·3000	+ 225·55−	− 10137	− 14100·92	4359
7·6	+ 536·9113	+ 386·71	− 9038	− 15960·07	− 7494
7·8	+ 588·4587	+ 566·91	− 7729	− 17992·72	− 11341
8·0	+ 643·0009	+ 766·62	− 6205−	− 20208·60	− 15973
8·2	+ 700·5957	+ 986·29	− 4458	− 22617·58	− 21465−
8·4	+ 761·2997	+ 1226·36	− 2484	− 25229·71	− 27891
8·6	+ 825·1685−	+ 1487·24	− 276	− 28055·24	− 35328
8·8	+ 892·2567	+ 1769·33	+ 2170	− 31104·56	− 43856
9·0	+ 962·6179	+ 2073·01	+ 4860	− 34388·26	− 53554
9·2	+ 1036·3050−	+ 2398·65−	+ 7799	− 37917·06	− 64504
9·4	+ 1113·3698	+ 2746·60	+ 10992	− 41701·88	− 76789
9·6	+ 1193·8636	+ 3117·21	+ 14444	− 45753·77	− 90492
9·8	+ 1277·8367	+ 3510·81	+ 18158	− 50083·96	− 105699
10·0	+ 1365·3388	+ 3927·70	+ 22140	− 54703·82	− 122497

TABLE II. *Table of Reciprocals of Powers of n.*

n	$n^{-.5}$	n^{-1} ·0	$n^{-1.5}$ ·00	n^{-2} ·000	$n^{-2.5}$ ·0000	n^{-3} ·00000
50	·1414 2136	200 0000	282 8427	400 0000	565 6854	800 0000
51	·1400 2801	196 0784	274 5647	384 4675+	538 3622	753 8579
52	·1386 7505−	192 3077	266 6828·	369 8225−	512 8515+	711 1971
53	·1373 6056	188 6792	259 1709	355 9986	489 0017	671 6954
54	·1360 8276	185 1852	252 0051	342 9355+	466 6761	635 0658
55	·1348 3997	181 8182	245 1636	330 5785+	445 7520	601 0518
56	·1336 3062	178 5714	238 6261	318 8776	426 1181	569 4242
57	·1324 5324	175 4386	232 3741	307 7870	407 6739	539 9772
58	·1313 0643	172 4138	226 3904	297 2652	390 3283	512 5261
59	·1301 8891	169 4915+	220 6592	287 2738	373 9986	486 9047
60	·1290 9944	166 6667	215 1657	277 7778	358 6096	462 9630
61	1280 3688	163 9344	209 8965+	268 7450−	344 0927	440 5655+
62	·1270 0013	161 2903	204 8389	260 1457	330 3853	419 5898
63	·1259 8816	158 7302	199 9812	251 9526	317 4305−	399 9248
64	·1250 0000	156 2500	195 3125	244 1406	305 1758	381 4697
65	·1240 3473	153 8462	190 8227	236 6864	293 5733	364 1329
66	·1230 9149	151 5152	186 5023	229 5684	282 5792	347 8309
67	·1221 6941	149 2537	182 3425−	222 7668	272 1529	332 4877
68	·1212 6781	147 0588	178 3350+	216 2630	262 2574	318 0338
69	·1203 8585+	144 9275+	174 4723	210 0399	252 8583	304 4057
70	·1195 2286	142 8571	170 7469	204 0816	243 9242	291 5452
71	·1186 7817	140 8451	167 1523	198 3733	235 4258	279 3991
72	·1178 5113	138 8889	163 6821	192 9012	227 3363	267 9184
73	·1170 4115−	136 9863	160 3303	187 6525−	219 6306	257 0582
74	·1162 4764	135 1351	157 0914	182 6150+	212 2857	246 7771
75	·1154 7005+	133 3333	153 9601	177 7778	205 2801	237 0370
76	·1147 0787	131 5789	150 9314	173 1302	198 5940	227 8029
77	·1139 6058	129 8701	148 0007	168 6625+	192 2088	219 0422
78	·1132 2770	128 2051	145 1637	164 3655+	186 1073	210 7251
79	·1125 0879	126 5823	142 4162	160 2307	180 2737	202 8237
80	·1118 0340	125 0000	139 7542	156 2500	174 6928	195 3125
81	·1111 1111	123 4568	137 1742	152 4158	169 3509	188 1676
82	·1104 3153	121 9512	134 6726	148 7210	164 2349	181 3671
83	·1097 6426	120 4819	132 2461	145 1589	159 3326	174 8903
84	·1091 0895−	119 0476	129 8916	141 7234	154 6329	168 7183
85	·1084 6523	117 6471	127 6062	138 4083	150 1249	162 8333
86	·1078 3277	116 2791	125 3869	135 2082	145 7988	157 2189
87	·1072 1125+	114 9425+	123 2313	132 1178	141 6452	151 8596
88	·1066 0036	113 6364	121 1368	129 1322	137 6554	146 7412
89	·1059 9979	112 3596	119 1009	126 2467	133 8212	141 8502
90	·1054 0926	111 1111	117 1214	123 4568	130 1349	137 1742
91	·1048 2848	109 8901	115 1961	120 7584	126 5892	132 7015−
92	·1042 5721	108 6957	113 3231	118 1474	123 1772	128 4211
93	·1036 9517	107 5269	111 5002	115 6203	119 8927	124 3229
94	·1031 4212	106 3830	109 7257	113 1734	116 7294	120 3972
95	·1025 9784	105 2632	107 9977	110 8033	113 6818	116 6351
96	·1020 6207	104 1667	106 3147	108 5069	110 7444	113 0281
97	·1015 3462	103 0928	104 6749	106 2812	107 9122	109 5683
98	·1010 1525+	102 0408	103 0768	104 1233	105 1804	106 2482
99	·1005 0378	101 0101	101 5190	102 0304	102 5444	103 0610
100	·1000 0000	100 0000	100 0000	100 0000	100 0000	100 0000

TABLE II (*continued*). *Table of Reciprocals of Powers of n.*

n	$n^{-.5}$ ·0	n^{-1} ·00	$n^{-1.5}$ ·000	n^{-2} ·0000	$n^{-2.5}$ ·00000	n^{-3} ·000000
101	995 0372	990 0990	985 1853	980 2960	975 4310	970 5901
103	985 3293	970 8738	956 6304	942 5959	928 7673	915 1417
105	975 9001	952 3810	929 4286	907 0295−	885 1701	863 8376
107	966 7365−	934 5794	903 4920	873 4387	844 3851	816 2979
109	957 8263	917 4312	878 7397	841 6800	806 1832	772 1835−
111	949 1580	900 9009	855 0973	811 6224	770 3579	731 1914
113	940 7209	884 9558	832 4964	783 1467	736 7224	693 0502
115	932 5048	869 5652	810 8737	756 1437	705 1076	657 5162
117	924 5003	854 7009	790 1713	730 5136	675 3601	624 3706
119	916 6985−	840 3361	770 3348	706 1648	647 3402	593 4158
121	909 0909	826 4463	751 3148	683 0135−	620 9213	564 4739
123	901 6696	813 0081	733 0647	660 9822	595 9876	537 3839
125	894 4272	800 0000	715 5418	640 0000	572 4334	512 0000
127	887 3565+	787 4016	698 7059	620 0012	550 1621	488 1900
129	880 4509	775 1938	682 5201	600 9254	529 0853	465 8337
131	873 7041	763 3588	666 9497	582 7166	509 1219	444 8219
133	867 1100	751 8797	651 9624	565 3231	490 1973	425 0549
135	860 6630	740 7407	637 5281	548 6968	472 2430	406 4421
137	854 3577	729 9270	623 6187	532 7934	455 1961	388 9003
139	848 1889	719 4245−	610 2079	517 5716	438 9985−	372 3536
141	842 1519	709 2199	597 2709	502 9928	423 5964	356 7325−
143	836 2420	699 3007	584 7846	489 0215−	408 9403	341 9731
145	830 4548	689 6552	572 7275−	475 6243	394 9844	328 0167
147	824 7861	680 2721	561 0790	462 7701	381 6863	314 8096
149	819 2319	671 1409	549 8201	450 4302	369 0068	302 3021
151	813 7885−	662 2517	538 9328	438 5773	356 9091	290 4485+
153	808 4521	653 5948	528 4001	427 1861	345 3595−	279 2066
155	803 2193	645 1613	518 2060	416 2331	334 3265−	268 5375−
157	798 0869	636 9427	508 3356	405 6960	323 7807	258 4051
159	793 0516	628 9308	498 7746	395 5540	313 6947	248 7761
161	788 1104	621 1180	489 5096	385 7876	304 0432	239 6196
163	783 2604	613 4969	480 5279	376 3785−	294 8024	230 9070
165	778 4989	606 0606	471 8175+	367 3095−	285 9500+	222 6118
167	773 8232	598 8024	463 3672	358 5643	277 4654	214 7092
169	769 2308	591 7160	455 1662	350 1278	269 3291	207 1762
171	764 7191	584 7953	447 2042	341 9856	261 5229	199 9916
173	760 2859	578 0347	439 4716	334 1241	254 0298	193 1353
175	755 9289	571 4286	431 9594	326 5306	246 8339	186 5889
177	751 6460	564 9718	424 6588	319 1931	239 9202	180 3351
179	747 4351	558 6592	417 5615−	312 1001	233 2746	174 3576
181	743 2941	552 4862	410 6598	305 2410	226 8838	168 6414
183	739 2213	546 4481	403 9461	298 6055+	220 7355+	163 1724
185	735 2146	540 5405+	397 4133	292 1841	214 8180	157 9373
187	731 2724	534 7594	391 0548	285 9676	209 1202	152 9238
189	727 3930	529 1005+	384 8640	279 9474	203 6318	148 1203
191	723 5746	523 5602	378 8349	274 1153	198 3429	143 5159
193	719 8158	518 1347	372 9615+	268 4636	193 2443	139 1003
195	716 1149	512 8205+	367 2384	262 9849	188 3274	134 8640
197	712 4705−	507 6142	361 6601	257 6722	183 5838	130 7981
199	708 8812	502 5126	356 2217	252 5189	179 0059	126 8939
200	707 1068	500 0000	353 5534	250 0000	176 7767	125 0000

ON PARTIAL MULTIPLE CORRELATION COEFFICIENTS IN A UNIVERSE OF MANIFOLD CHARACTERISTICS.

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THE following symbols and terminology will be used:

The subscripts 0, 1, ... n, u, v, w etc. denote the manifold of correlated characteristics. r_{st} is the correlation coefficient of the sth and tth characteristics, and the standard deviation of these will be represented by σ_s and σ_t. All the characteristics will be supposed measured from their individual mean values. We shall use Δ to signify the determinant formed of all the correlation coefficients of the manifold, i.e.

$$\Delta = \begin{vmatrix} 1, & r_{10}, & r_{20} \dots r_{n0}, \dots r_{u0}, & r_{v0}, \dots \\ r_{01}, & 1, & r_{21} \dots r_{n1}, \dots r_{u1}, & r_{v1}, \dots \\ r_{02}, & r_{12}, & 1 \dots r_{n2}, \dots r_{u2}, & r_{v2}, \dots \\ & & & \\ r_{0n}, & r_{1n}, & r_{2n} \dots 1, \dots r_{un}, & r_{vn}, \\ & & & \\ r_{0u}, & r_{1u}, & r_{2u} \dots r_{nu}, \dots 1, & r_{vu}, \dots \\ r_{0v}, & r_{1v}, & r_{2v} \dots r_{nv}, \dots r_{uv}, & 1, \dots \end{vmatrix} \quad .(1).$$

First minors will be denoted in the usual way by Δ_{uv}, or Δ_{st}, corresponding to the constituents r_{uv}, or r_{st}. Second minors or the minors of these minors will be denoted by Δ_{uvst}, and still higher order minors in like fashion.

A partial correlation coefficient between the characteristics 0 and 1 for 2, 3 ... n constant will be denoted by $\rho_{01.23\dots n}$, and the reader will doubtless bear in mind the reduction formula:

$$\rho_{01.23\dots n} = \frac{\rho_{01.3\dots n} - \rho_{02.3\dots n}\,\rho_{12.3\dots n}}{\sqrt{(1-\rho^2_{02.3\dots n})(1-\rho^2_{12.3\dots n})}} \quad \dots\dots\dots\dots(2).$$

Further:

$$\rho_{01.23\dots n} = -\frac{\Delta_{01}}{\sqrt{\Delta_{00}\Delta_{11}}} \quad \dots\dots\dots\dots(3).$$

We shall use $R_{0.123\dots n}$ to denote the multiple correlation coefficient of the characteristic 0 on the characteristics 1, 2, 3 ... n. This multiple correlation coefficient is obtained by considering the linear function of 1, 2, 3 ... n which gives the maximum correlation coefficient with the characteristic 0. It is known* that:

$$1 - R^2_{0.123\dots n\dots uv\dots} = \frac{\Delta}{\Delta_{00}} \quad \dots\dots\dots\dots(4).$$

* See Pearson, *Biometrika*, Vol. VIII. p. 439.

Now if we are going to treat of the sub-universe in which the characteristics 1 to n vary but $u, v, w \ldots$ are constant, it is, I think, clear that we must use in Δ and Δ_{00} the partial correlation coefficients $\rho_{st.uvw\ldots}$ instead of r_{st}; we shall then have a partial multiple correlation coefficient, for it depends on correlation coefficients for the universe in which $u, v, w \ldots$ are fixed. We shall denote such a partial multiple correlation coefficient by:

$$R_{0.123\ldots n\,|\,uvw\ldots\,|}.$$

Now it is not needful to evaluate $\bar{\Delta}$ and $\bar{\Delta}_{00}$, where the rules indicate that the total correlation coefficients have been replaced by partials for $u, v, w \ldots$ constant. It will be adequate to write down any formula for $R_{0.123\ldots n}$, if we remember that in expressing it in terms of partial correlation coefficients we are to add the characteristics $u, v, w \ldots$. We must not, however, use any partial which invades the universe $u, v, w \ldots$, as this would be treating those characteristics as variable in the partial multiple coefficient.

The useful formula for the present purpose is*:

$$1 - R^2_{0.1.2\ldots n} = (1 - \rho^2_{01.23\ldots n})(1 - \rho^2_{02.3\ldots n})(1 - \rho^2_{03.4\ldots n})(1 - \rho^2_{04.5\ldots n}) \ldots (1 - r^2_{0n}) \qquad \text{(5)}.$$

Writing this in our present notation we have:

$$1 - R^2_{0.12\ldots n\,|\,uvw\ldots\,|} = (1 - \rho^2_{01.23\ldots n\,|\,uvw\ldots\,|})(1 - \rho^2_{02.3\ldots n\,|\,uvw\ldots\,|})$$
$$(1 - \rho^2_{03.4\ldots n\,|\,uvw\ldots\,|}) \ldots (1 - \rho^2_{0n\,|\,uvw\ldots\,|}) \qquad \text{(6)}.$$

Here the vertical rules in the partial multiple coefficient are of significance, but may be dropped in the partial correlation coefficients on the right-hand side of the equation.

But: $$\rho_{01.23\ldots nuvw\ldots} = -\frac{\Delta_{01}}{\sqrt{\Delta_{00}\Delta_{11}}} \text{ by (3)},$$

where Δ is the determinant for all characteristics. Hence:

$$1 - \rho^2_{01.23\ldots nvuw\ldots} = 1 - \frac{\Delta^2_{01}}{\Delta_{00}\Delta_{11}} = \frac{\Delta_{00}\Delta_{11} - \Delta^2_{01}}{\Delta_{00}\Delta_{11}},$$

or, by a well-known property of determinants:

$$= \frac{\Delta \,.\, \Delta_{0011}}{\Delta_{00}\Delta_{11}} \qquad \text{(7)}.$$

Similarly:

$$1 - \rho^2_{02.3\ldots nuvw\ldots} = \frac{\Delta_{11}\Delta_{110022}}{\Delta_{1100}\Delta_{1122}}$$

$$1 - \rho^2_{03.4\ldots nvuw\ldots} = \frac{\Delta_{1122}\Delta_{11220033}}{\Delta_{112200}\Delta_{112233}},$$

and so on, to

$$1 - \rho^2_{0n.uvw\ldots} = \frac{\Delta_{1122\ldots n-1n-1}\Delta_{1122\ldots n-1,n-1.00nn}}{\Delta_{1122\ldots n-1n-1.00}\Delta_{1122\ldots n-1,n-1.nn}}.$$

* See Yule, *R. S. Proc.* A, Vol. LXXIX. p. 189. Pearson, *R. S. Proc.* A, Vol. XCI. p. 496.

Thus taking the combined product*:

$$1 - R^2_{0.12\ldots n|uvw\ldots|} = \frac{\Delta}{\Delta_{00}} \times \frac{\Delta_{001122\ldots nn}}{\Delta_{1122\ldots nn}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(8).$$

But by (4) Δ/Δ_{00} is $1 - R^2_{0.12\ldots nuvw\ldots}$, i.e. the *total* multiple correlation coefficient, i.e. that without vertical rules to the *uvw* ... subscripts, and further $\Delta_{1122\ldots nn}$ is the correlation coefficient determinant Δ less the first n rows and n columns, i.e.

$$= \begin{vmatrix} 1, & r_{0u}, & r_{0v}, & r_{0w} \cdots \\ r_{u0}, & 1, & r_{uv}, & r_{uw} \cdots \\ r_{v0}, & r_{vu}, & 1 & r_{vw} \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix} = \Delta', \text{ say.}$$

Thus:

$$\frac{\Delta_{1122\ldots nn}}{\Delta_{001122\ldots nn}} = 1 - R^2_{0.uvw\ldots} \text{ by (4).}$$

Accordingly:

$$1 - R^2_{0.123\ldots n|vuw\ldots|} = \frac{1 - R^2_{0.123\ldots nuvw}}{1 - R^2_{0.uvw\ldots}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(9).$$

It is thus possible to express any partial multiple correlation coefficient in terms of two total multiple correlation coefficients, namely that of the first characteristic on the universe of all the other characteristics, and that of the first characteristic on the sub-universe of the characteristics treated as constant in the partial multiple coefficient. We have thus three distinct methods of reaching partial multiple correlation coefficients as follows:

(*a*) as functions of total multiple coefficients by (9),

(*b*) as functions of total correlation coefficients, i.e. by (8)

$$1 - R^2_{0.12\ldots n|uvw\ldots|} = \frac{\Delta}{\Delta_{00}} \times \frac{\Delta'_{00}}{\Delta'},$$

(*c*) as functions of the partial correlation coefficients of successively lower orders as in (6), or again determinantally by partial correlation coefficients all of the same order, and that order the number of variables in $|uvw\ldots|$. Thus:

$$1 - R^2_{0.123\ldots n|uvw..|} = \frac{\begin{vmatrix} 1 & , \rho_{01.uvw\ldots}, & \rho_{02.uvw..}, & \cdots \rho_{0n.uvw..} \\ \rho_{10.uvw\ldots}, & 1 & , \rho_{12.uvw\ldots}, & \cdots \rho_{1n.uvw..} \\ \rho_{20.uvw\ldots}, & \rho_{21.uvw\ldots}, & 1 & , \cdots \rho_{2n.uvw..} \\ & & & \\ \rho_{n0.uvw\ldots}, & \rho_{n1.uvw\ldots}, & \rho_{n2.uvw\ldots}, & \cdots \quad 1 \end{vmatrix}}{\begin{vmatrix} 1 & , \rho_{12.uvw\ldots}, & \cdots \rho_{1n.uvw\ldots} \\ \rho_{21.uvw\ldots}, & 1 & , \cdots \rho_{2n.uvw\ldots} \\ \cdots & \cdots & \cdots \\ \rho_{n1.uvw\ldots}, & \rho_{n2.uvw\ldots}, & \cdots \quad 1 \end{vmatrix}} \quad \ldots\ldots\ldots\ldots\ldots\ldots(10).$$

* The order of pairs of subscripts may of course be interchanged.

In practice which of the three methods will be best depends upon what coefficients are already calculated, or what it may be desirable to calculate.

As we know from (4) that it is impossible for a multiple correlation coefficient to be perfect, unless the discriminant Δ vanishes, i.e. unless the characteristic 0 is an absolute linear function of the remaining characteristics, so we know from (5) that this involves one of the partial correlation coefficients of one or other order being perfect. A similar property holds for the partial multiple coefficient by (6), it cannot be perfect unless one of the partial correlation coefficients involving the constancy of at least $uvw\ldots$ (it may include some or all of the other variates) is also perfect.

Some of the simpler partial multiple correlation coefficients will now be considered individually.

(*a*) $R_{0.1|2|}$.

We have by (9):
$$1 - R^2_{0.1|2|} = \frac{1 - R^2_{0.12}}{1 - R^2_{0.2}},$$
$$1 - R^2_{0.12} = \frac{\Delta}{\Delta_{00}} = (1 - r^2_{01} - r^2_{12} - r^2_{20} + 2r_{01}r_{12}r_{20})/(1 - r^2_{12}),$$
$$1 - R^2_{0.2} = 1 - r^2_{20}.$$

Hence:
$$R^2_{0.1|2|} = 1 - \frac{1 - r^2_{01} - r^2_{12} - r^2_{20} + 2r_{01}r_{12}r_{20}}{(1 - r^2_{12})(1 - r^2_{20})}$$
$$= \frac{(r_{01} - r_{12}r_{02})^2}{(1 - r^2_{12})(1 - r^2_{20})} = \rho^2_{01.2},$$

in other words the multiple correlation coefficient of one variable on a second for a third constant is equal to the partial correlation coefficient of the first two variables for a constant third. This result is deduced at once from formula (6) for we have only one factor on the right-hand side, i.e.
$$1 - R^2_{0.1|2|} = (1 - \rho^2_{01.2}),$$
leading to
$$R^2_{0.1|2|} = \rho^2_{01.2}.$$

Lastly if we take (10) we have again
$$1 - R^2_{0.1|2|} = \begin{vmatrix} 1 & \rho_{01.2} \\ \rho_{01.2} & 1 \end{vmatrix} \div |1| = 1 - \rho^2_{01.2}$$
i.e. $R^2_{0.1|2|} = \rho^2_{01.2}$ as before.

(*b*) $R_{0.1|23|}$.

Here again we have only one factor in (6) and deduce at once
$$1 - R^2_{0.1|23|} = (1 - \rho^2_{01.23}),$$
or $R_{0.1|23|}$ is of the same numerical magnitude as $\rho_{01.23}$, a result which follows at once also from (10).

If we use (8) we have
$$1 - R^2_{0.1|23|} = \frac{\Delta}{\Delta_{00}} \times \frac{\Delta_{0011}}{\Delta_{11}} = \frac{\Delta}{\Delta_{00}}\,\frac{\Delta_{00}\Delta_{11} - \Delta^2_{01}}{\Delta_{11}\Delta}$$
$$= 1 - \frac{\Delta^2_{01}}{\Delta_{00}\Delta_{11}} = 1 - \rho^2_{01.23},$$

by aid of (3) and (7). Or, we deduce, if we wish to express $R_{0.1|23|}$ in total correlation coefficients, that

$$R^2_{0.1|23|} = \frac{\{r_{01}(1-r^2_{23}) - r_{12}(r_{02}-r_{13}r_{03}) - r_{13}(r_{03}-r_{02}r_{23})\}^2}{\{1-r^2_{02}-r^2_{03}-r^2_{23}+2r_{02}r_{03}r_{23}\}\{1-r^2_{12}-r^2_{13}-r^2_{23}+2r_{12}r_{13}r_{23}\}}.$$

It is quite clear that the result which we have reached in both cases (*a*) and (*b*) admits of extension by reason of (6) to every partial multiple coefficient in which there is only a single non-fixed variable, i.e. we have generally

$$R^2_{0.1|uvw...|} = \rho^2_{01.uvw...},$$

or every partial correlation coefficient, whatever number of variables be fixed, can be considered as a multiple correlation coefficient of a single changing variate.

(*c*) $R_{0.12|3|}$.

From (6) we have:

$$1 - R^2_{0.12|3|} = (1-\rho^2_{01.23})(1-\rho^2_{02.3}),$$

whence substituting for $\rho_{01.23}$ in first order partials by aid of (2) we have:

$$1 - R^2_{0.12|3|} = \frac{1-\rho^2_{01.3}-\rho^2_{12.3}-\rho^2_{02.3}+2\rho_{02.3}\rho_{12.3}\rho_{01.3}}{1-\rho^2_{12.3}},$$

which we should have obtained directly from (10). This gives:

$$R^2_{0.12|3|} = \frac{\rho^2_{02.3}+\rho^2_{01.3}-2\rho_{12.3}\rho_{01.3}\rho_{02.3}}{1-\rho^2_{12.3}},$$

which we should have obtained from the usual expression for $R^2_{0.12}$, i.e.

$$\frac{r^2_{02}+r^2_{01}-2r_{12}r_{02}r_{01}}{1-r^2_{12}}$$

by adding the partial subscript 3.

Again either by substituting the total coefficients in this result, or directly by observing that:

$$1 - R^2_{0.12|3|} = \frac{1-R^2_{0.123}}{1-R^2_{0.3}} = \frac{\Delta}{\Delta_{00}}\,\frac{1}{1-r^2_{03}},$$

we have

$$R^2_{0.12|3|} = \frac{\left[\begin{array}{c} r^2_{01}+r^2_{02}+r^2_{13}(r^2_{03}-r^2_{02})+r^2_{23}(r^2_{03}-r^2_{01})-2r^2_{03}r_{12}r_{13}r_{23} \\ -2(r_{01}r_{03}r_{13}+r_{02}r_{03}r_{23}+r_{01}r_{02}r_{12}) \\ +2(r_{01}r_{02}r_{13}r_{23}+r_{01}r_{03}r_{12}r_{23}+r_{02}r_{03}r_{12}r_{13}) \end{array}\right]}{(1-r^2_{03})(1-r^2_{12}-r^2_{13}-r^2_{23}+2r_{12}r_{13}r_{23})}$$

(*d*) $R_{0.12|uvw...|}$.

In the same way we have:

$$R^2_{0.12|uvw...|} = \frac{\rho^2_{01.uvw...}+\rho^2_{02.uvw...}-2\rho_{12.uvw...}\rho_{01.uvw...}\rho_{02.uvw}}{1-\rho^2_{12.uvw...}},$$

but it is best to work out the numerical values of the partial correlation coefficients rather than substitute their algebraic values in terms of total correlation coefficients. It is clear that even such quantities as $R_{0.12|34|}$ get very complicated if

expressed algebraically in terms of total correlation coefficients. It will be found simplest to evaluate them numerically, by (8) and (9):

$$1 - R^2_{0.12|34|} = \frac{1 - R^2_{0.1234}}{1 - R^2_{0.34}} = \frac{\Delta}{\Delta_{00}} \cdot \frac{\Delta_{00\,11\,22}}{\Delta_{11\,22}},$$

where $\quad \Delta_{00\,11\,22} = 1 - r^2_{34}, \quad \Delta_{11\,22} = 1 - r^2_{03} - r^2_{04} - r^2_{34} + 2r_{03}r_{04}r_{34},$

and can be easily found, while Δ and Δ_{00} being determinants of the fifth and fourth orders are most simply obtained numerically by continuous reduction to lower orders.

Or again we may proceed by (6) writing

$$1 - R^2_{0.12|34|} = (1 - \rho^2_{01.234})(1 - \rho^2_{02.34}),$$

which involves the calculation of one second and one third order partial correlation coefficient. This method is quite simple, if these coefficients have already been found, otherwise we shall need to compute three first order and six second order partials before we reach the desired third order coefficient. The process will be numerically longer than the numerical calculation of the determinants, but it of course gives additional information.

Applications. The possible applications—always assuming an approach to linearity in our associations—appear to be numerous. For example:

(i) What is the total influence on the London-Berlin rate of exchange of the London-Amsterdam and Berlin-Amsterdam rates, if we make the New York rates on London, Berlin and Amsterdam, respectively, constant?

(ii) What is the total influence of the two parents on their offspring, if we make any character in the four grandparents constant?

(iii) What is the total influence of p environmental factors, if we retain q hereditary factors at the same intensity, and *vice versâ*?

This paper was developed six years ago, but was not published then, in order that numerical illustrations might be provided of the value of partial multiple coefficients of correlation. Considerable work has been done towards applying the new formulae to data relating to the foreign exchanges; but it is uncertain when this work may be completed as the author's time is much occupied. On the recommendation of the Editor of *Biometrika*, the theory alone is now published, as such coefficients are needed for immediate use in practical statistics.

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ON CRITERIA FOR FACTORISING CORRELATED VARIABLES.

By STUART C. DODD, Ph.D., Princeton University. Fellow of the National Research Council and Rockefeller Foundation, U.S.A.

(1) Let us suppose that the system of correlated variables $x_1, x_2, \dots x_t$ are functions of the variables $v_1, v_2, \dots v_m$, where the number of the latter is less than that of the former, i.e. $m < t$. Mathematically we may say that

$$\left.\begin{array}{l} x_1 = f_1(v_1, v_2, \dots v_m) \\ x_2 = f_2(v_1, v_2, \dots v_m) \\ x_3 = f_3(v_1, v_2, \dots v_m) \\ \\ x_t = f_t(v_1, v_2, \dots v_m) \end{array}\right\} \qquad \dots\dots (1).$$

Theoretically we can eliminate the v's from these equations and we shall be left with $t-m$ equations of condition, say:

$$\left.\begin{array}{l} x_{m+1} = \phi_{m+1}(x_1, x_2, \dots x_m) \\ x_{m+2} = \phi_{m+2}(x_1, x_2, \dots x_m) \\ \\ x_t \quad = \phi_t \quad (x_1, x_2, \dots x_m) \end{array}\right\} \qquad (2).$$

Statistically $x_{m+1}, x_{m+2}, \dots x_t$ will be absolutely determined by $x_1, x_2, \dots x_m$; or, looked at geometrically, the regression surface for x_{m+1} on $x_1, x_2, \dots x_m$ contains all the points, i.e. there is no variability in any array of x_{m+1} for given $x_1, x_2, \dots x_m$. This amounts to saying statistically that the multiple correlation ratio of x_{m+1} on $x_1, x_2, \dots x_m$ is perfect or:

$$\eta^2_{x_{m+1}\cdot x_1, x_2, \dots x_m} = 1 \dots\dots (3).$$

Now if in the population sampled such a relation holds, it must also hold for every sample from that population. For every individual randomly selected can only have variates lying on the surface

$$x_{m+1} = \phi_{m+1}(x_1, x_2, \dots x_m),$$

and thus there can be no variation of x_{m+1} for given values of $x_1, x_2, \dots x_m$. In other words every truly random sample will also give

$$\eta^2_{x_{m+1}\cdot x_1, x_2, \dots x_m} = 1,$$

and there is no variation of this correlation ratio with sampling, i.e. this mth order multiple correlation ratio has a zero "probable error." Accordingly if $\eta^2_{x_{m+1}\cdot x_1, x_2, \dots x_m}$ be not unity for any sample, then either the original series is not factorisable into m factors, or the deviation from unity is not to be estimated by the "probable error" of random sampling, for the "probable error" is essentially zero, but only by some measure of the accuracy of the observations on which the data depend.

All tests by means of the standard deviations of random sampling are illusory, when we are seeking to determine whether a given correlation ratio (or a correlation coefficient) is perfect in the population sampled. If it be perfect in that population it will be perfect in the sample, for the standard deviation in samples will be zero. Errors of observation are of course a different matter, but their extent and variation are not controlled by the same laws as those of random sampling, and it is not legitimate to apply the latter laws to treatment of such errors.

In the above discussion we have taken the variate x_{m+1} and supposed it expressed as a unique function of $x_1, x_2, \ldots x_m$ so that $\eta^2_{x_{m+1}\cdot x_1, x_2, \ldots x_m}$ is unity. But clearly we are dealing here with any $m+1$ of the t variates, and we must therefore conclude that it is a needful condition of t variables factorising into m other variables that all the possible mth order multiple correlation ratios should be perfect. There are

$$t!/m!\,(t-m-1)!$$

such correlation ratios, and every one of these should be perfect in the sample. It does not, however, follow that there are this number of conditions to be independently satisfied. As a matter of fact there are only $t-m$. For, speaking statistically, if

$$\eta^2_{x_{m+1}\cdot x_1, x_2, \ldots x_m} = 1,$$

then we may interchange any of the $x_1, \ldots x_m$ variates with x_{m+1}, and the perfect association will still be maintained. In other words the perfect association indicated by a certain number of multiple correlation ratios carries with it the perfect association of a number of other multiple correlation ratios.

An elementary illustration of this may be given for a simple case. Let us suppose that three variates x_1, x_2 and x_3 are linear functions of v_1 and v_2. Then it follows that any x variate is a linear function of the other two. Accordingly the three multiple correlation coefficients

$$\rho_{1.23},\ \rho_{2.31} \text{ and } \rho_{3.12}$$

are perfect, or we have:

$$\left.\begin{aligned}
\rho^2_{1.23} &= \frac{r^2_{12} + r^2_{13} - 2r_{12}r_{13}r_{23}}{1 - r^2_{23}} = 1\\
\rho^2_{2.31} &= \frac{r^2_{23} + r^2_{21} - 2r_{23}r_{21}r_{31}}{1 - r^2_{31}} = 1\\
\rho^2_{3.12} &= \frac{r^2_{32} + r^2_{31} - 2r_{32}r_{31}r_{12}}{1 - r^2_{12}} = 1
\end{aligned}\right\} \quad \ldots\ldots(4).$$

These give each of them

$$1 - r^2_{12} - r^2_{23} - r^2_{31} + 2r_{12}r_{23}r_{31} = 0 = \begin{vmatrix} 1, & r_{12}, & r_{13} \\ r_{21}, & 1, & r_{23} \\ r_{31}, & r_{32}, & 1 \end{vmatrix} = 0 \quad \ldots\ldots\ldots(5),$$

or the discriminant is zero, which is the essential condition of the three equations not being independent.

Now the "probable error" of a multiple correlation coefficient for a system of linearly related variates is well known to be $\cdot 67449\,(1-\tilde{\rho}^2_{1.23})/\sqrt{N}$, where N is the size of the sample and $\tilde{\rho}^2_{1.23}$ the value in the sampled population. Accordingly if $\tilde{\rho}_{1.23}$ is perfect, i.e. unity, the probable error is zero, or the value in the sample can have no variation. Hence we conclude generally that since when the discriminant vanishes there is no variation in the discriminant:

(i) The discriminant must always factorise the standard deviation of the discriminant of samples.

(ii) The difference between the value of the discriminant in the sampled population and its mean value in samples must contain the discriminant in the sampled population as a factor.

The latter conclusion follows from the fact that it must always be possible to draw a sample having its discriminant equal to that of the population value, hence since when the discriminant of the sampled population vanishes there is no variation of the discriminant of the samples, the mean value of the discriminant in samples must take the population value, i.e. if D be the discriminant in the sampled population, $\bar{\Delta}$ the mean value of the discriminant in samples, Δ its value in a single sample and σ_Δ the standard deviation of Δ, then

$$\bar{\Delta} = D + \chi_1 D \quad \text{.................................(6)},$$

$$\sigma_\Delta = \chi_2 D \quad \text{...(7)},$$

where χ_1 and χ_2 are certain functions of the correlation coefficients.

(2) We may illustrate this by determining χ_1 and χ_2 directly in a simple case*. Let $\bar{r}_{ts}$ be the mean value of r_{ts} in samples and δr_{ts} in any sample be measured from $\bar{r}_{ts}$. Then if ρ_{ts} be the value of the correlation in the sampled population for the tth and sth variates we know that†

$$\bar{r}_{ts} = \rho_{ts}\left(1 - \frac{1-\rho^2_{ts}}{2N}\right) \quad \text{..............................(8)},$$

if we neglect terms of the order $\frac{1}{N^2}$ and higher orders.

It simplifies matters if we do not put

$$r_{ts} = r_{st},$$

but only in our final result.

Now let $\bar{\Delta}$ be the mean value of Δ in samples, then $\bar{\Delta}$ is not equal to D, nor on the other hand is it equal to $\tilde{D}$, where we write

$$\tilde{D} \text{ for } \begin{vmatrix} 1, & \bar{r}_{12}, & \dots & \bar{r}_{1n} \\ \bar{r}_{21}, & 1, & \dots & \bar{r}_{2n} \\ \vdots & & \dots & \\ \bar{r}_{n1}, & \bar{r}_{n2}, & \dots & 1 \end{vmatrix} \quad \text{..............................(9)}.$$

* It is needful to warn the reader that this is only a very special case, as it has been too often supposed that the vanishing or non-vanishing of the discriminantal determinant is an adequate criterion of factorising.

† H. E. Soper, *Biometrika*, Vol. IX. p. 105.

We put $r_{st} = \bar{r}_{st} + \delta r_{st}$ and note that Δ is a linear function of r_{st}. Accordingly if $\tilde{D}_{st}$ be the (s, t) minor of $\tilde{D}$:

$$\tilde{D} + \delta\Delta = \tilde{D} + \sum_{s,t} \tilde{D}_{st}\,\delta r_{st} + \text{product terms},$$

and we shall write

$$\delta\Delta = \Delta - \tilde{D} = \sum_{s,t} \tilde{D}_{st}\,\delta r_{st} + \text{product terms} \quad \text{..............(10).}$$

Here $\delta\Delta$ is not measured from its mean value $\bar{\Delta}$, but from $\tilde{D}$. Squaring and taking mean values, which mean values we shall denote by curled brackets, we have

$$\{(\delta\Delta)^2\} = \sum_{s,t} \tilde{D}^2_{st}\,\{\delta^2 r_{st}\} + 2 \sum_{s,t,u,v}{}' \tilde{D}_{st}\,\tilde{D}_{uv}\,\{\delta r_{st}\,\delta r_{uv}\} + \text{higher terms ...(11)},$$

where Σ denotes a sum for all values of s, t, (s not equal to t) and Σ' denotes a sum for all values of s, t, u, v, where s and t are not equal at the same time to u and v respectively. Now it has been shown by Pearson and Filon* that we have for material following a normal distribution:

$$\{\delta^2 r_{st}\} = (1 - \rho^2_{st})^2/N \text{ ....(12)},$$

where N is the size of the sample, and

$$\{\delta r_{st}\,\delta r_{uv}\} = \frac{1}{2N}\left[(\rho_{su} - \rho_{st}\rho_{ut})(\rho_{tv} - \rho_{tu}\rho_{vu}) + (\rho_{sv} - \rho_{uv}\rho_{su})(\rho_{ut} - \rho_{st}\rho_{su})\right.$$
$$\left. + (\rho_{su} - \rho_{sv}\rho_{uv})(\rho_{tv} - \rho_{st}\rho_{sv}) + (\rho_{sv} - \rho_{ut}\rho_{tv})(\rho_{tu} - \rho_{tv}\rho_{uv})\right] \text{ ......(13).}$$

But if we put $u = s$, $v = t$ in this expression, remembering that $\rho_{\kappa\kappa} = 1$, we find that it reduces to

$$\frac{1}{2N}\left[(1 - \rho^2_{st})(1 - \rho^2_{st}) + (1 - \rho^2_{st})(1 - \rho^2_{st})\right],$$

or

$$\{\delta^2 r_{st}\} = \frac{1}{N}(1 - \rho^2_{st})^2,$$

as before.

We can accordingly combine the Σ and the Σ' summations and write

$$\{(\delta\Delta)^2\} = \frac{1}{2N} \sum_{s,t,u,v} \tilde{D}_{st}\,\tilde{D}_{uv}\left[(\ldots)(\ldots) + (\ldots)(\ldots) + (\ldots)(\ldots) + (\ldots)(\ldots)\right]$$
$$+ \text{higher terms ....(14)},$$

the eight factors in round brackets being as above, and Σ being a sum which extends to *all* values of s, t, u, v, including the cases where $s = u$ and $t = v$†. Here to the first order of approximation we may write D_{st} for $\tilde{D}_{st}$, etc.

Consider the first pair in round brackets first:

$$\sum_{s,t,u,v} D_{st}\,D_{uv}\,(\rho_{su} - \rho_{st}\rho_{ut})(\rho_{tv} - \rho_{tu}\rho_{vu}) \text{ ....(15).}$$

This equals:

$$\sum_{t,u} \sum_{s} D_{st}(\rho_{su} - \rho_{st}\rho_{ut}) \sum_{v} D_{uv}(\rho_{tv} - \rho_{tu}\rho_{vu}) \text{ ....(16).}$$

* *Phil. Trans.* Vol. 191 A, pp. 259 and 262 (1898).

† The additional 2 in the denominator is introduced so as to generalise Σ the sum, and render u, v not controlled by s, t. It is equivalent to writing $ab + bc + ca = \frac{1}{2}(ba + cb + ac + ab + bc + ca)$, where every pair is taken in both orders.

But
$$\sum_s D_{st}\rho_{st} = D,\ \sum_v D_{uv}\rho_{uv} = D \quad\text{......................(17)},$$

$$\left.\begin{array}{l}\sum_s D_{st}\rho_{su} = 0, \text{ unless } u = t \text{ when it equals } D\\ \sum_v D_{uv}\rho_{tv} = 0, \text{ unless } u = t \text{ when it equals } D\end{array}\right\} \quad\text{..............(18)}.$$

If ϵ_{ut} be Kronecker's symbol, $= 0$ when u is not equal to t and $= 1$ when $u = t$, we have our expression in (16)

$$= D^2 \sum_{t,u} (\epsilon_{ut} - \rho_{tu})^2 \quad\text{............................(18a)},$$

$$= D^2 \sum_{t,u}{}'' (\rho^2_{tu}) \quad\text{.....................................(19)},$$

where Σ'' denotes a sum for all values of t and u except those for which $t = u$.

We proceed in exactly the same way for the other pairs of factors. In the second pair we sum first over the suffixes v and t; in the third over u and t, and the fourth over s and u. Accordingly there results:

$$\{(\delta\Delta)^2\} = \frac{1}{2N} D^2\, 4\sum_{t,u}{}'' (\rho^2_{tu}).$$

Any correlation coefficient will occur twice, i.e. as ρ^2_{tu} and as ρ^2_{ut}. Accordingly we find

$$\{(\delta\Delta)^2\} = \frac{4D^2\,(\text{sum of squares of correlation coefficients})}{N} \quad\text{......(20)}.$$

We have not, however, yet established that this is the standard deviation of Δ, i.e. σ_Δ^2. The above is the mean square deviation of Δ from $\tilde{D}$ and we have for σ_Δ^2:

$$\sigma_\Delta^2 = \{(\delta\Delta)^2\} - (\{\delta\Delta\})^2 \quad\text{.........................(21)}.$$

It is needful accordingly to find how the mean value of Δ in samples, i.e. $\bar{\Delta}$, differs from the population-value D and the value $\tilde{D}$ of the determinant with the mean correlation coefficients, like $\bar{r}_{ts}$, inserted.

Now for any sample:

$$\Delta = \tilde{D} + \sum_{s,t} \tilde{D}_{st}\,\delta r_{st} + \tfrac{1}{2}\sum_{s,t,u,v} \tilde{D}_{stuv}\,\delta r_{st}\,\delta r_{uv} + \text{higher order products} \quad\text{......(22)},$$

where the odd subscripts refer to rows and the even to columns, and we retain initially the distinction between r_{ij} and r_{ji}. The only limitation is that s and t shall not be equal to t and v respectively. The $\frac{1}{2}$ is again introduced to allow u, v freedom from s, t. (See footnote, p. 48.)

When we take mean values the linear terms vanish and we have:

$$\{\delta\Delta\} = \Delta - \tilde{D} = \frac{1}{4N}\sum_{s,t,u,v} D_{stuv}[(\ldots)(\ldots) + (\ldots)(\ldots) + (\ldots)(\ldots) + (\ldots)(\ldots)]^* + \text{higher terms} \quad\text{.................(23)},$$

* We will represent the four summations as Q_1, Q_2, Q_3 and Q_4.

where the curved brackets contain the same factors as in Equation (13). We shall evaluate the four series in order:

$$Q_1 = \sum_{s,t,u,v} D_{stuv}(\rho_{su} - \rho_{st}\rho_{tu})(\rho_{tv} - \rho_{tu}\rho_{uv})$$

$$= \sum_{\substack{t,u,v\\t\neq v}} (\rho_{tv} - \rho_{tu}\rho_{uv}) \sum_{\substack{s\\s\neq u}}' D_{stuv}(\rho_{su} - \rho_{st}\rho_{tu})$$

$$= \sum_{\substack{t,u,v\\t\neq v}} (\rho_{tv} - \rho_{tu}\rho_{uv}) D_{uv}(\epsilon_{tu} - \rho_{tu}) \quad \ldots\ldots\ldots\ldots\ldots(24),$$

where ϵ_{tu} is, as before, Kronecker's symbol,

$$= \sum_{t,u} (\epsilon_{tu} - \rho_{tu}) \sum_{\substack{v\\v\neq t}} D_{uv}(\rho_{tv} - \rho_{tu}\rho_{uv})$$

$$= \sum_{t,u} (\epsilon_{tu} - \rho_{tu})\left[\sum_{v} D_{uv}(\rho_{tv} - \rho_{tu}\rho_{uv}) - D_{ut}(1 - \rho^2_{tu})\right]$$

$$= \sum_{t,u} (\epsilon_{tu} - \rho_{tu})\left[(\epsilon_{tu} - \rho_{tu}) D - D_{ut}(1 - \rho^2_{tu})\right]$$

$$= D \sum_{\substack{t,u\\t\neq u}} (\rho^2_{tu}) + \sum_{t,u} \rho_{tu} D_{ut}(1 - \rho^2_{tu})$$

$$= D \sum_{\substack{t,u\\t\neq u}} (\rho^2_{tu}) + nD - \sum_{\substack{t,u\\t\neq u}} D_{ut}\rho^3_{ut} \quad \ldots\ldots\ldots\ldots\ldots(24\,a).$$

The second term

$$Q_2 = \sum_{\substack{s,t,u,v\\s\neq u,\, t\neq v}} D_{stuv}(\rho_{sv} - \rho_{su}\rho_{uv})(\rho_{ut} - \rho_{us}\rho_{st})$$

$$= \sum_{\substack{t,s,u\\s\neq u}} (\rho_{ut} - \rho_{us}\rho_{st}) \sum_{\substack{v\\v\neq t}} D_{stuv}(\rho_{sv} - \rho_{su}\rho_{uv})$$

$$= \sum_{\substack{t,s,u\\s\neq u}} (\rho_{ut} - \rho_{us}\rho_{st})(-D_{ut} - \rho_{su} D_{st}),$$

because $D_{stuv} = -D_{svut}$,

$$= -\sum_{s,t} (D - \rho_{st} D_{st}) + \sum_{s,t} \rho_{st}(D\epsilon_{st} - D_{st})$$
$$- \sum_{u,t} \rho_{ut}(D\epsilon_{ut} - D_{ut}) + \sum_{\substack{s,u\\s\neq u}} (\rho^2_{su}) D$$

$$= -n(n-1) D + D \sum_{\substack{s,u\\s\neq u}} \rho^2_{su} \quad \ldots\ldots\ldots\ldots\ldots(25).$$

The third term

$$Q_3 = \sum_{\substack{s,t,u,v\\s\neq u,\, t\neq v}} D_{stuv}(\rho_{su} - \rho_{sv}\rho_{vu})(\rho_{tv} - \rho_{ts}\rho_{sv})$$

$$= \sum_{\substack{s,t,v\\t\neq v}} (\rho_{tv} - \rho_{ts}\rho_{sv}) D_{st}(\epsilon_{sv} - \rho_{sv})$$

$$= \sum_{s,v} (\epsilon_{sv} - \rho_{sv})\left[D(\epsilon_{sv} - \rho_{sv}) - D_{sv}(1 - \rho^2_{sv})\right]$$

$$= D \sum_{\substack{s,v\\s\neq v}} (\rho^2_{sv}) + nD - \sum_{\substack{s,v\\s\neq v}} (D_{sv}\rho^3_{sv}) \quad \ldots\ldots\ldots\ldots\ldots(26).$$

Finally:

$$Q_4 = \sum_{\substack{s,t,u,v\\ s\neq u,\, t\neq v}} D_{stuv}(\rho_{sv} - \rho_{st}\rho_{tv})(\rho_{tu} - \rho_{tv}\rho_{vu})$$

$$= \sum_{\substack{s,t,v\\ t\neq v}} (\rho_{sv} - \rho_{st}\rho_{tv}) \sum_{\substack{u\\ u\neq s}} D_{stuv}(\rho_{tu} - \rho_{tv}\rho_{vu})$$

$$= \sum_{\substack{s,t,v\\ t\neq v}} (\rho_{sv} - \rho_{st}\rho_{tv})[-D_{sv} - \rho_{tv} D_{st}]$$

$$= -n(n-1)D + D \sum_{\substack{t,v\\ t\neq v}} (\rho^2_{tv}) \quad \text{.................................(27).}$$

Substituting Q_1, Q_2, Q_3, Q_4 in

$$\{\delta\Delta\} = \{\Delta\} - \tilde{D} = \frac{1}{4N}(Q_1 + Q_2 + Q_3 + Q_4)$$

$$= \frac{1}{4N}\Big[D \sum_{\substack{t,u\\ t\neq u}} (\rho^2_{tu}) + nD - \sum_{\substack{t,u\\ t\neq u}} (D_{ut}\rho^3_{ut})$$

$$- n(n-1)D + D \sum_{\substack{s,u\\ s\neq u}} (\rho^2_{su})$$

$$+ D \sum_{\substack{s,v\\ s\neq v}} (\rho^2_{sv}) + nD - \sum_{\substack{s,v\\ s\neq v}} (D_{sv}\rho^3_{sv})$$

$$- n(n-1)D + D \sum_{\substack{t,v\\ t\neq v}} (\rho^2_{tv})\Big] \quad \text{.......................(28),}$$

we see that there are in all four complete summations of the ρ^2_{ij}, and that in each ρ^2_{ji} can occur as well as ρ^2_{ij} owing to our generalisation of the original sums. In the same way we have in the cubic terms $D_{ut}\rho^3_{ut}$ and $D_{tu}\rho^3_{tu}$. Thus we have

$$\{\delta\Delta\} = \frac{1}{4N}\Big[D\, 4\sum_{\substack{t,u\\ t\neq u}} (\rho^2_{tu}) + 2nD - 2n(n-1)D - 2\sum_{\substack{t,u\\ t\neq u}} (D_{ut}\rho^3_{ut})\Big] \quad \text{.........(29)}$$

Now: $\tilde{D} = |\bar{r}_{ut}|$, but, as Soper has shown,

$$\bar{r}_{ut} = \rho_{ut}\left(1 - \frac{1-\rho^2_{ut}}{2N}\right) + \text{higher order terms} \quad \text{...............(30).}$$

Hence:

$$\tilde{D} = |\rho_{ut}| - \sum_{\substack{u,t\\ u\neq t}} \frac{[\rho_{ut}(1-\rho^2_{ut})D_{ut}]}{2N} + \text{etc.}$$

$$= D - \frac{nD}{2N} + \frac{1}{2N}\sum_{\substack{u,t\\ u\neq t}} (D_{ut}\rho^3_{ut}) + \text{etc.} \quad \text{.................(31).}$$

Clearly, in terms divided by N, $\tilde{D}$ or $\tilde{D}_{ut}$ may be replaced to our order of approximation by D or D_{ut}. Accordingly

$$\{\delta\Delta\} = \{\Delta\} - \tilde{D} = D - \tilde{D} + \frac{1}{4N}\Big[D\, 4\sum_{\substack{t,u\\ t\neq u}} (\rho^2_{tu}) - 2n(n-1)D\Big] \quad \text{......(32).}$$

Or, disregarding the double occurrence of ρ^2_{tu} and ρ^2_{ut} we have as far as the order $\frac{1}{N}$:

$$\{\Delta\} = D + \frac{1}{N}[2D \text{ (sum of squares of correlation coefficients)} - \tfrac{1}{2}Dn(n-1)] \quad \text{..............(33).}$$

Now the above equations show that $\{\delta\Delta\}$ is of the order $1/N$ and accordingly its square is of the order $1/N^2$. Hence unless in our evaluation of $\{(\delta\Delta)^2\}$ we go to terms of the order $1/N^2$, we are not justified in retaining the term $(\{\delta\Delta\})^2$ in equation (21).

Thus to the order $1/N$:

$\{\Delta\}$ = mean discriminant in samples

$$= D + \frac{D}{N}[2 \text{ (sum of squared correlation coefficients)} - \tfrac{1}{2}n(n-1)] \quad \text{...(34)*},$$

$$\sigma_\Delta{}^2 = \frac{4D^2}{N} \text{ (sum of squared correlation coefficients)} \quad \text{..........................(35).}$$

It is clear from these equations, that to the above order, when $D = 0$ in the sampled population, $\sigma_\Delta = 0$ in the samples, and every sample must take for its Δ also the zero value. But it might be argued that the mean and standard deviation of Δ depend, when $D = 0$, on the terms in $1/N^2$ and higher powers, and that if these could be worked out they would show that Δ is subject to variation although $D = 0$. Such terms however, albeit they might be hard to compute, must also have D for a factor, and vanish like the terms in $1/N$ for $D = 0$. This follows at once from the previous and more general method of treating the problem at the beginning of this paper, and the present inadequate consideration of a special case has only been included because it represents the line of approach adopted by certain psychologists.

It is clear that no deviation from zero in the discriminant of a sample can be attributed to errors of random sampling. Either the discriminant does not vanish in the population sampled or, if it does, the deviation from zero in the sample must be due to inaccuracy of observation, and the laws of such inaccuracies are not those of random sampling. The obvious course is to repeat the observations with greater care; if on reduction the discriminant shows no closer approach to zero, it is fairly certain that it cannot be zero in the sampled population.

This study owes much to the assistance of Mr Philip Hall and of Professor Karl Pearson, in whose laboratory it was made.

* This agrees with the value given by Isserlis for the special case of $n=3$. See *Phil. Mag.* Sept. 1917, pp. 205—220. His formula on p. 211 is correct, but on p. 213 there is a misprint in equation (11), i.e. 2Δ for $4\Delta^2$.

UEBER DIE ANWENDUNG DER DIFFERENZENMETHODE ("VARIATE DIFFERENCE METHOD") BEI REIHENAUSGLEICHUNGEN, STABILITÄTSUNTERSUCHUNGEN UND KORRELATIONSMESSUNGEN.

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II.

(12) "Die Lexis'sche Dispersionstheorie gipfelt, bekanntlich, in dem Begriff der normalen Stabilität (bezw. Dispersion). Als normal stabil werden Reihen von statistischen Zahlen bezeichnet, deren Schwankungen um ihren Mittelwert den Erwartungen entsprechen, welche sich aus dem Schema einer unveränderlichen Wahrscheinlichkeit bei gegenseitiger Unabhängigkeit aller Versuche ergeben. Sind die Schwankungen geringer, als es dem Schema der normalen Dispersion entsprechen würde, so ist die Dispersion unternormal (bezw. die Stabilität der Reihe übernormal). Als Kriterium zur Unterscheidung der normalen Dispersion gilt für Lexis die Größe des sogenannten Divergenzkoeffizienten Q^2*." Prof. L. v. Bortkiewicz hat die Begriffe "normale," "übernormale," "unternormale Dispersion" von dem speziellen Falle der als Häufigkeiten erscheinenden statistischen Zahlen, welchen Lexis im Auge hatte, auf beliebige statistische Durchschnittszahlen ausgedehnt. "Bezeichnet man durch das Zeichen E die mathematische Erwartung einer zufälligen variablen Größe, durch $x_{i,j}$ den zufälligen Wert, welchen die Größe x beim j-ten Versuche der i-ten Serie annimmt, und setzt

$$Ex^h = m_h; \quad E(x-m_1)^h = \mu_h; \quad x_{i(n)} = \frac{1}{n}\sum_{j=1}^{n} x_{i,j}; \quad x_{(rn)} = \frac{1}{rn}\sum_{i=1}^{r}\sum_{j=1}^{n} x_{i,j},$$

so haben die Größen:

$$Q^2 = \frac{\frac{1}{r}\sum_{i=1}^{r}\left[x_{i(n)} - m_1\right]^2}{\frac{1}{n}\mu_2} \quad \text{.............................(34)},$$

$$\mathfrak{Q}^2 = \frac{\frac{1}{r-1}\sum_{i=1}^{r}\left[x_{i(n)} - x_{(rn)}\right]^2}{\frac{1}{n(rn-1)}\sum_{i=1}^{r}\sum_{j=1}^{n}\left[x_{i,j} - x_{(rn)}\right]^2} \quad \text{....................(35)},$$

genau dieselbe Bedeutung, wie die Lexis'schen Q^2 in dem oben erwähnten speziellen Falle†."

Q^2 kann nur dann rechnerisch bestimmt werden, wenn μ_2 und m_1 *a priori* bekannt sind. $\mathfrak{Q}^2$ wird aus einer gegebenen statistischen Reihe direkt berechnet.

* Al. A. Tchouproff, "Ist die normale Stabilität empirisch nachweisbar?" S. 369.

† Ebenda, S. 371.

Hauptsächlich durch Prof. Al. A. Tchouproff ist es jedoch nachgewiesen worden, daß der Koeffizient $\mathfrak{Q}^2$ nicht nur im Falle einer unveränderlichen Grundwahrscheinlichkeit bei gegenseitiger Unabhängigkeit aller Versuche, sondern auch in verschiedenen anderen Fällen, zum Beispiel bei "uniform verbundenen Reihen," genau den Wert $+1$ annimmt. Es entsteht daher die ganz natürliche Frage: welche formale Eigenschaften muß eine statistische Reihe besitzen, damit sich aus ihr genau $EQ^2=1$ ergibt, und wie sollen, ferner, diejenigen Reihen beschaffen sein, welche ein $EQ^2>1$ oder <1 aufweisen? Das ist, im Grunde genommen, ganz dieselbe Frage, die wir uns in den vorhergehenden §§ bezüglich der Differenzenmethode vorgelegt haben. Wir glauben uns daher auch berechtigt, hier, entgegen den traditionellen Annahmen, dieselben vereinfachenden Bedingungen einzusetzen, wie oben, das heißt, die einschränkenden Bedingungen des § 10 (*Biometrika*, Vol. XVIII., S. 315).

Es sei ferner zur Vereinfachung angenommen, daß alle $x_{i,j}$ fortlaufend von 1 bis nr numeriert sind. Auch r_j behalte die ihm in Formel (30) zugemessene Bedeutung.

Da EQ^2, bekanntlich, gleich dem Quotienten aus den mathematischen Erwartungen seines Zählers und Nenners angenommen werden kann, so erhalten wir—nach einigen Umformungen—aus (34):

$$EQ^2=1+\frac{2}{n}\sum_{j=1}^{n-1}(n-j)\,r_j \qquad\text{.........................(36).}$$

Damit nun $EQ^2=1$, *bei beliebigem* n (von $n=2$ angefangen), müssen, offenbar, alle r_j gleich 0 werden. Somit hätten wir dann, bei Kenntnis der wahren Werte μ_2 und m_1, dieselben Bedingungen für eine "normal stabile" Reihe, wie für eine "Gruppe R" erlangt. Ist jedoch μ_2 unbekannt, so müssen wir zur Formel (35) $\mathfrak{Q}^2$ übergehen, bei welcher die Bedingungsgleichungen für $E\mathfrak{Q}^2=1$ noch bedeutend komplizierter ausfallen, als bei Anwendung der Differenzenmethode*.

* Wie zu erwarten stand, erweist es sich aber, das $E\mathfrak{Q}^2=1$ auch wenn $r_1=r_2=\ldots=r_{nr-1}\neq 0$, das heißt, auch im Falle einer uniform verbundenen Reihe.

Um bei $E\mathfrak{Q}^2$ zu einfacheren Formeln zu gelangen, müßte man zum Teil das Schema des § 10 umändern und die Bedingung einführen: $E(x_i-x_{(nr)})(x_{i+j}-x_{(nr)})$ sei von i unabhängig und nur von j abhängig, so daß

$$R_j=\frac{E(x_i-x_{(nr)})(x_{i+j}-x_{(nr)})}{\frac{nr}{nr-1}E(x_i-x_{(nr)})^2}.$$

Dann kommt man zur Formel:

$$E\mathfrak{Q}^2=\frac{nr-1}{nr-n}+\frac{2r}{n(r-1)}\sum_{j=1}^{n-1}(n-j)\,R_j \qquad\text{...................................(37).}$$

Da $R_j=-\frac{1}{nr}$ nicht nur für eine unverbundene, sondern auch für eine uniform verbundene Reihe, so wird in beiden Fällen, wie auch zu erwarten stand, die Formel eben $E\mathfrak{Q}^2=1$ ergeben.

Als empirische Annäherungsformel für R_j kann R_j' dienen, welches sich aus Formel (29 a) bestimmt, wo $N=nr$ zu setzen ist.

Es ist aber zu beachten, daß gerade zwischen EQ^2_{k+1} und $E\sigma_k'^2$ gewisse Analogien gefunden werden können, denn beide sind lineare Funktionen von nur k Unbekannten: $r_1, r_2, \ldots r_k$, während in einem jeden $E\mathfrak{Q}^2$ bei beliebigem n *alle* $nr-1$ Unbekannten von r_1 bis r_{nr-1} auftreten müssen. Das hängt damit zusammen, daß im Falle eines Konstanten Ex, ganz gleich ob es bekannt ist oder nicht, man immer:

$$x_i-x_{i+j}=(x_i-Ex)-(x_{i+j}-Ex)\ \text{setzen kann.}$$

Wenn man sich verabredet, durch ein Subskriptum bei Q^2 die Zahl der in jede Serie eintretenden Reihenglieder anzugeben, so kann aus Formel (36) leicht folgende Formel abgeleitet werden:

$$E[Q^2_{n+1} - Q_n^2] = EQ^2_{n+1} - EQ_n^2 = \frac{2}{n(n+1)} \sum_{i=1}^{n} ir_i \;\ldots\ldots\ldots\ldots(38).$$

Drückt man diese Formel, gleich Formel (33 *a*), in umgekehrten Differenzen der Reihe (31) aus, so erhält man nach einigen Umformungen:

$$E[Q^2_{n+1} - Q^2_n] = \frac{2}{n(n+1)} [C^{n-1}_{n+1} r_1 - 2C^{n-2}_{n+1} \nabla' r_1 + 3C^{n-3}_{n+1} \nabla'' r_1 - 4C^{n-4}_{n+1} \nabla''' r_1 + \ldots$$
$$+ (-1)^{n-1} nC^0_{n+1} \nabla^{(n-1)} r_1] \ldots(39).$$

Aus Formeln (36), (38) und (39) ist es möglich, 3 Systeme aufzubauen, welche ihrerseits den beiden Systemen (33) und (33 *a*) analog sind.

Wir werden hier nur einige der einfachsten Fälle kurz besprechen.

Aus (36) erhalten wir die Bedingungen, unter welchen für eine *bestimmte Zahl* n der eine Serie bildenden Beobachtungen sich genau $EQ^2 = 1$ ergibt. Wir ersehen, daß dazu die einzige Gleichung: $\sum_{j=1}^{n-1} (n-j) r_j = 0$ genügt. Ist außerdem noch $\sum_{i=1}^{n} ir_i = 0$, so wird auch EQ^2_{n+1} genau den Wert $+1$ annehmen. Zwei Gleichungen genügen aber nicht zur Bestimmung von n Unbekannten, wenn n größer als 2 ist, und die weiteren Eigenschaften einer "normal stabilen" Reihe bleiben also in dieser Einstellung recht unbestimmt. Am plausibelsten ist noch die Annahme eines symmetrischen Aufbaues der Reihe $r_1, r_2, \ldots r_{n-1}$, so daß etwa $r_1 = r_{n-1}, r_2 = r_{n-2}, r_3 = r_{n-3}$, etc. wobei aber $\sum_{i=1}^{n-1} r_i = 0$ und $r_n = 0$ sein müßten. Sind auch alle *weiteren* $EQ^2_{n+2}, EQ^2_{n+3}, \ldots$ u.s.w. gleich 1, so ergibt das nur die weitere ebenfalls plausible Bedingung: $r_{n+1} = r_{n+2} = r_{n+3} = \ldots = 0$.

Für den Fall:

$$1 < EQ_2^2 < EQ_3^2 < EQ_4^2 < \ldots$$

ergeben sich, andererseits, aus (38) folgende Bedingungen:

$$r_1 > 0; \quad r_1 + 2r_2 > 0; \quad r_1 + 2r_2 + 3r_3 > 0, \ldots \sum_{i=1}^{n} ir_i > 0.$$

Diese Bedingungen sind liberaler, als die entsprechenden und ihnen verwandten Ungleichungen für eine "G-Gruppe" (siehe § 10). Es genügt schon, zum Beispiel, wenn alle $r_i > 0$ sind. Die Reihe der EQ^2 wächst am raschesten, wenn alle r_j gleich $+1$ ausfallen. Dann ist $EQ^2_{n+1} - EQ^2_n = 1$ bei beliebigem n und folglich: $EQ^2_{n+1} = n + 1$. Unter denselben Bedingungen ist aber $E\sigma_k'^2$, wie aus (32) ersichtlich, *ein Minimum* und gleich 0 ebenfalls bei beliebigen k.

Soll $1 > EQ_2^2 > EQ_3^2 > \ldots$ sein, so erhalten wir folgende Bedingungsungleichungen:

$$r_1 < 0; \quad r_1 + 2r_2 < 0; \quad r_1 + 2r_2 + 3r_3 < 0, \ldots \sum_{i=1}^{n} ir_i < 0.$$

Dazu genügt, daß alle $r_j < 0$ sind. Diese dürften aber dann, in der Regel, recht klein ausfallen. Denn ist, zum Beispiel, die statistische Reihe uniform (und also alle r_j einander gleich), so erhält man, dank dem Umstande, daß EQ^2 nicht kleiner als 0 sein kann, aus (36) die folgende untere Grenze: $r_{\text{minim.}} = -\frac{1}{n-1}$. Es müssen hierbei aber alle r_j bei $j \geqq n$ gleich Null werden. Sind jedoch diese bis r_{nr-1} uniform, so ist ihre minimale Grenze nur $-\frac{1}{nr-1}$.

Wenn wir annehmen, daß die Zeichen vor r_j regelmäßig alternieren, so erhalten wir bei $r_j = (-1)^j$ ("maximal-schädliche Z-Gruppe") aus (36):

$$EQ_n^2 = 0, \text{ bei geradem } n, \text{ und } EQ_n^2 = \frac{1}{n}, \text{ bei ungeradem } n.$$

($E\sigma_k'^2$ ist aber bei denselben Bedingungen ein Maximum und gleich $\frac{4^k}{C_{2k}^k}\mu_2$ oder, angenähert, $\mu_2\sqrt{\pi k}$.)

Im allgemeinen sind jedoch die Bedingungen für eine übernormal stabile Reihe in dieser Einstellung etwas strenger, als diejenigen für eine "Z-Gruppe."

Freilich, begnügt man sich damit, daß nur immer $EQ^2_n < 1$ sein soll, so erhält man auch aus (36) sehr coulante Bedingungen.

Aus obigen Ausführungen ist es schon ersichtlich, daß die Einteilung in Gruppen "G," "R" und "Z" unstreitbar in einem gewissen verwandtschaftlichen Verhältnis zu der Lexis'schen Klassifikation steht. Es wäre wohl möglich, die Differenzenmethode so zu modifizieren, daß ihre 3 Gruppen genau mit den Lexis'schen übereinstimmen. Da aber hierbei jede durch Differenzieren zu bearbeitende Reihe stark verkürzt würde, also dem praktischen Gebrauch auch größere Schwierigkeiten in den Weg legen müßte, so kann dieser Fall hier übergangen werden. Es ist auch nicht recht ersichtlich, warum gerade diese formale Klassifikation alle übrigen ihr verwandten so sehr überragen sollte.

Ein einzelnes Q^2 dürfte, im allgemeinen, keinesfalls als ein sicheres Maß der Stabilität einer statistischen Reihe angesehen werden, denn, wie wir sehen, ist seine Größe manchmal geradezu durch den Umfang von n bestimmt. Es wäre also angezeigt, wenn das gegebene statistische Material es zuläßt, immer eine ganze Reihe von Q^2 für verschiedene n zu berechnen, was freilich den Nachteil hätte, einen beträchtlichen Mehraufwand an Arbeit zu erfordern.

Ganz anders stellt sich die Sache, wenn die Reihe $x_1, x_2, \ldots$ als eine Summe einer unternormal stabilen, einer normal stabilen und einer übernormal stabilen Komponente angesehen werden kann. Dann wird, je größer n, desto mehr die unternormale Komponente die beiden andern überwuchern. Will man hier mit Hilfe des Kriteriums Q^2 die Anwesenheit einer übernormal stabilen Komponente feststellen, so ist es dasselbe, als wenn man die Anwesenheit eines bestimmten Giftes in einem gegebenen Stoffe dadurch zu konstatieren sucht, daß man ihn gleich am Anfang mit einem Reaktiv bearbeitet, welches notorisch dieses Gift vollkommen zerstört.

Unbeschadet der großen geschichtlichen Rolle der Lexis-Bortkiewicz'schen Kriterien Q^2 und $\mathfrak{Q}^2$, ist es, vielleicht, an der Zeit, die Frage aufzuwerfen, ob es nicht in manchen Fällen angebracht wäre, Q^2 und besonders $\mathfrak{Q}^2$ durch andere Mittelwerte zu ersetzen. Uns, wenigstens, erscheint es, daß die Differenzenreihe $\sigma_2'^2 - \sigma_1'^2$, $\sigma_3'^2 - \sigma_2'^2$, u.s.w. viel eher die Anwesenheit einer der "übernormal stabilen" verwandten Komponente in einer evolutorischen Reihe festzustellen erlaubt, als sogar eine Anzahl von verschiedenen $\mathfrak{Q}^2$*.

Die dadurch erzielten Vorteile wären die folgenden:

(1) Die Differenzen $\sigma_{i+1}'^2 - \sigma_i'^2$ eliminieren automatisch den Einfluß der für die Lexis'sche Methode so fatalen "glatten" Komponente; sie finden dabei ihre Analogie nicht im empirischen $\mathfrak{Q}^2$, sondern im mehr apriorischen Q^2, da durch Anwendung der endlichen Differenzen alle x_i auf die Form $x_i - Ex$ zurückgeführt werden. (2) Ihre obere und untere Grenze sind für jedes k genau bestimmt, und daher können sie, mit entsprechenden Koeffizienten multipliziert, bis zu einem gewissen Grade als Maß des "Einflusses" der Komponenten "Z" und "G" in der k-ten Differenz angesehen werden. (3) Sie sind für die "Zeitfolge zufälliger Ereignisse" ihrer Struktur nach empfindlicher als Q^2 oder $\mathfrak{Q}^2$. (4) Der Übergang von der Differenz k zur Differenz $k+1$ ist immer leicht zu bewerkstelligen, derjenige aber von der Grundzahl n in einer Serie zu $n+1$—nur dann, wenn sich N durch n und $n+1$ genau teilen läßt†.

(13) Es wird nicht selten darauf hingewiesen, daß die Differenzenmethode durch die Anwesenheit der "Z"-Komponente kompromettiert werde. *Es ist mir bis jetzt trotz eifrigen Suchens nicht gelungen, eine statistische Reihe zu finden, welche wirklich unstreitbar eine solche Komponente aufzuweisen hätte.* Damit ist aber noch durchaus nicht gesagt, daß derartige Komponenten überhaupt nicht existieren. Jedenfalls, würde die Feststellung ihres öfteren Auftretens, dank ihrer Verwandtschaft mit den Lexis'schen übernormal stabilen Reihen, eine interessante wissenschaftliche Entdeckung ausmachen, welche die dazu verwandte Arbeit vollauf

* Will man an einer gewissen Analogie mit Q^2 festhalten, so kann man auch die Größen $\frac{\sigma_k'^2}{\sigma'^2}$ oder die Koeffizienten $O_i = \frac{\sigma_i'^2 - \sigma'^2_{i-1}}{\sigma'^2_{i-1}}$ bestimmen. Letztere entsprechen genau den Kriterien $\frac{{}_0D_i}{2i-1}$ und $\frac{{}_kD_{i-k}}{2i-1}$ in "Über ein neues Verfahren."

† Im ersten Teil der vorliegenden Arbeit, welche im Januar 1926 für den Druck abgeschlossen wurde, konnte die G. U. Yule'sche Veröffentlichung "Why do we sometimes get nonsense-correlations between time series? A study in sampling and the nature of time series" (*Journ. Roy. Stat. Soc.*, LXXXIX, 1926, Part I) nicht mehr berücksichtigt werden. Ohne hier auf den Inhalt dieser interessanten Veröffentlichung weiter einzugehen, möchte ich nur darauf hinweisen, daß Yule's "serial correlations" meinen Koeffizienten $r_1, r_2, \ldots r_k$ genau entsprechen, wenn man letztere aus der Sprache der mathematischen Erwartungen in die sonst in England übliche mathematische Ausdrucksweise übersetzt. Seine "random" und "conjunct series" reihen sich in Lexis' "normal stabile" und "unternormal stabile Reihen" ein. Die "disjunct" und "oscillatory series" entsprechen aber beide gewissen Spezialfällen der Lexis'schen "ubernormal stabilen Reihen." Einen Nachteil der Yule'schen Klassifikation bildet der Umstand, daß sie nicht erschöpfend ist.

Ich benutzte die Koeffizienten $r_1, r_2, \ldots r_k$ zur Bestimmung der Reihentypen "Z," "R," "G" und "S" schon in meiner Arbeit "Über ein neues Verfahren" etc., also noch im Jahre 1923 (*Biometrika*, Vol. XV., parts 1 and 2).

rechtfertigen würde. Eventuell würde sie zu einer Art "*Neo-Queteletismus*" führen können.

Was nun speziell die Yule'schen *regelmäßigen* wellenartigen Komponenten mit wenigen Gliedern in einer vollen Periode anbetrifft, so kann gerade deren "schädlicher" Einfluß, wenn man will, spielend leicht beseitigt werden.

Besitzt nämlich die Reihe $u_1, u_2, \ldots u_N$ eine solche Komponente: $z_1, z_2, \ldots z_N$, wobei offenbar jedes $z_i = z_{i+j}$, wenn j die (konstante) Zahl der Glieder in einer vollen Periode bedeutet, so braucht man nur die Differenzen:

$$u_{1,j} = u_1 - u_{1+j},\quad u_{2,j} = u_2 - u_{2+j},\quad u_{3,j} = u_3 - u_{3+j},\ \ldots\ u_{i,j} = u_i - u_{i+j},$$

zu bilden, um die Komponente Z vollkommen verschwinden zu lassen. Die Reihe: $u_{1,j}, u_{2,j}, \ldots u_{N-j,j}$, in welcher die Ordnung der Parabel schon um 1 verringert ist, kann dann fast genau nach den gewöhnlichen Regeln der Differenzenmethode bearbeitet werden, denn es ist, zum Beispiel, leicht zu beweisen, daß

$$\frac{E\sum\limits_{h=1}^{N-j-i} \Delta^i u^2_{h,j}}{2C^i_{2i}(N-j-i)} = \mu_2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (40),$$

wenn nur $i \leq j-1$.

Statt dieses Ausdruckes kann man auch den mit ihm identischen

$$\frac{E\sum\limits_{h=1}^{N-j-i} (\Delta^i x_h - \Delta^i x_{h+j})^2}{2C^i_{2i}(N-j-i)} = \mu_2$$

benutzen, natürlich, ebenfalls nur bis $i \leq j-1$. Ist j klein und läßt es nur geringe Differenzenordnungen zu, so kann man j verdoppeln oder verdreifachen, in dem man zu den Differenzen $u_{i,2j} = u_i - u_{i+2j}$, oder $u_{i,3j} = u_i - u_{i+3j}$ übergeht. Es können auch andere analoge Verfahren angegeben werden, zum Beispiel, das endliche Differenzieren der Summen $\sum\limits_{i=1}^{j} x_i,\ \sum\limits_{i=j+1}^{2j} x_i,\ \sum\limits_{i=2j+1}^{3j} x_i$, u.s.w.

Numerische Beispiele.

(14) Es wurden 320 mal je 6 bulgarische Münzen zu 1 Leva geworfen und die Zahl der ausgefallenen Wappen notiert. Die gegenseitige Unabhängigkeit der einzelnen Versuche wurde durch tüchtiges Schütteln und hohes Werfen der Münzen zu erreichen gesucht. Die Ergebnisse des Experimentes sind in folgender Zahlenreihe dargestellt, wo jede zehn Ziffern durch einen vertikalen Strich von einander getrennt sind:

2, 3, 1, 4, 2, 1, 3, 2, 3, 3,	1, 2, 1, 4, 3, 3, 2, 3, 4, 4,	2, 1, 5, 3, 2, 4, 2, 4, 2, 2,	4, 6, 3, 3, 5, 2, 4, 4, 4, 3,
2, 4, 6, 0, 3, 1, 4, 6, 2, 3,	2, 3, 4, 2, 2, 1, 3, 5, 3, 6,	2, 3, 4, 3, 2, 4, 4, 5, 3, 3,	3, 4, 2, 1, 3, 4, 3, 2, 2, 2,
3, 3, 3, 4, 4, 3, 2, 3, 4, 3,	3, 2, 1, 3, 3, 3, 2, 3, 3, 3,	1, 2, 4, 3, 3, 3, 3, 4, 3, 3,	1, 3, 4, 3, 1, 2, 2, 3, 1, 3,
3, 4, 3, 1, 5, 1, 3, 3, 5, 2,	5, 4, 3, 4, 3, 4, 3, 4, 1, 2,	3, 3, 5, 3, 6, 3, 1, 4, 4, 2,	3, 1, 0, 2, 2, 4, 2, 1, 2, 2,
5, 2, 3, 5, 2, 3, 4, 4, 5, 3,	2, 3, 3, 3, 1, 2, 4, 4, 2, 4,	1, 3, 3, 5, 3, 4, 3, 2, 1, 3,	2, 4, 3, 3, 4, 4, 3, 4, 3, 4,
3, 1, 5, 3, 1, 2, 6, 2, 4, 1,	2, 2, 3, 1, 4, 2, 4, 3, 3, 3,	1, 4, 4, 3, 2, 5, 5, 4, 1, 4,	2, 6, 1, 3, 3, 6, 3, 2, 4, 3,
3, 5, 4, 3, 2, 1, 1, 4, 4, 5,	2, 2, 3, 4, 4, 4, 4, 1, 2, 4,	3, 3, 2, 3, 2, 5, 2, 2, 3, 5,	4, 4, 4, 3, 4, 3, 5, 3, 5, 5,
2, 5, 3, 5, 3, 4, 4, 2, 4, 2,	2, 3, 1, 4, 5, 1, 1, 3, 4, 4,	2, 2, 1, 4, 2, 3, 3, 3, 3, 2,	5, 4, 2, 2, 3, 3, 2, 4, 3, 3.

Es ergab sich also:

Zahl 0, Zahl 1, Zahl 2, Zahl 3, Zahl 4, Zahl 5, Zahl 6,

2 mal, 36 mal, 70 mal, 106 mal, 72 mal, 26 mal, 8 mal,

während die Theorie erfordert:

5 mal, 30 mal, 75 mal, 100 mal, 75 mal, 30 mal, 5 mal.

Die Übereinstimmung ist befriedigend.

Da das Verteilungsgesetz der Zahlen hier binomial ist, so ist m_1 die mathematische Erwartung eines Versuches gleich:

$$m_1 = \tfrac{1}{64}\,.\,0 + \tfrac{6}{64}\,.\,1 + \tfrac{15}{64}\,.\,2 + \tfrac{20}{64}\,.\,3 + \tfrac{15}{64}\,.\,4 + \tfrac{6}{64}\,.\,5 + \tfrac{1}{64}\,.\,6 = 3.$$

Das arithmetische Mittel aller Versuche ergab ebenfalls genau 3.

Die apriorische Streuung ist:

$$\mu_2 = \tfrac{1}{64}(-3)^2 + \tfrac{6}{64}(-2)^2 + \tfrac{15}{64}(-1)^2 + \tfrac{20}{64}0^2 + \tfrac{15}{64}(+1)^2 + \tfrac{6}{64}(+2)^2 + \tfrac{1}{64}(+3)^2 = 1{\cdot}5.$$

$$\sigma_0'^2 = \frac{\sum\limits_{i=1}^{320}(x_i-3)^2}{N}$$ ergab ebenfalls genau 1·5.

Schließlich ist

$$E(x_i - m_1)^4 = \mu_4$$
$$= \tfrac{1}{64}(-3)^4 + \tfrac{6}{64}(-2)^4 + \tfrac{15}{64}(-1)^4 + \tfrac{20}{64}0^4 + \tfrac{15}{64}(+1)^4 + \tfrac{6}{64}(+2)^4 + \tfrac{1}{64}(+3)^4 = 6,$$

während $\frac{1}{320}\sum\limits_{i=1}^{320}(x_i-3)^4 = 6{\cdot}075.$

Berechnet man jetzt nach Formel (11) die empirischen Streuungen $\sigma_i'^2$, so erhält man folgende Werte:

$$\frac{1}{2\times 319}\sum_{i=1}^{319}\Delta' x_i^2 = 1{\cdot}5596; \qquad \frac{1}{6\times 318}\sum_{i=1}^{318}\Delta'' x_i^2 = 1{\cdot}5718;$$

$$\frac{1}{20\times 317}\sum_{i=1}^{317}\Delta''' x_i^2 = 1{\cdot}5852; \qquad \frac{1}{70\times 316}\sum_{i=1}^{316}\Delta^{\text{iv}} x_i^2 = 1{\cdot}5988;$$

$$\frac{1}{252\times 315}\sum_{i=1}^{315}\Delta^{\text{v}} x_i^2 = 1{\cdot}6088; \qquad \frac{1}{924\times 314}\sum_{i=1}^{314}\Delta^{\text{vi}} x_i^2 = 1{\cdot}6161.$$

Die Reihe der $\sigma_j'^2$ ergibt also ein langsames Steigen. Um festzustellen, ob nicht diese Erscheinung als Symptom der Anwesenheit einer "Z"-Komponente angesehen werden muß, berechnen wir die mittleren Fehler der Differenzen $\sigma_i'^2 - \sigma_{i-1}'^2$, welche sich aus Tabelle II ergeben. Da $N = 320$ und

$$\mu_4 - 3\mu_2^2 = 6 - 3\,(1{\cdot}5)^2 = -0{\cdot}75,$$

also sehr klein ist, so können wir ruhig die angenäherten Formeln der letzten Kolonne rechts gebrauchen, indem wir hier die Werte $\mu_2^2 = 2{\cdot}25$ und $N = 320$ einsetzen, und aus dem Ganzen Quadratwurzeln ziehen. Wir erhalten dann folgende Tabelle:

	$\sigma_1'^2-\sigma_0'^2$	$\sigma_2'^2-\sigma_1'^2$	$\sigma_3'^2-\sigma_2'^2$	$\sigma_4'^2-\sigma_3'^2$	$\sigma_5'^2-\sigma_4'^2$	$\sigma_6'^2-\sigma_5'^2$
Differenzen	0·0596	0·0122	0·0134	0·0136	0·0100	0·0073
Mittl. Fehler	0·084	0·040	0·028	0·022	0·018	0·016
Verhältnis $\frac{\text{Differenz}}{\text{Mittl. Fehler}}$	0·71	0·31	0·48	0·62	0·56	0·46

Die Differenzen sind beträchtlich kleiner als ihre mittleren Fehler, und, mit einer einzigen Ausnahme, sogar kleiner als die "wahrscheinlichen" Fehler. Man kann also ruhig bei der Annahme bleiben, die Reihe gehöre zur "*R*"-Gruppe. Diese Annahme kann auch dadurch unterstützt werden, daß man aus Tabelle I die mittleren Fehler für die empirischen Streuungen bestimmt. Es ergibt sich, zum Beispiel, für $\sigma_0'^2$, $1{\cdot}500 \pm 0{\cdot}108$ und, für $\sigma_6'^2$, $1{\cdot}500 \pm 0{\cdot}207$. Auch hier ist die Übereinstimmung sehr befriedigend.

Berechnet man schließlich den Divergenzkoeffizienten $\mathfrak{D}^2$, so erhält man:

$$\mathfrak{D}_2^2 = 0{\cdot}975\,; \quad \mathfrak{D}_{10}^2 = 1{\cdot}254\,; \quad \mathfrak{D}_{20}^2 = 0{\cdot}921 \text{ und } \mathfrak{D}_{40}^2 = 0{\cdot}663\,;$$

ihre mittleren Fehler, nach der Formel $\sqrt{\frac{2}{r-1}}$ berechnet, ergeben in derselben Reihenfolge: $\pm 0{\cdot}079$; $\pm 0{\cdot}254$; $\pm 0{\cdot}365$; $\pm 0{\cdot}535$. Die mathematischen Erwartungen aller dieser $\mathfrak{D}^2$ können also sehr wohl gleich 1 sein.

TABELLE III.

No.	σ'^2	$\sigma_0'^2$	$\sigma_1'^2$	$\sigma_2'^2$	$\sigma_3'^2$	$\sigma_4'^2$	$\sigma_5'^2$	$\sigma_6'^2$	$\sigma_7'^2$	$\sigma_8'^2$	$\sigma_9'^2$
1	0·933	1·2	1·389	1·729	1·957	2·012	1·821	1·392	0·842	0·334	0·000
2	1·344	1·3	0·833	0·896	1·021	1·107	1·118	1·075	1·030	1·016	1·041
3	1·567	1·5	2·111	2·563	2·764	2·636	2·367	2·135	1·984	1·867	1·742
4	1·289	1·8	1·722	1·938	1·914	1·869	1·995	2·352	2·838	3·258	3·441
5	3·878	3·5	4·833	5·354	5·936	6·369	6·568	6·656	6·704	6·709	6·659
6	2·322	2·1	1·556	1·333	1·179	0·943	0·721	0·613	0·597	0·606	0·594
7	0·900	0·9	0·722	0·646	0·657	0·655	0·656	0·699	0·771	0·835	0·864
8	0·933	1·0	0·722	0·521	0·307	0·150	0·091	0·074	0·054	0·029	0·015
9	0·400	0·4	0·333	0·229	0·129	0·074	0·053	0·039	0·027	0·022	0·022
10	0·489	0·6	0·444	0·396	0·414	0·452	0·456	0·434	0·417	0·420	0·444
11	0·767	0·7	0·444	0·354	0·371	0·341	0·262	0·175	0·100	0·040	0·001
12	1·112	1·5	1·111	0·875	0·736	0·643	0·596	0·615	0·679	0·737	0·774
13	2·000	1·8	3·056	3·625	4·079	4·786	5·806	7·003	8·158	9·057	9·567
14	1·344	1·3	0·944	1·083	1·200	1·248	1·276	1·311	1·357	1·420	1·499
15	2·044	2·0	2·389	2·500	2·629	2·817	2·883	2·666	2·213	1·710	1·348
16	1·211	2·3	1·056	0·833	0·829	0·912	1·018	1·176	1·360	1·512	1·590
17	1·378	1·6	1·667	1·438	1·164	1·088	1·147	1·188	1·141	1·049	0·969
18	1·067	1·0	1·000	0·813	0·564	0·364	0·283	0·265	0·235	0·190	0·163
19	1·511	1·4	1·111	0·958	0·993	1·131	1·287	1·412	1·497	1·550	1·590
20	0·489	0·6	0·556	0·521	0·436	0·348	0·275	0·205	0·135	0·076	0·045
21	3·067	2·8	4·111	4·479	4·479	4·079	3·414	2·662	1·986	1·542	1·423
22	0·900	0·9	1·278	1·792	2·336	2·931	3·555	4·158	4·691	5·115	5·392
23	2·233	2·1	2·167	1·583	0·993	0·719	0·714	0·833	0·990	1·119	1·185
24	2·678	2·5	3·833	4·188	3·886	3·343	3·049	3·131	3·410	3·685	3·856
25	2·178	2·0	1·000	0·604	0·471	0·417	0·329	0·221	0·132	0·078	0·058
26	1·333	1·2	0·889	0·583	0·500	0·510	0·465	0·363	0·246	0·152	0·101
27	1·333	1·2	1·444	1·500	1·757	2·121	2·587	3·150	3·725	4·182	4·428
28	0·667	1·6	0·833	1·125	1·429	1·693	1·895	2·040	2·150	2·250	2·350
29	1·378	1·4	1·889	2·146	2·221	2·117	1·868	1·539	1·197	0·899	0·689
30	2·178	2·0	2·000	1·771	1·543	1·229	0·849	0·597	0·584	0·686	0·719
31	0·722	0·9	0·889	1·104	1·343	1·562	1·688	1·699	1·635	1·563	1·533
32	0·989	0·9	0·667	0·563	0·586	0·564	0·491	0·382	0·265	0·172	0·128
Arithm. Mittel:	1·458	1·5	1·531	1·564	1·588	1·601	1·612	1·633	1·661	1·684	1·695

Um zu erproben, ob die Formeln der §§ 6 und 7 auch auf kleine N angewandt werden können, wollen wir jetzt die ganze Reihe der 320 Versuche in 32 Gruppen zu 10 Gliedern* teilen und für jede derselben die empirischen Streuungen $\sigma_k'^2$ von $k=1$ bis $k=9$ bestimmen (im letzten Grenzfall besteht die Formel nur aus einem einzigen Gliede). Der Vollständigkeit wegen berechnen wir auch, nach Formeln (9) und (10), die Werte von σ'^2 und $\sigma_0'^2$. Dann erhalten wir Tabelle III auf S. 60.

Wenn wir mit dem Auge die Zahlenkolonnen für jedes einzelne k durchlaufen, so erhalten wir ein sehr buntes Bild: es ist nichts von einer Konstanz dieser Werte zu bemerken, obgleich sie doch alle eigentlich als Annäherungen an dieselbe Größe 1·5 zu betrachten sind. Berechnen wir jedoch für jede Kolonne ihr arithmetisches Mittel (siehe letzte Zeile der Tabelle), so unterscheiden sich diese Mittel verhältnismäßig nur wenig von den entsprechenden für alle 320 Glieder gefundenen Werten $\sigma_k'^2$ (s. oben). Stimmen nun die Abweichungen der Zahlen unserer Tabelle von ihrer gemeinsamen mathematischen Erwartung 1·5 mit den Formeln der Theorie überein? Um diese Frage zu beantworten, wollen wir für jede Kolonne der Tabelle III ihren mittleren Fehler bestimmen. Da uns μ_2 bekannt ist, so können wir ersteren nach der Formel $\sqrt{\dfrac{\sum\limits_{i=1}^{32}(\sigma_{j,i}'^2-1{\cdot}5)^2}{32}}$ berechnen.

Andererseits, besitzen wir in Formel (12) die von der Theorie für eine "R"-Gruppe vorgeschriebenen Werte der entsprechenden Streuungen. Wir gewinnen letztere, wenn wir in den Formeln der Tabelle I die Werte: $\mu_4=6$, $\mu_2^2=2{\cdot}25$ und $N=10$ einsetzen und aus den gefundenen Größen Quadratwurzeln ziehen. Die nach der ersten (empirischen) und der zweiten (apriorischen) Methode bestimmten mittleren Fehler für die Koeffizienten $\sigma_0'^2$, $\sigma_1'^2$, ... sind in folgender Tabelle wiedergegeben (weiter als bis $k=5$ dürfen wir bei $N=10$ die Formel (12) nicht anwenden).

Bei $k=$	0	1	2	3	4	5
Empirische mittlere Fehler ...	0·680	1·080	1·251	1·357	1·469	1·541
Apriorische mittlere Fehler ...	0·612	0·802	0·968	1·118	1·269	1·430
Verhältnis $\dfrac{\text{Empir. mittl. Fehler}}{\text{Aprior. mittl. Fehler}}$	1·111	1·347	1·292	1·214	1·158	1·078

Die Übereinstimmung der Theorie mit der Empirie ist ganz befriedigend, besonders wenn man berücksichtigt, daß bei einer Versuchszahl 32 der mittlere Fehler des mittleren Fehlers $\sqrt{\dfrac{\sum\limits_{i=1}^{32}(\sigma_i'^2-1{\cdot}5)^2}{.32}}$ selbst durchaus keine verschwindend kleine Größe auszumachen braucht.

Wenn wir jetzt einzeln jede horizontale Zeile der Tabelle III betrachten, so ergibt sich ein viel weniger buntes Bild, da mit steigendem k die Korrelationen

* Sie sind in der Tabelle Seite 58 durch vertikale Striche gekennzeichnet.

zwischen jedem $\sigma_k'^2$ und dem ihm folgenden $\sigma_{k+1}'^2$ sich immer bemerkbarer machen. Besonders deutlich werden die Zusammenhänge, wenn man die Differenzen

$$\sigma_{k+1}'^2 - \sigma_k'^2$$

berechnet.

TABELLE IV.

No.	$\sigma_1'^2 - \sigma_0'^2$	$\sigma_2'^2 - \sigma_1'^2$	$\sigma_3'^2 - \sigma_2'^2$	$\sigma_4'^2 - \sigma_3'^2$	$\sigma_5'^2 - \sigma_4'^2$	$\sigma_6'^2 - \sigma_5'^2$
1	+0·189	+0·340	+0·228	+0·055	−0·191	−0·429
2	−0·467	+0·053	+0·125	+0·086	+0·011	−0·043
3	+0·611	+0·452	+0·201	−0·128	−0·269	−0·232
4	−0·078	+0·216	−0·024	−0·045	+0·126	+0·357
5	+1·533	+0·521	+0·582	+0·433	+0·199	+0·088
6	−0·544	−0·223	−0·154	−0·236	−0·222	−0·108
7	−0·178	−0·076	+0·011	−0·002	+0·001	+0·043
8	−0·278	−0·201	−0·214	−0·157	−0·059	−0·017
9	−0·067	−0·104	−0·100	−0·055	−0·021	−0·014
10	−0·156	−0·048	+0·018	+0·038	+0·004	−0·022
11	−0·256	−0·090	+0·017	−0·030	−0·079	−0·087
12	−0·389	−0·236	−0·139	−0·093	−0·047	+0·019
13	+1·256	+0·569	+0·454	+0·707	+1·020	+1·197
14	−0·356	+0·139	+0·117	+0·048	+0·028	+0·035
15	+0·389	+0·111	+0·129	+0·188	+0·066	−0·217
16	−1·244	−0·223	−0·004	+0·083	+0·106	+0·158
17	+0·067	−0·229	−0·274	−0·076	+0·059	+0·041
18	0·000	−0·187	−0·249	−0·200	−0·081	−0·018
19	−0·289	−0·153	+0·035	+0·138	+0·156	+0·125
20	−0·044	−0·035	−0·085	−0·088	−0·073	−0·070
21	+1·311	+0·368	−0·000	−0·400	−0·665	−0·752
22	+0·378	+0·514	+0·544	+0·595	+0·624	+0·603
23	+0·067	−0·584	−0·590	−0·274	−0·005	+0·119
24	+1·333	+0·355	−0·302	−0·543	−0·294	+0·082
25	−1·000	−0·396	−0·133	−0·054	−0·088	−0·108
26	−0·311	−0·306	−0·083	+0·010	−0·045	−0·102
27	+0·244	+0·056	+0·257	+0·364	+0·466	+0·563
28	−0·767	+0·292	+0·304	+0·264	+0·202	+0·145
29	+0·489	+0·257	+0·075	−0·104	−0·249	−0·329
30	0·000	−0·229	−0·228	−0·314	−0·380	−0·252
31	−0·011	+0·215	+0·239	+0·219	+0·126	+0·011
32	−0·233	−0·104	+0·023	−0·022	−0·073	−0·109

Wir können hier abermals die Übereinstimmung der gefundenen Werte mit den Voraussetzungen der Theorie durch Berechnung der mittleren Fehler prüfen.

Da $E(\sigma'^2_{k+1} - \sigma_k'^2) = 0$ für jedes k, so berechnet sich der empirische mittlere Fehler für diese Differenz aus der Formel: $\sqrt{\dfrac{\sum\limits_{i=1}^{32} (\sigma'^2_{k+1,i} - \sigma'^2_{k,i})^2}{32}}$, der apriorische ist aber durch die Formeln der Tabelle II gegeben, wobei wiederum $\mu_2 = 6$, $\mu_2^2 = 2{\cdot}25$ und $N = 10$ ist. Bei so kleinem N müssen wir, natürlich, die bis zur Ordnung $\dfrac{1}{N^3}$ korrekte Formel gebrauchen.

Wir kommen dann zu folgenden Resultaten.

Bei $=k$	1	2	3	4	5
Empirischer mittlerer Fehler ...	0·63	0·29	0·25	0·26	0·29
Apriorischer mittlerer Fehler ...	0·52	0·32	0·29	0·30	0·33
Verhältnis $\frac{\text{Empir. mittl. Fehler}}{\text{Aprior. mittl. Fehler}}$	1·21	0·91	0·86	0·87	0·88

Auch hier ist die Übereinstimmung befriedigend.

Daraus, daß sich unsere Formeln auch in diesem Falle bewähren, folgt jedoch durchaus nicht, daß bei so kleinem N die Anwendung der Differenzenmethode ratsam, ja überhaupt möglich ist. Es genügt eigentlich schon, einfach darauf hinzuweisen, daß hier bei $\mu_2 = 1{\cdot}5$ und $N = 10$ der mittlere Fehler von $\sigma_k'^2$ bereits in der ersten Differenz 0·8 übersteigt. Dazu kommt noch der Umstand, daß wir bei unserem Experiment über eine vollkommene Kenntnis der ihm zu Grunde liegenden Wahrscheinlichkeiten verfügten und daher auch die Größen m_1, μ_2 und μ_4 bestimmen konnten. Im allgemeinen, ist es jedoch nicht der Fall. Der gewöhnliche Kunstgriff, in den Formeln m_1 durch $x_{(N)}$, μ_2 durch σ'^2 u.s.w. zu ersetzen, geht bei so kleinem N nicht an, wie man das schon aus dem Vergleich der Kolonnen 1 und 2 in Tabelle III ersehen kann, wenn man dabei bedenkt, daß alle Zahlen eigentlich aus einer Annäherungsformel zum Bestimmen der "wahren Größe" $\mu_2 = 1{\cdot}5$ hervorgegangen sind. Vollends können die Grenzwerte, denen die Größen $\sigma_k'^2$ bei wachsendem k *scheinbar* zustreben, nicht als Annäherungen an den wahren Wert von μ_2 gelten. Im Gegenteil, gerade die Betrachtung der Tabellen III und IV lehrt uns, wie wenig man eigentlich bei kleinem N aus dem raschen Anwachsen der empirischen Streuungen oder aus einem ebensolchen Abfallen derselben auf die Unzulässigkeit der Grundhypothese, die Reihe x gehöre zur "R-Gruppe," schließen kann. Die Sache ist nämlich die, daß bei ganz kleinem N es durchaus nicht sehr unwahrscheinlich ist, daß N unmittelbar einander nachfolgende *x zufallig* so ausfallen, daß sie durch eine Parabel einer der ersten Ordnungen oder durch eine kurzperiodische Sinus-Kurve darstellbar sind, *obwohl sie mit ihnen absolut nichts zu tun haben.* Es genügt sogar schon, wenn nur ein Teil der Glieder einer gegebenen Gruppe diese Bedingungen erfüllt. So, zum Beispiel, ist das *maximale* $\sigma_9'^2$ der Tabelle (Gruppe Nr. 13) aus folgender ursprünglichen Zahlenkonstellation entstanden: 3, 4, 3, 1, 5, 1, 3, 3, 5, 2. Ausschlaggebend war, daß die 8 Glieder: Nr. 3 bis Nr. 7 und Nr. 8 bis Nr. 10 ganz "Z-Gruppen"-mäßig ausfielen. Im Gegenteil, besteht die ein rasches Abfallen der $\sigma_k'^2$ ergebende Gruppe Nr. 25 aus folgenden Zahlen: 3, 5, 4, 3, 2, 1, 1, 4, 4, 5. Hier waren Nr. 2 bis Nr. 6 ausschlaggebend. Und überhaupt, je näher k an N herankommt, desto mehr Wahrscheinlichkeit besteht dafür, daß die Glieder der Reihe schon durch eine Parabel k-ter Ordnung mehr oder weniger genau auszudrücken sind*. *Jede* lange statistische Reihe, wenn man sie in ganz kurze Abschnitte zerlegt, wird bald

* Eben diese Eigenschaften einer kurzen Reihe (die auch so große Werte ihrer mittleren Fehler verursachen) bewirkten die äußerste Asymmetrie der Verteilung der $\sigma_k'^2$ in jeder Kolonne der Tabelle III, welche mit wachsendem k nur immer größer wird. An die Frage nach dem Verteilungsgesetz dieser Größen bei verschiedenen N bin ich nicht herangetreten.

"glatt," bald "zackig" erscheinen. Daraus folgt aber durchaus nicht, daß sie, als Ganzes, zur Gruppe "*G*" oder "*Z*" gehört.

Die einzige Möglichkeit, sich so kurzer Reihen mit Vorteil zu bedienen, ergibt sich dann, wenn man nicht über eine einzige, sondern über eine ganze Gruppe einander analoger kurzer Reihen verfügt, wie es, zum Beispiel, in der landwirtschaftlichen Statistik vorkommt (Ernteerträge in verschiedenen Bezirken eines Landes etc.)*.

Diese eigentlich recht elementaren Ausführungen sind dadurch hervorgerufen, daß die hauptsächliche Eigentümlichkeit der Differenzenmethode, die Feststellung der Tatsache, ob die Reihe der empirischen Streuungen oder der Korrelationskoeffizienten "stabil" geworden ist, verbunden mit der großen und immer größer werdenden positiven Korrelation zwischen den Nachbargliedern dieser Reihen,—durch ihre oberflächliche Analogie mit der Feststellung der Konvergenz *mathematischer* Reihen, leicht Unheil stiften kann (und wohl auch gestiftet hat). Der *scheinbare* "Grenzwert," welchen die Reihe der $\sigma_k'^2$ "zustrebt," ist nicht der "wahre Wert" μ_2, wie ihn die Naturforscher verstehen, sondern nur der "*vermutliche,*" der "*präsumptive*" Wert, und er kann viel weiter von μ_2 entfernt sein, als viele der $\sigma_k'^2$. "Der Präsumptivwert kann arithmetisch ganz genau sein. Wenn aus einer geschlossenen Urne bei 100 Versuchen 20 mal eine weiße Kugel erscheint, so ist die präsumptive Größe der apriorischen Wahrscheinlichkeit, eine weiße Kugel aus der Urne zu ziehen, ganz genau gleich $\frac{1}{5}$; daraus folgt aber durchaus nicht, daß in der Urne wirklich $\frac{1}{5}$ der Kugeln weiß ist. Der Präsumptivwert mißt nicht eine apriorische Größe, wie man eine physische Konstante mißt: mit einer Genauigkeit bis zu der oder der Dezimalstelle. Er ist selber eine apriorische Größe, gleich einer Nummer auf einem aus einem geschlossenem Kasten gezogenem Zettel, welche als Repräsentant des arithmetischen Mittels aller Zahlen im Kasten angesehen wird. Die Zuverlässigkeit eines "Präsumptivwert" bestimmt sich aus seinem mittleren Fehler, und nicht aus der Zahl seiner berechneten Dezimalen. Der Präsumptivwert ist selber eine zufällige Variable, welche verschiedene Werte mit bestimmten Wahrscheinlichkeiten annehmen kann. Darin besteht eben das *quid proprium* dieses eigentümlichen Begriffes der stochastischen Theorie der Statistik†."

Niemand regt sich besonders darüber auf, daß ein nach 10 bis 20 Gruppenpaaren berechneter Korrelationskoeffizient beträchtlich von der ihm zu Grunde liegenden mathematischen Erwartung abweichen kann. Ebenso wenig verwunderlich ist auch die mangelhafte Konstanz der empirischen "Differenzen-Streuungen" solcher Reihen. Das "Gesetz der Großen Zahlen" ist eben nicht zu umgehen! Da hierbei der mittlere Fehler der Streuung mit jeder endlichen Differenz immer *größer* wird, so wird auch kein vernünftiger Mensch die Differenzenmethode dort

* Das war eben der Weg, den M. Vinogradowa und N. Tschetwerikoff in ihren Arbeiten seinerzeit eingeschlagen haben.

† Al. A. Tchouproff, "Die Grundaufgaben der stochastischen Theorie der Statistik," in den Moskauer *Statistischen Nachrichten*, 1925, S. 86—87 (russisch). Siehe auch Al. A. Tchouproff, *Grundbegriffe und Grundprobleme der Korrelationstheorie*, Leipzig-Berlin, 1925, S. 78—79.

anwenden, wo er dazu nicht durch die Anwesenheit einer stärkeren "glatten" Komponente *gezwungen* ist. Daraus folgt ferner, daß wenn die Differenzen schon die Annahme zulassen, diese Komponente sei eliminiert, im Allgemeinen, der erste "stabile Wert" auch der beste sein wird, weil mit dem kleinsten mittleren Fehler beheftete. Die Anwendung *hoher* Differenzen hat nur bei *langen* statistischen Reihen rechten Sinn.

Wollen wir jetzt in unserem Experiment die Grundzahl verdoppeln, das heißt mit anderen Worten, die Koeffizienten $\sigma_k'^2$ für 16 Gruppen zu je 20 Gliedern berechnen, so erhalten wir folgende Tabelle, die schon auf den ersten Blick ein viel stabileres Verhalten der einzelnen Glieder aufweist.

TABELLE V.

(Gruppen zu 20 Gliedern.)

Nr.	σ'^2	$\sigma_0'^2$	$\sigma_1'^2$	$\sigma_2'^2$	$\sigma_3'^2$	$\sigma_4'^2$	$\sigma_5'^2$	$\sigma_6'^2$
1	1·103	1·25	1·158	1·287	1·377	1·403	1·360	1·282
2	1·671	1·65	1·921	2·037	2·012	1·884	1·760	1·704
3	3·095	2·80	3·053	3·046	3·132	3·172	3·062	2·820
4	0·997	0·95	0·684	0·528	0·459	0·433	0·431	0·455
5	0·516	0·50	0·368	0·296	0·265	0·254	0·246	0·238
6	0·990	1·10	0·842	0·732	0·662	0·596	0·550	0·533
7	1·608	1·55	2·132	2·574	2·871	3·124	3·382	3·602
8	2·134	2·15	1·658	1·648	1·650	1·613	1·528	1·380
9	1·326	1·30	1·290	1·046	0·788	0·638	0·591	0·558
10	1·042	1·00	0·816	0·824	0·906	0·973	1·007	1·009
11	1·882	1·85	2·579	2·944	3·129	3·237	3·411	3·704
12	2·326	2·30	2·947	3·130	3·382	3·739	4·205	4·774
13	1·674	1·60	1·132	0·759	0·629	0·621	0·620	0·617
14	1·211	1·40	1·105	1·259	1·409	1·524	1·597	1·627
15	1·779	1·70	1·842	1·787	1·706	1·610	1·486	1·339
16	0·905	0·90	0·974	1·037	1·082	1·088	1·023	0·896
Arithm. Mittel:	1·516	1·50	1·531	1·558	1·591	1·619	1·641	1·659

Verdoppeln wir die Grundzahl nochmals, so kommen wir zu folgender

TABELLE VI.

(je 40 Glieder in einer Gruppe)

Nr.	σ'^2	$\sigma_0'^2$	$\sigma_1'^2$	$\sigma_2'^2$	$\sigma_3'^2$	$\sigma_4'^2$	$\sigma_5'^2$	$\sigma_6'^2$
1	1·477	1·450	1·551	1·597	1·592	1·557	1·493	1·415
2	1·922	1·875	2·026	2·018	2·073	2·124	2·094	1·986
3	0·756	0·800	0·641	0·544	0·470	0·421	0·395	0·383
4	1·887	1·850	1·859	2·004	2·112	2·205	2·291	2·351
5	1·156	1·150	1·141	1·105	1·089	1·112	1·152	1·184
6	2·128	2·075	2·744	3·004	3·150	3·262	3·427	3·666
7	1·446	1·500	1·103	1·000	0·985	1·004	1·021	1·031
8	1·331	1·300	1·423	1·373	1·316	1·254	1·178	1·092
Arithm. Mittel	1·513	1·500	1·561	1·581	1·599	1·617	1·631	1·639

Setzen wir diese Verdoppelung fort, so erhalten wir noch folgende Resultate.

TABELLE VII.

Nr.	σ'^2	$\sigma_0'^2$	$\sigma_1'^2$	$\sigma_2'^2$	$\sigma_3'^2$	$\sigma_4'^2$	$\sigma_5'^2$	$\sigma_6'^2$
	(je 80 Glieder in einer Gruppe)							
1	1·682	1·663	1·772	1·780	1·773	1·757	1·726	1·690
2	1·311	1·325	1·234	1·252	1·276	1·301	1·327	1·354
3	1·625	1·613	1·924	2·013	2·080	2·163	2·265	2·372
4	1·402	1·400	1·304	1·252	1·238	1·243	1 251	1·258
Arithm. Mittel	1·505	1·500	1·559	1·574	1·592	1·616	1·642	1·669
	(je 160 Glieder in einer Gruppe)							
1	1·492	1·494	1·497	1·499	1·498	1·492	1·480	1·466
2	1·504	1·506	1·604	1·617	1·640	1·679	1·722	1·761
Arithm. Mittel	1·498	1·500	1·550	1·558	1·569	1·585	1·601	1·614
	(eine Gruppe aus 320 Gliedern)							
1	1·505	1·500	1·560	1·572	1·585	1·599	1·609	1·616

Die Daten der Tabellen III und V bis VII sind in den Diagrammen Nr. 1 bis 4 dargestellt.

Eine jede Reihe $\sigma_0'^2$, $\sigma_1'^2$, $\sigma_2'^2$, ... ist durch eine gebrochene mehr oder weniger horizontale Linie gegeben, deren Nr. an beiden Enden derselben ersichtlich ist. Durch die punktierte gerade Linie ist 1·5, die apriorische Streuung, angedeutet.

Aus den Diagrammen* ersieht man klar, wie mit jeder Verdoppelung der Grundzahl die Gesammtheit der Linien, welche man mit einem ungefähr wagerecht gelegten Bündel gedroschenen Strohes vergleichen kann, immer "glatter" wird und die einzelnen Linien näher zu einander rücken. Die *scheinbaren* Grenzwerte, zu denen letztere "streben," nähern sich immer mehr $\mu_2 = 1·5$, und die Differenzen $\sigma'^2_{k+1} - \sigma_k'^2$ werden immer kleiner. Dabei behalten aber alle Bündel ein nach rechts hin auseinanderstrebendes Aussehen, was mit der schrittweisen Vergrößerung der mittleren Fehler direkt zusammenhängt. Es sei noch erwähnt, daß mit wachsender Grundzahl die Größen σ'^2 sich immer mehr den $\sigma_0'^2$ nähern und daher diese immer besser ersetzen können. Die Diagramme sind recht instruktiv.

Hätten wir die Bedingungen des Experimentes geändert und den Zahlenreihen noch ein "*Z*"-Element, zum Beispiel die Komponente $+1, -1, +1, -1, +1, \ldots$, zugegeben, so würden die rechten Enden der meisten "Strohhalme" in unserem Diagramm eine Biegung nach oben erhalten. Hätten wir, im Gegenteil, den

[* We regret that owing to Professor Anderson's diagrams not being sent in a manner suited to photographic reproduction, they had to be redrawn, and the scales modified to suit the page size of this Journal. EDITORS.]

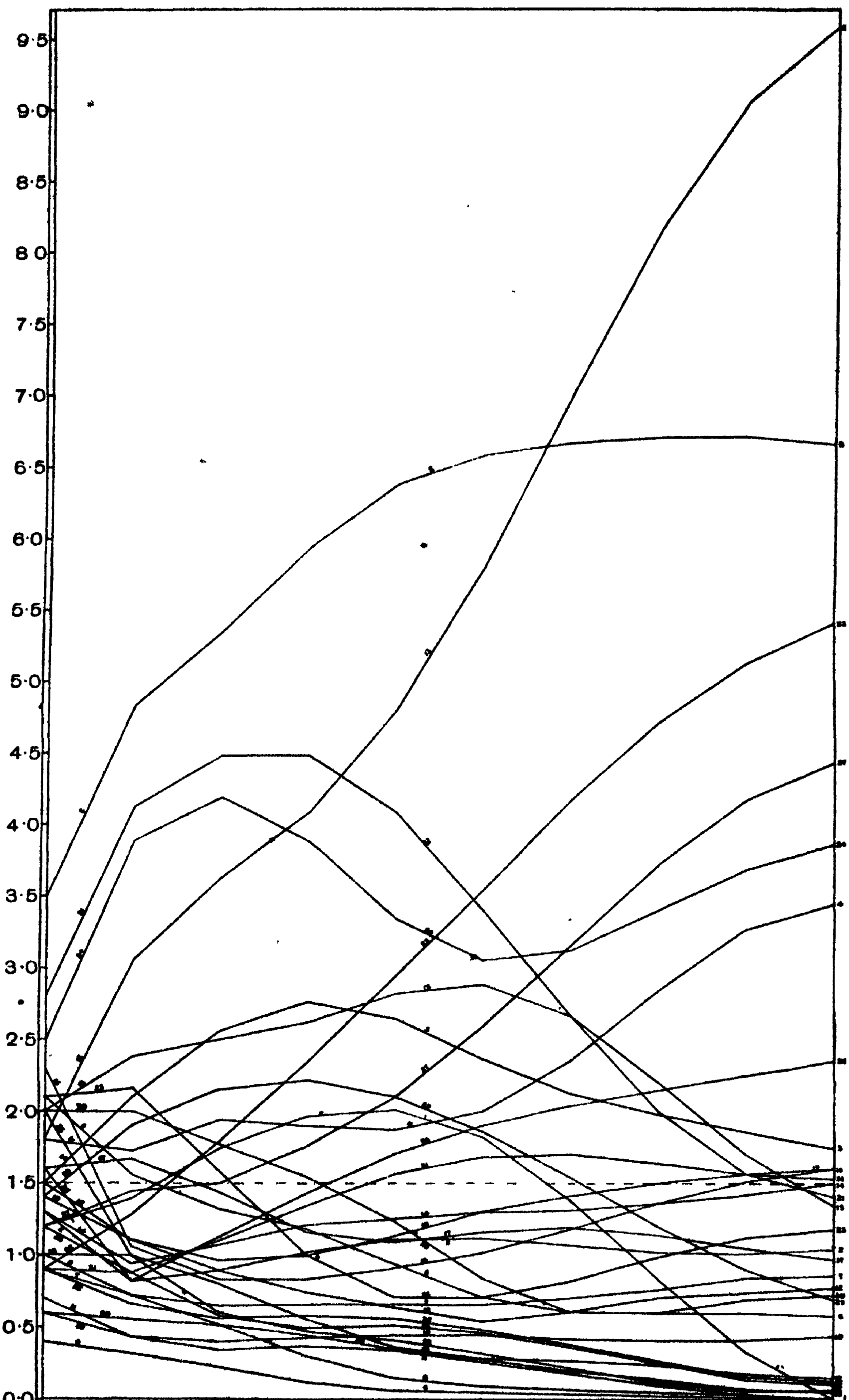

Diagramm No. 1. Empirische Streuungen für 32 Gruppen zu 10 Gliedern bei $k=0, 1, 2, 3, 4, 5, 6, 7, 8, 9$.

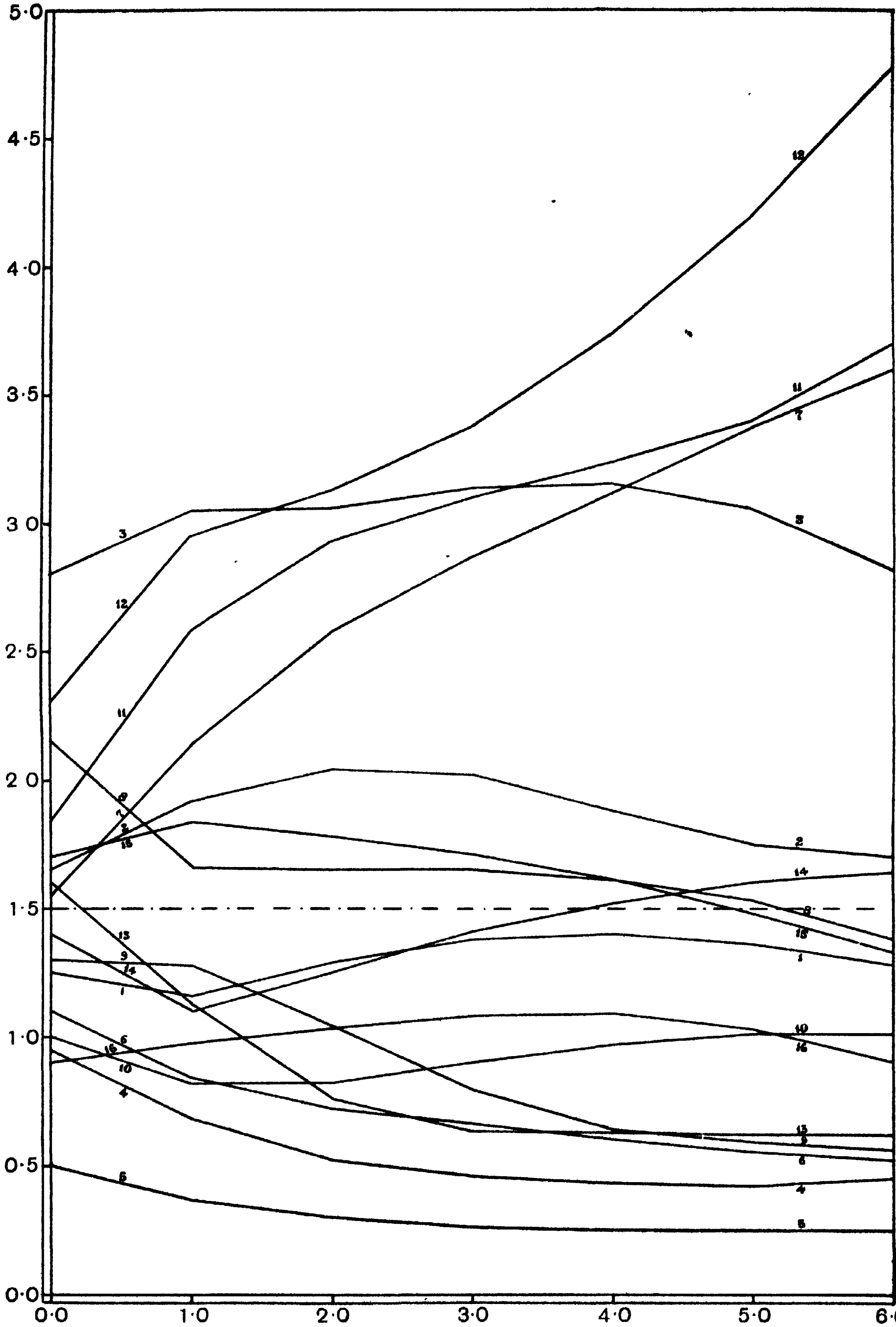

Diagramm No. 2. Empirische Streuungen für 16 Gruppen zu 20 Gliedern bei $k=0, 1, 2, 3, 4, 5, 6$.

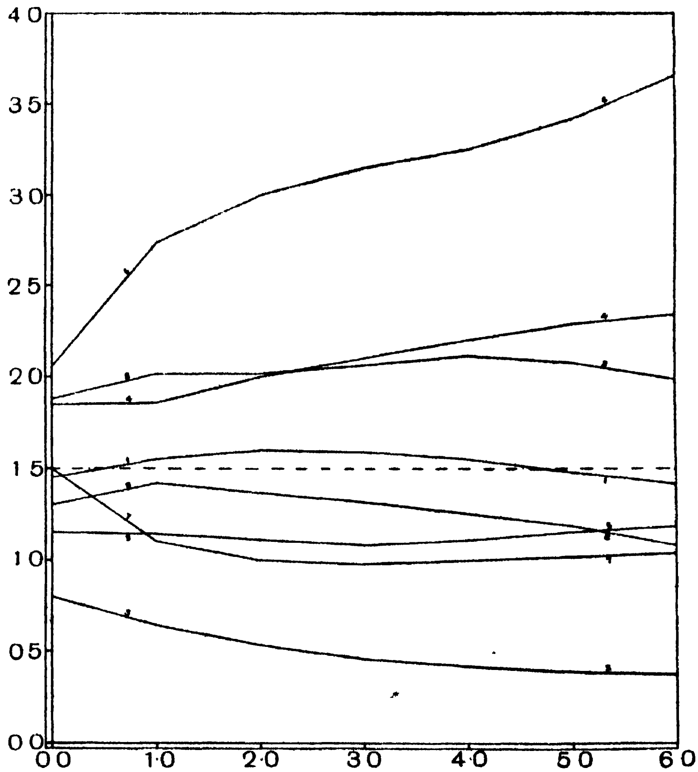

Diagramm No. 3. Empirische Streuungen für 8 Gruppen zu 40 Gliedern bei $k=0, 1, 2, 3, 4, 5, 6$.

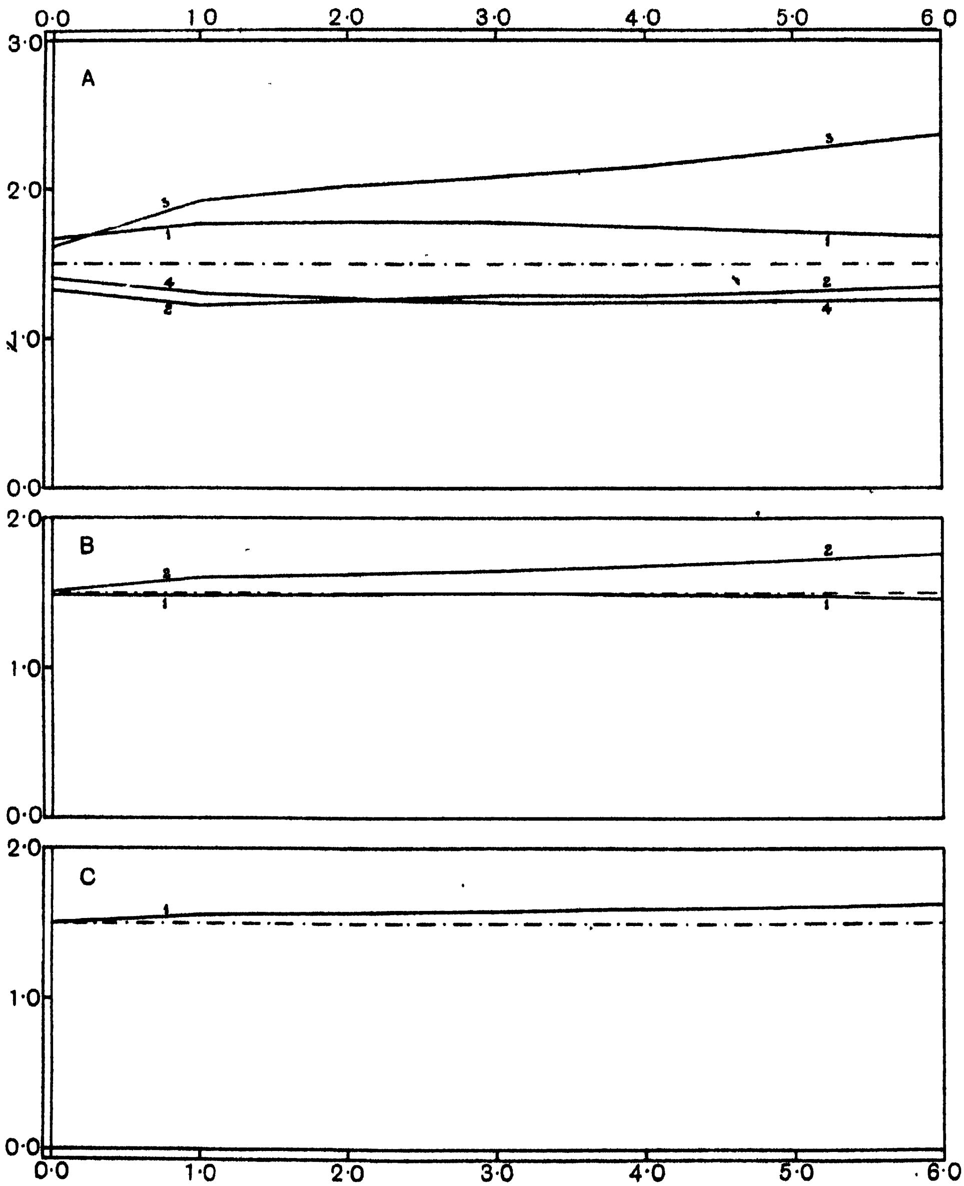

Diagramm No. 4. A. Empirische Streuungen für 4 Gruppen zu 80 Gliedern bei $k=0, 1, 2, 3, 4, 5, 6$.
B. Empirische Streuungen für 2 Gruppen zu 160 Gliedern bei $k=0, 1, 2, 3, 4, 5, 6$.
C. Empirische Streuungen für 1 Gruppe zu 320 Gliedern bei $k=0, 1, 2, 3, 4, 5, 6$.

Zahlenreihen eine "glatte Komponente" zugeteilt, so würden die linken Enden der "Strohhalme" in die Höhe gebogen werden; vereinigten wir beide Verfahren, so würden beide Enden des "Bündels" im Vergleich zu dessen Mitte heraufrücken, ohne daß aber in einem dieser 3 Fälle diese "Bündel" dadurch ein weniger zerzaustes Aussehen erhalten hätten.

(15) Bilden wir jetzt aus den Zahlenwerten unseres Experiments: $x_1, x_2, x_3, \ldots x_{320}$, zwei mit einander korrelierte und aus je 160 Gliedern bestehende Reihen: $x_1 + x_2, x_3 + x_4, x_5 + x_6, \ldots x_{319} + x_{320}$, einerseits, und $x_1, x_3, x_5, \ldots x_{319}$,—andererseits. Die mathematische Erwartung eines Gliedes der ersten Reihe ist 6, diejenige der zweiten ist 3. Die mathematische Erwartung des Produktes $(x_i - 3)(x_i + x_{i+1} - 6)$ ist genau 1·5. Die apriorische Streuung der ersten Reihe ist 3, der zweiten: 1·5. Wir erhalten also für den apriorischen Korrelationskoeffizienten $r_{1/1}$ zwischen x_i und $(x_i + x_{i+1})$ die Größe $\frac{+1{\cdot}5}{\sqrt{3 \times 1{\cdot}5}} = \frac{+1}{\sqrt{2}} = 0{\cdot}707$.

Annäherungswerte für dieselbe ergeben sich aus Formel (25) und (26): ρ_0 und $\rho_{0(k)}$. Diese ergeben:

$$\rho_0 = \frac{+1{\cdot}506}{\sqrt{2{\cdot}925 \times 1{\cdot}544}} = 0{\cdot}709; \quad \rho_{0(1)} = \frac{+1{\cdot}736}{\sqrt{3{\cdot}261 \times 1{\cdot}645}} = 0{\cdot}750;$$

$$\rho_{0(2)} = \frac{+1{\cdot}932}{\sqrt{3{\cdot}542 \times 1{\cdot}751}} = 0{\cdot}776; \quad \rho_{0(3)} = \frac{+2{\cdot}081}{\sqrt{3{\cdot}729 \times 1{\cdot}834}} = 0{\cdot}796 \text{ u.s.w.}$$

Die Reihe der ρ ist scheinbar nicht stabil (obgleich ihre Abweichungen vom apriorischen $r_{1/1}$, an dessen mittl. Fehler gemessen, durchaus zulässig sind). Ist ein solches Verhalten mit der "R-Gruppen-Hypothese" noch zu vereinbaren? Um diese Frage zu beantworten, geht es nicht an, nur die Differenzen $\rho_{0(1)} - \rho_0$, $\rho_{0(2)} - \rho_{0(1)}$ u.s.w. zu betrachten, denn es ist leicht ersichtlich, daß die Zähler dieser Koeffizienten, ihrer Struktur nach, sich oft in derselben Richtung bewegen werden wie die Nenner, und daß, folglich, obige Differenzen die Sachlage eher noch in einem zu günstigen Lichte erscheinen lassen können (Wir berücksichtigen hierbei auch garnicht die Ungenauigkeit, welche dadurch entsteht, daß $E\rho_{0(k)}$ und $E\rho_0$ nur in ersten Annäherung gleich $r_{1/1}$ gesetzt werden können). So ist $\rho_{0(3)}$ nur um 12·3 % größer als $\rho_{0(1)}$, während deren Zähler sich um ganze 38·2 % von einander unterscheiden.

Schon allein aus diesem Grunde, von rechnerischen Schwierigkeiten garnicht zu reden, ist es daher geboten, die Differenzen der Zähler und beider Elemente der Nenner einzeln an Hand der Formeln der Tabelle II zu prüfen, für die Zähler natürlich—unter Berücksichtigung der Formel (22). Da N ziemlich groß ist, so können wir hierbei die Annäherungsformel bis zur Ordnung $\frac{1}{N}$ gebrauchen, welche noch den Vorteil besitzt, daß sie uns vom Verteilungsgesetz der beiden Reihen unabhängig macht.

Wir erhalten dann folgende Resultate:

	Bei $k=1$	$k=2$	$k=3$
$p_k' - p'_{k-1}$ …	0·230	0·196	0·149
Mittlerer Fehler …	0·146	0·069	0·053
Linke $\sigma_k'^2 - \sigma'^2_{k-1}$ …	0·336	0·281	0·187
Mittlerer Fehler …	0·238	0·113	0·079
Rechte $\sigma_k'^2 - \sigma'^2_{k-1}$ …	0·101	0·106	0·083
Mittlerer Fehler …	0·119	0·057	0·040

Die Differenzen übersteigen ihre mittleren Fehler ungefähr ums doppelte bis dreifache. Das ist *recht viel*, aber immerhin noch im Bereiche der Möglichkeit. Bei "normaler Verteilung" ist die Wahrscheinlichkeit dafür, daß eine zufällige Variable von ihrer mathematischen Erwartung *mehr* als um den zweifachen mittleren Fehler abweicht, größer als $\frac{1}{22}$, und um mehr als den dreifachen—immerhin noch etwa $\frac{1}{370}$. Somit steht unser Fall wohl ganz nahe an der Grenze des Zulässigen, erlaubt es aber doch, die "R-Gruppen-Hypothese" einigermaßen aufrecht zu erhalten.

Der Fall ist insofern lehrreich, als er, erstens, augenfällig darlegt, daß es die Grundhypothese der Differenzenrechnung garnicht erfordert, daß die einzelnen $\rho_{0(k)}$ oder $\sigma_k'^2$ einander *sehr nahe* kommen; und zweitens, sehen wir hier wieder, daß mit steigender Differenz die erreichten Resultate sich von den ihnen zu Grunde liegenden mathematischen Erwartungen oftmals nur entfernen, statt ihnen näherzukommen.

Der Vollständigkeit wegen wollen wir auch bei diesem Experiment die beiden Reihen in Gruppen zu je 20, 40 und 80 Gliedern zerlegen und für jede derselben die Werte ρ_0, $\rho_{0(k)}$, p_k', $\sigma_k'^2$ berechnen. Wir erhalten dann folgende Tabelle.

Anmerkung. Die Koeffizienten $\rho_{0(k)}$ sind direkt nach der Formel

$$\frac{\sum\limits_{i=1}^{N-k} \Delta^k x_i \Delta^k y_i}{\sqrt{\sum\limits_{i=1}^{N-k} \Delta^k x_i^2 \, . \sum\limits_{i=1}^{N-k} \Delta^k y_i^2}}$$

berechnet und bis zur 3ten Dezimalen korrekt.

Das Bild ähnelt demjenigen der Tabellen V bis VII. Es fällt aber wiederum auf, daß die Korrelationskoeffizienten selbst viel weniger schwanken, als ihre Bestandteile einzeln genommen. Schon bei $N=20$ wird die Tatsache offensichtlich, daß eine beträchtliche positive Korrelation zwischen beiden Reihen existiert, besonders, wenn man berücksichtigt, daß $\frac{1-\rho_0^2}{\sqrt{N}}$ hier 0·112 bedeutet.

(16) Korrelieren wir jetzt die frühere Summenreihe $x_1+x_2, x_3+x_4, \ldots x_{319}+x_{320}$, mit einer aus denselben Grundzahlen anders zusammengestellten Reihe: x_2+x_3, $x_4+x_5, \ldots x_{318}+{}_{319}, x_{320}+x_1$. Bezeichnen wir, zur Vereinfachung, die Glieder der ersten durch: $z_1, z_2, \ldots z_{160}$, und die der zweiten durch $y_1, y_2, \ldots y_{160}$.

TABELLE VIII.

Nr.	ρ_0	$\rho_{0(1)}$	$\rho_{0(2)}$	$\rho_{0(3)}$
		Gruppen zu 20 Gliedern		
1	$\frac{+1{\cdot}75}{\sqrt{3{\cdot}30\times1{\cdot}55}}=0{\cdot}774$	$\frac{+1{\cdot}66}{\sqrt{3{\cdot}84\times1{\cdot}26}}=0{\cdot}753$	$\frac{+2{\cdot}04}{\sqrt{4{\cdot}60\times1{\cdot}42}}=0{\cdot}798$	$\frac{+2{\cdot}41}{\sqrt{5{\cdot}16\times1{\cdot}59}}=0{\cdot}842$
2	$\frac{+0{\cdot}95}{\sqrt{3{\cdot}55\times1{\cdot}05}}=0{\cdot}492$	$\frac{+1{\cdot}66}{\sqrt{4{\cdot}68\times1{\cdot}47}}=0{\cdot}631$	$\frac{+2{\cdot}17}{\sqrt{5{\cdot}32\times1{\cdot}55}}=0{\cdot}755$	$\frac{+2{\cdot}52}{\sqrt{5{\cdot}89\times1{\cdot}51}}=0{\cdot}848$
3	$\frac{+1{\cdot}55}{\sqrt{2{\cdot}00\times1{\cdot}35}}=0{\cdot}943$	$\frac{+1{\cdot}71}{\sqrt{2{\cdot}16\times1{\cdot}53}}=0{\cdot}943$	$\frac{+2{\cdot}08}{\sqrt{2{\cdot}57\times1{\cdot}86}}=0{\cdot}952$	$\frac{+2{\cdot}38}{\sqrt{2{\cdot}91\times2{\cdot}13}}=0{\cdot}954$
4	$\frac{+2{\cdot}30}{\sqrt{3{\cdot}70\times2{\cdot}30}}=0{\cdot}788$	$\frac{+1{\cdot}79}{\sqrt{2{\cdot}71\times2{\cdot}03}}=0{\cdot}764$	$\frac{+1{\cdot}81}{\sqrt{2{\cdot}57\times2{\cdot}02}}=0{\cdot}792$	$\frac{+1{\cdot}81}{\sqrt{2{\cdot}47\times2{\cdot}01}}=0{\cdot}814$
5	$\frac{+1{\cdot}40}{\sqrt{2{\cdot}40\times1{\cdot}35}}=0{\cdot}778$	$\frac{+1{\cdot}55}{\sqrt{2{\cdot}68\times1{\cdot}32}}=0{\cdot}826$	$\frac{+1{\cdot}61}{\sqrt{2{\cdot}82\times1{\cdot}24}}=0{\cdot}861$	$\frac{+1{\cdot}69}{\sqrt{2{\cdot}87\times1{\cdot}25}}=0{\cdot}891$
6	$\frac{+1{\cdot}55}{\sqrt{3{\cdot}15\times2{\cdot}05}}=0{\cdot}610$	$\frac{+2{\cdot}97}{\sqrt{4{\cdot}66\times2{\cdot}76}}=0{\cdot}829$	$\frac{+3{\cdot}55}{\sqrt{5{\cdot}63\times3{\cdot}19}}=0{\cdot}836$	$\frac{+3{\cdot}86}{\sqrt{6{\cdot}13\times3{\cdot}47}}=0{\cdot}837$
7	$\frac{+1{\cdot}50}{\sqrt{3{\cdot}50\times1{\cdot}25}}=0{\cdot}717$	$\frac{+0{\cdot}92}{\sqrt{3{\cdot}11\times0{\cdot}79}}=0{\cdot}588$	$\frac{+0{\cdot}91}{\sqrt{3{\cdot}10\times0{\cdot}77}}=0{\cdot}588$	$\frac{+0{\cdot}88}{\sqrt{3{\cdot}11\times0{\cdot}74}}=0{\cdot}583$
8	$\frac{+1{\cdot}05}{\sqrt{1{\cdot}80\times1{\cdot}45}}=0{\cdot}650$	$\frac{+1{\cdot}55}{\sqrt{2{\cdot}34\times1{\cdot}87}}=0{\cdot}742$	$\frac{+1{\cdot}92}{\sqrt{2{\cdot}85\times2{\cdot}13}}=0{\cdot}779$	$\frac{+2{\cdot}24}{\sqrt{3{\cdot}26\times2{\cdot}37}}=0{\cdot}805$
	Arithm. Mittel: 0·719	0·759	0·795	0·822
		Gruppen zu 40 Gliedern		
1	$\frac{+1{\cdot}35}{\sqrt{3{\cdot}43\times1{\cdot}30}}=0{\cdot}640$	$\frac{+1{\cdot}64}{\sqrt{4{\cdot}17\times1{\cdot}38}}=0{\cdot}683$	$\frac{+2{\cdot}02}{\sqrt{4{\cdot}72\times1{\cdot}56}}=0{\cdot}740$	$\frac{+2{\cdot}32}{\sqrt{5{\cdot}09\times1{\cdot}75}}=0{\cdot}779$
2	$\frac{+1{\cdot}93}{\sqrt{2{\cdot}85\times1{\cdot}83}}=0{\cdot}844$	$\frac{+1{\cdot}78}{\sqrt{2{\cdot}49\times1{\cdot}78}}=0{\cdot}846$	$\frac{+1{\cdot}95}{\sqrt{2{\cdot}67\times1{\cdot}89}}=0{\cdot}866$	$\frac{+2{\cdot}10}{\sqrt{2{\cdot}83\times1{\cdot}99}}=0{\cdot}885$
3	$\frac{+1{\cdot}48}{\sqrt{2{\cdot}78\times1{\cdot}70}}=0{\cdot}679$	$\frac{+2{\cdot}21}{\sqrt{3{\cdot}69\times1{\cdot}99}}=0{\cdot}814$	$\frac{+2{\cdot}50}{\sqrt{4{\cdot}26\times2{\cdot}12}}=0{\cdot}834$	$\frac{+2{\cdot}76}{\sqrt{4{\cdot}64\times2{\cdot}26}}=0{\cdot}850$
4	$\frac{+1{\cdot}28}{\sqrt{2{\cdot}65\times1{\cdot}35}}=0{\cdot}674$	$\frac{+1{\cdot}32}{\sqrt{2{\cdot}77\times1{\cdot}41}}=0{\cdot}668$	$\frac{+1{\cdot}47}{\sqrt{3{\cdot}00\times1{\cdot}48}}=0{\cdot}699$	$\frac{+1{\cdot}58}{\sqrt{3{\cdot}14\times1{\cdot}53}}=0{\cdot}719$
	Arithm. Mittel: 0·709	0·753	0·785	0·808
		Gruppen zu 80 Gliedern		
1	$\frac{+1{\cdot}64}{\sqrt{3{\cdot}14\times1{\cdot}56}}=0{\cdot}740$	$\frac{+1{\cdot}70}{\sqrt{3{\cdot}31\times1{\cdot}57}}=0{\cdot}747$	$\frac{+1{\cdot}95}{\sqrt{3{\cdot}61\times1{\cdot}70}}=0{\cdot}785$	$\frac{+2{\cdot}14}{\sqrt{3{\cdot}82\times1{\cdot}81}}=0{\cdot}813$
2	$\frac{+1{\cdot}38}{\sqrt{2{\cdot}71\times1{\cdot}53}}=0{\cdot}676$	$\frac{+1{\cdot}73}{\sqrt{3{\cdot}20\times1{\cdot}68}}=0{\cdot}748$	$\frac{+1{\cdot}93}{\sqrt{3{\cdot}54\times1{\cdot}77}}=0{\cdot}772$	$\frac{+2{\cdot}10}{\sqrt{3{\cdot}77\times1{\cdot}86}}=0{\cdot}792$
	Arithm. Mittel: 0·708	0·747	0·778	0·796

Da in unserem Falle:

$$E(z_i-Ez)^2=E(y_i-Ey)^2=2E(x_i-Ex)^2=3\,;$$
$$E(z_i-Ez)(y_i-Ey)=E(x_i-Ex)^2=1{\cdot}5\,;$$
$$E(z_i-Ez)(y_{i-1}-Ey)=E(x_i-Ex)^2=1{\cdot}5\,;$$

und alle übrigen $E(z_i - Ez)(y_{i+j} - Ey) = 0$, so erhalten wir aus Formeln (24)—(25):

$$r_{1/1} = r_{1/1}^{(-1)} = \frac{+1\cdot 5}{\sqrt{3 \times 3}} = +0\cdot 50,$$

alle übrigen $r_{1/1}^{(j)}$ sind aber gleich Null. Als empirische Annäherungen zu diesen apriorischen Koeffizienten dienen die Größen ${}_0\rho_j$ (siehe Formeln (24)—(25)). Rechnen wir unsere beiden Zahlenreihen wirklich durch, so erhalten wir für letztere folgende Werte:

$${}_0\rho_2 = +0\cdot 1074\,; \quad {}_0\rho_1 = +0\cdot 0044\,;$$

$${}_0\rho_0 = \frac{+1\cdot 250}{\sqrt{2\cdot 925 \times 2\cdot 850}} = +0\cdot 4329\,; \quad {}_0\rho_{-1} = \frac{+1\cdot 465}{\sqrt{2\cdot 925 \times 2\cdot 850}} \doteq +0\cdot 5075\,;$$

$${}_0\rho_{-2} = -0\cdot 1096\,; \quad {}_0\rho_{-3} = +0\cdot 2758\,; \quad {}_0\rho_{-4} = -0\cdot 0511.$$

Ausgenommen ${}_0\rho_{-3}$, unterscheiden sich die ${}_0\rho_j$ von den entsprechenden $r_{1/1}^{(j)}$ um einen Betrag, der den mittleren Fehler der letzteren auch bei Annahme eines "normalen" Verteilungsgesetzes kaum übersteigt.

Damit die Reihe ${}_0\rho_0$, $\rho_{0(1)}$, $\rho_{0(2)}$, ... konstant bleibe, ist es, bekanntlich, erforderlich, daß alle $r_{1/1}^{(j)}$ gleich 0 seien, und nur $r_{1/1}$ davon differiere*. Diese Bedingung ist in unserem Falle nicht erfüllt, da auch $r_{1/1}^{(-1)} = 0\cdot 50$, und daher steht es nicht zu erwarten, daß die Reihe der $\rho_{0(k)}$ konstante Werte ergebe. Wir erhalten auch wirklich:

$$\rho_{0(1)} = \frac{+0\cdot 513}{\sqrt{3\cdot 261 \times 2\cdot 997}} = +0\cdot 1640\,; \quad \rho_{0(2)} = \frac{+0\cdot 271}{\sqrt{3\cdot 542 \times 3\cdot 091}} = +0\cdot 0819.$$

Die Differenzen der Zähler von $\rho_{0(1)}$ und ${}_0\rho_0$ und von $\rho_{0(2)}$ und $\rho_{0(1)}$ ergeben $-0\cdot 737$ und $-0\cdot 242$, also schon das vierfache und dreifache ihrer mittl. Fehler ($0\cdot 188$ und $0\cdot 088$)!

Die Größe $p'_{j(k)}$ der Zähler von $\rho_{j(k)}$ (vergl. Formeln (21) und (26)) kann mit guter Annäherung durch die empirische Formel (28) bestimmt werden.

Hieraus ergab sich für $\rho_{0(1)}$ der Zähler $+0\cdot 51$, während auch direkte Berechnung genau zu $+0\cdot 51$ führt; für $\rho_{0(2)}$ ergab sich der Zähler $+0\cdot 26$ und direkt $+0\cdot 27$; für $\rho_{-1(1)}$, $\rho_{-1(2)}$, $\rho_{-2(1)}$ und $\rho_{-2(2)}$ ergaben sich in derselben Reihenfolge die Zähler: nach Formel (28): $+1\cdot 00$; $+0\cdot 98$; $-1\cdot 45$; $-1\cdot 64$; und direkt berechnet: $+1\cdot 00$; $+0\cdot 99$; $-1\cdot 47$; $-1\cdot 69$. Die Übereinstimmung ist ganz befriedigend.

Umgekehrt, kann man die Größen ${}_0\rho_j$ (oder auch ρ_j) aus den gegebenen $\rho_{j(k)}$ nicht unmittelbar bestimmen, da es hierbei immer mehr Unbekannte als Gleichungen geben wird. Berücksichtigt man aber, daß in Formel (28) die Koeffizienten vor ρ_{k+j} desto geringere Werte erhalten, je größer k ist, und daß sie auch an und für sich in der Regel nicht groß sein werden, so ergibt es sich, daß man, ohne einen beträchtlichen Fehler zu begehen, eine Anzahl derselben gleich 0 setzen und dadurch die Zahl der Unbekannten mit der Zahl der Gleichungen identisch machen

* **Steht man von der Forderung $\rho_0 = \rho_{0(1)}$ ab, so genügt es, wenn alle $r_{1/1}^{(j)}$ (ausgenommen $r_{1/1}$) einander gleich seien: uniform verbundene Reihen.**

werden kann*. Setzen wir, zum Beispiel, ${}_0\rho_1 = {}_0\rho_2 = {}_0\rho_{-2} = 0$, so berechnen sich aus den Zählern der Koeffizienten $\rho_{0(1)}$ und $\rho_{0(2)}$ folgende 2 Werte: ${}_0\rho_0 = 0{\cdot}429$; ${}_0\rho_{-1} = 0{\cdot}503$; also, in diesem Fall, bis zur 2-ten Dezimalstelle genaue Resultate!

(17) *Zahl π.*

In seinem Artikel "Über normal stabile Korrelation†" hat Prof. Al. A. Tchouproff einige Divergenzkoeffizienten Q^2 für die ersten 675 der von Shanks bestimmten 707 Dezimalen dieser Zahl berechnet und hierbei bemerkt, daß letztere "doch eine übernormale Stabilität" anzudeuten scheinen. "Interessant ist auch der systematische Rückgang der Werte von Q^2 mit zunehmendem n" (Zahl der Glieder in einer Serie). Ich habe nun dieselben 675 ersten Dezimalen einer systematischeren Bearbeitung unterworfen und für $n = 2, 3, 4, 5, \ldots$ die entsprechenden Koeffizienten $\mathfrak{Q}^2$ gefunden. Teilte sich 675 nicht durch n, so wurde die erste sich durch n teilende größere oder kleinere Zahl der Dezimalen genommen. Sie ist überall als *linkes* Subskriptum angegeben (das rechte bedeutet, wie früher, n). Die Resultate waren die folgenden:

$${}_{676}\mathfrak{Q}_2^2 = 1{\cdot}034;\quad {}_{675}\mathfrak{Q}_3^2 = 0{\cdot}971;\quad {}_{676}\mathfrak{Q}_4^2 = 0{\cdot}844;\quad {}_{675}\mathfrak{Q}_5^2 = 0{\cdot}895;$$
$${}_{678}\mathfrak{Q}_6^2 = 0{\cdot}865;\quad {}_{672}\mathfrak{Q}_7^2 = 0{\cdot}778;\quad {}_{672}\mathfrak{Q}_8^2 = 0{\cdot}732;\quad {}_{675}\mathfrak{Q}_9^2 = 0{\cdot}786;$$
$${}_{680}\mathfrak{Q}_{10}^2 = 0{\cdot}801;\quad {}_{675}\mathfrak{Q}_{15}^2 = 0{\cdot}776;\quad {}_{672}\mathfrak{Q}_{16}^2 = 0{\cdot}658.$$

Hierzu kommen noch die von Tchouproff berechneten:

$${}_{675}Q_{45}^2 = 0{\cdot}51 \text{ und } {}_{675}Q_{75}^2 = 0{\cdot}42.$$

An ihrem mittleren Fehler gemessen, dessen Annäherungswert, bekanntlich, $\sqrt{\dfrac{2}{r-1}}$ ist, wobei $r = \dfrac{N}{n}$, sind die Abweichungen der $\mathfrak{Q}^2$ von 1 nicht übermäßig groß, es ist jedoch ihre fallende Tendenz bei steigendem n unverkennbar.

Um festzustellen, ob selbige nicht durch irgendwelche speziellen Zusammenhänge zwischen den Dezimalen hervorgerufen werden, berechnete ich nach Formel (29a) die ersten 8 Koeffizienten R_j'. Es stellte sich heraus:

$$R_1' = -0{\cdot}0296;\quad R_2' = -0{\cdot}0424;\quad R_3' = -0{\cdot}0473;\quad R_4' = -0{\cdot}0195;$$
$$R_5' = +0{\cdot}0402;\quad R_6' = +0{\cdot}0152;\quad R_7' = -0{\cdot}0219;\quad R_8' = -0{\cdot}0148.$$

Im Falle einer "normal stabilen" oder auch "uniformen" Reihe, würden diese Koeffizienten ungefähr $-\dfrac{1}{N} = -0{\cdot}0015$ ergeben. Aus Formel $\dfrac{1-r^2}{\sqrt{N}}$ erhält man für $r = -0{\cdot}0015$ und $N = 676$ den Wert: $\pm 0{\cdot}038$. In den Fällen also, für welche diese Formel den Wert des mittleren Fehlers ergibt, können die Koeffizienten R_j' sehr wohl mit der Hypothese einer "normal stabilen Reihe" vereinbart werden. Wie die apriorischen Koeffizienten r_j (Formel (30)) beschaffen sind, können wir,

* Verringert man die Zahl der Koeffizienten noch mehr, so kann man ihre "vorteilhaftesten" Werte auch nach der Methode der kleinsten Quadrate bestimmen. Nur muß man hierbei noch die Bedingungen einführen, daß keines der ${}_0\rho_j$ die Grenzen ± 1 überschreite.

† *Skandinavisk Aktuarietidskrift*, 1923.

natürlich, nicht genau bestimmen, doch ist es zu erwarten, daß sie jedenfalls nicht viel von 0 abweichen können, da sonst die R_j' größere Werte erhalten würden*.

Die fallende Tendenz der $\mathfrak{Q}^2$ und Q^2 *könnte* daher sehr wohl dadurch erklärt werden, daß die Mehrzahl der r_j *ganz kleine* negative Werte besitzen. Sind diese einander gleich, so verwandelt sich Formel (36) in $EQ^2 = 1 + (n-1)\,r$. Damit also $EQ_{16}^2 = 0{\cdot}658$ genügt in diesem Falle schon ein Wert von $r_j = -0{\cdot}0228$, welcher gut mit den Werten der empirischen R_j' korrespondiert†.

Wenden wir hier jetzt die Differenzenmethode an. Der Mittelwert der ersten 675 Dezimalen ist 4·420. Seine mathematische Erwartung, unter der Voraussetzung rein zufälliger Aufeinanderfolge der Ziffern, ist 4·5. Die Abweichung erreicht also nicht ihren theoretischen mittl. Fehler, der sich hier auf 0·11 stellt. Die empirische Streuung der Reihe ist 8·549, ihre mathematische Erwartung ist unter denselben Voraussetzungen gleich 8·25. Die Abweichung ist kaum um 7 % größer als ihr theoretischer mittlerer Fehler (0·28). Die Differenz zwischen $\sigma_1'^2$ und σ'^2 (vergl. Tabelle IX unten) beträgt demgegenüber nur 0·243 und ist somit beträchtlich kleiner als ihr theoretischer mittlerer Fehler, der sich aus Tabelle II bei $\mu_2 = 8{\cdot}25$ auf 0·318 stellt.

TABELLE IX.

Streuungen	Empirisch	Nach Formel (29) berechnet	Differenzen	Differenzen der empirischen Daten	Der nach Formel (29) berechneten	Mittlerer Fehler der Differenzen
$\sigma'^2 =$	8·549	—	$\sigma_1'^2 - \sigma'^2 =$	+0·243	+0·253	±0·318
$\sigma_1'^2 =$	8·792	8·802	$\sigma_2'^2 - \sigma_1'^2 =$	−0·026	−0·037	±0·148
$\sigma_2'^2 =$	8·766	8·765	$\sigma_3'^2 - \sigma_2'^2 =$	−0·008	−0·013	±0·107
$\sigma_3'^2 =$	8·758	8·752	$\sigma_4'^2 - \sigma_3'^2 =$	—	0·000	±0·082
$\sigma_4'^2 =$	—	8·752	$\sigma_5'^2 - \sigma_4'^2 =$	—	+0·002	±0·069
$\sigma_5'^2 =$	—	8·754	$\sigma_6'^2 - \sigma_5'^2 =$	—	+0·001	±0·059
$\sigma_6'^2 =$	—	8·755	$\sigma_7'^2 - \sigma_6'^2 =$	—	−0·003	±0·055
$\sigma_7'^2 =$	—	8·752	$\sigma_8'^2 - \sigma_7'^2 =$	—	−0·004	±0·049
$\sigma_8'^2 =$	—	8·748	$\sigma_9'^2 - \sigma_8'^2 =$	—	−0·006	±0·044
$\sigma_9'^2 =$	—	8·742	—	—	—	—

* Wären in unserem Falle die $\mathfrak{Q}_n^2$ ihren mathematischen Erwartungen genau gleich, so könnten wir aus Formel (38) leicht alle r_j berechnen. Definiert man nämlich $\Delta' Q_n^2 = Q_n^2 - Q^2_{n+1}$ und setzt $Q_1^2 = 1$, so erhält man hieraus leicht: $r_n = \frac{1}{2}\left[(n-1)\,\Delta' Q^2_{n-1} - (n+1)\,\Delta' Q_n^2\right]$. Wären, im Gegenteil, die R_j' ihren apriorischen Werten gleich, so könnten wir die Werte von EQ^2 nach Formel (37) genau bestimmen. Der erste Weg ist aber in unserem Falle nicht gangbar, da bei kleinen r_j auch kleine Abweichungen der Q^2 von ihren mathematischen Erwartungen einen starken Einfluß auf die gefundenen Werte ausüben. So erhalten wir hier: $r_1 = +0{\cdot}034$, $r_2 = -0{\cdot}112$, $r_3 = -0{\cdot}191$, $r_4 = +0{\cdot}318$, $r_5 = -0{\cdot}192$, $r_6 = -0{\cdot}230$, $r_7 = +0{\cdot}077$ u.s.w. Der zweite Weg führt insofern zu besseren Resultaten, als die berechneten Q^2, wenn sie auch von den in Wirklichkeit gefundenen nicht unerheblich abweichen, eine regelmäßig abfallende Reihe bilden. Es ergeben sich nämlich folgende Werte: $Q_2^2 = 0{\cdot}972$; $Q_3^2 = 0{\cdot}935$; $Q_4^2 = 0{\cdot}893$; $Q_5^2 = 0{\cdot}861$; $Q_6^2 = 0{\cdot}853$; $Q_7^2 = 0{\cdot}853$; $Q_8^2 = 0{\cdot}847$; $Q_9^2 = 0{\cdot}839$ u.s.w. Bringt man diese Data zusammen mit den empirischen $\mathfrak{Q}^2$ aufs Diagramm, so erhält man hieraus den Eindruck, als ob letztere um erstere etwa wie eine "zufällige" um ihre "evolutorische Komponente" schwanken.

Die Abweichungen entstehen, nämlich, dadurch, daß ein jedes R_j' im Zähler alle $(N-j)$ Produkte $(x_i - x_{(N)})(x_{i+j} - x_{(N)})$ enthält, während im Zähler von $\mathfrak{Q}^2$ nur ein Teil dieser Produkte auftritt. In dieser Hinsicht dürften die berechneten Q^2 vielleicht besser sein als die empirischen.

† Unter denselben Annahmen würden wir aus (37) den Wert $R_j = -0{\cdot}0238$ erhalten.

Die Differenzen zwischen $\sigma_2'^2$ und $\sigma_1'^2$, $\sigma_3'^2$ und $\sigma_2'^2$ u.s.w. sind vollends verschwindend klein gegenüber ihren mittleren Fehlern.

Um der mühevollen Berechnung der Koeffizienten $\sigma_k'^2$ für höhere Differenzen zu entgehen, habe ich deren angenäherte Werte nach Formel (29) bestimmt. Sie dürften in unserem Falle bis zur 2-ten Dezimalstelle korrekt sein. Die maximale Größe, welche $\sigma_k'^2 - \sigma'^2_{k-1}$ erreichen kann, ist $+\frac{\sigma'^2_{k-1}}{2k-1}$. Folglich, kann der Ausdruck $\frac{(2k-1)(\sigma_k'^2 - \sigma'^2_{k-1})}{\sigma'^2_{k-1}}$ als ein gewisses Maß des Einflusses einer "Z"-Komponente betrachtet werden. Sein maximaler Wert ist $+1$, der Wert 0 entspricht dem Falle $\sigma_k'^2 = \sigma'^2_{k-1}$; bei $\sigma'^2_{k-1} > \sigma_k'^2$ verliert er aber seinen Sinn, da sein Minimum im Falle $\sigma_k'^2 = 0$ gleich $(2k-1)$ wird. Bei $k=1$ auf unser Beispiel angewandt, ergibt dieser Koeffizient den sehr kleinen Wert $+0{\cdot}028$. Also wiederum kein klares Anzeichen der Anwesenheit einer Komponente "Z."

Wie wir sehen, widersprechen die nach der Bortkiewicz'schen Methode und die nach der Differenzenmethode gefundenen Resultate einander nicht direkt. Doch dürfte der allgemeine Eindruck eines möglicherweise ein wenig übernormal stabilen Charakters der Reihe, welchen erstere hinterläßt, durch die zweite eher widerlegt werden.

(18) *Weizenpreise in Berlin, New York und Chicago.*

Wenn man die Bewegung der Weizenpreise an den wichtigsten Getreidebörsen mit einander vergleicht, so ist deren beträchtliche positive Korrelation augenfällig. Sie hängt, natürlich, damit zusammen, daß die wichtigsten preisbildenden Momente überall dieselben bleiben: Welternte, Weltnachfrage, Kaufkraft des Goldes, u.s.w. Will man daher den Einfluß, welchen das Getriebe einer Börse auf dasjenige einer anderen ausübt, messen, das heißt mit anderen Worten, will man die Zusammenhänge zwischen Börsengeschäften verschiedener Getreidehandelsplätze untersuchen, so darf man nur die nach Abzug der "Säkularen," "Konjunktur-" und "Saison-Komponenten" etc. verbleibenden "restlichen Komponenten" der verschiedenen Preiskurven mit einander vergleichen. Die Anwendung der Differenzenmethode ist also hier geradezu geboten*.

Vergleicht man europäische mit amerikanischen Preisen, so gestaltet sich die Sachlage insofern interessant, als dank dem Zeitunterschiede (zum Beispiel, 5 Stunden zwischen "Greenwich time" und "Eastern time" der Vereinigten Staaten) die europäischen Getreidebörsen eines gewissen Tages in der Regel schon geschlossen sein werden ehe die amerikanischen überhaupt eröffnet wurden. Daher

* Wie es scheint, könnte die Differenzenmethode gerade bei Preisuntersuchungen sich als nützlich erweisen. Es wäre, zum Beispiel, recht pikant, wenn es sich wirklich nachweisen ließe, daß die einzelnen Preiskurven in der Regel ein "Z"-Element enthalten, denn das würde auf eine gewisse Tendenz zum "dynamischen Gleichgewicht" der Preise hinweisen. Besonders dürften solche Feststellungen diejenige neueste Richtung in der Preistheorie angehen, welche dem Empirismus in einer mehr statistischen Form huldigt, wie etwa eine Reihe Amerikaner, den Franzosen F. Simiand, die Russen P. Struve, St. Cohn und a. m.

können europäische Preisnotierungen auf amerikanische Börsengeschäfte noch am selben Kalendertage einwirken, während der amerikanische Einfluß sich in Europa erst am darauffolgenden Tage bemerkbar machen kann. Es besteht also die Möglichkeit, diese gegenseitigen Einflüsse besonders zu messen und mit einander zu vergleichen. Derartige Untersuchungen erfordern aber recht umfangreiches Material, denn die zu vergleichenden Reihen müssen jedenfalls so lang sein, daß schon ein Korrelationskoeffizient von, sagen wir, + 0·05 oder + 0·10 einigermaßen "significant" wird, das heißt, die Zahl der Reihenglieder müßte nicht weniger als etwa 1000—1500, betragen.

Daher sollen die jetzt nachfolgenden Ausführungen keinesfalls für eine Untersuchung der Frage, sondern nur für ein einfaches Rechenexempel gelten.

Aus den offiziellen "Vierteljahrsheften zur Statistik des deutschen Reiches," 19. Jahrgang, 1910, wurden *die Weizenpreise für* 1000 *kg. in Mark* in Berlin, New York und Chicago für die Zeit vom 8. September bis zum 24. Dezember des Jahres 1908 entnommen und nach der Differenzenmethode bearbeitet. Da 15 Sonntage und noch 4 andere Tage ausfielen, an denen wenn auch nur eine der 3 Börsen geschlossen war (30. IX, 3. XI, 18. XI, 26. XI), so ergaben sich im Ganzen 3 Reihen zu je 89 Gliedern. Bezeichnet man Berlin durch s und New York durch s', so erhält man aus Formeln (25) und (26) folgende Werte:

$$\rho_0 = \frac{+4{\cdot}663}{\sqrt{5{\cdot}996 \times 16{\cdot}140}} = +0{\cdot}474; \quad \rho_{-1} = \frac{+5{\cdot}963}{\sqrt{5{\cdot}996 \times 16{\cdot}140}} = +0{\cdot}606;$$

$$\rho_{0\,(1)} = \frac{-0{\cdot}079}{\sqrt{0{\cdot}762 \times 1{\cdot}021}} = -0{\cdot}089; \quad \rho_{-1\,(1)} = \frac{+0{\cdot}378}{\sqrt{0{\cdot}762 \times 1{\cdot}021}} = +0{\cdot}429;$$

$$\rho_{0\,(2)} = \frac{-0{\cdot}201}{\sqrt{0{\cdot}531 \times 0{\cdot}728}} = -0{\cdot}323; \quad \rho_{-1\,(2)} = \frac{+0{\cdot}257}{\sqrt{0{\cdot}531 \times 0{\cdot}728}} = +0{\cdot}413;$$

$$\rho_{0\,(3)} = \frac{-0{\cdot}215}{\sqrt{0{\cdot}491 \times 0{\cdot}659}} = -0{\cdot}377; \quad \rho_{-1\,(3)} = \frac{+0{\cdot}257}{\sqrt{0{\cdot}491 \times 0{\cdot}659}} = +0{\cdot}452;$$

$$\rho_{0\,(4)} = \frac{-0{\cdot}223}{\sqrt{0{\cdot}475 \times 0{\cdot}629}} = -0{\cdot}407; \quad \rho_{-1\,(4)} = \frac{+0{\cdot}262}{\sqrt{0{\cdot}475 \times 0{\cdot}629}} = +0{\cdot}478;$$

$$\rho_{0\,(5)} = \frac{-0{\cdot}230}{\sqrt{0{\cdot}465 \times 0{\cdot}612}} = -0{\cdot}431; \quad \rho_{-1\,(5)} = \frac{+0{\cdot}262}{\sqrt{0{\cdot}465 \times 0{\cdot}612}} = +0{\cdot}491;$$

$$\rho_{0\,(6)} = \frac{-0{\cdot}219}{\sqrt{0{\cdot}459 \times 0{\cdot}601}} = -0{\cdot}417; \quad \rho_{-1\,(6)} = \frac{+0{\cdot}261}{\sqrt{0{\cdot}459 \times 0{\cdot}601}} = +0{\cdot}497.$$

Die erste zu lösende Frage besteht darin, von welcher endlichen Differenz angefangen man die verschiedenen evolutorischen Komponenten als praktisch eliminiert ansehen darf. Dazu sollen wieder die ersten Differenzen der empirischen Streuungen mit ihren mittleren Fehlern verglichen werden. Letztere berechnen sich aus Tabelle II.

Die uns unbekannten apriorischen μ_2 wollen wir hierbei für $\sigma_1'^2 - \sigma'^2$ durch $\sigma_1'^2$, für $\sigma_2'^2 - \sigma_1'^2$ durch $\sigma_2'^2$, für $\sigma_k'^2 - \sigma'^2_{k-1}$ durch $\sigma_k'^2$ ersetzen. Wir erhalten dann folgende Tabelle.

Berlin.

Für	$k=1$	2	3	4	5	6
Differenz $\sigma_k'^2 - \sigma'^2_{k-1}$ ...	$-$ 5·234	$-$0·231	$-$0·040	$-$0·026	$-$0·010	$-$0·006
Mittlerer Fehler ...	$\pm$ 0·081	$\pm$0·027	$\pm$0·018	$\pm$0·013	$\pm$0·011	$\pm$0·010
Verhältnis: $\frac{\text{Differenz}}{\text{Mittl. Fehler}}$	$\pm$64·6	$\pm$8·6	$\pm$2·2	$\pm$2·0	$\pm$0·9	$\pm$0·6

New York.

Für	$k=1$	2	3	4	5	6
Differenz $\sigma_k'^2 - \sigma'^2_{k-1}$ ...	$-$ 15·119	$-$0·293	$-$0·069	$-$0·030	$-$0·017	$-$0·011
Mittlerer Fehler ...	$\pm$ 0·108	$\pm$0·037	$\pm$0·024	$\pm$0·018	$\pm$0·015	$\pm$0·013
Verhältnis: $\frac{\text{Differenz}}{\text{Mittl. Fehler}}$	$\pm$140·0	$\pm$7·9	$\pm$2·9	$\pm$1·7	$\pm$1·1	$\pm$0·8

Man kann also annehmen, daß in der 5. und 6. Differenz schon keine bemerkbaren Reste evolutorischer Elemente mehr enthalten sind. Eine "Z"-Komponente ist nicht nachzuweisen. Die Differenzen der Zähler zusammen mit ihren mittlern Fehlern, ergeben für $k=5$ und 6 folgende Werte.

	$k=5$	$k=6$		$k=5$	$k=6$
Differenz $p_0'{}_{(k)} - p_0'{}_{(k-1)}$ ...	$-$0·007	+0·011	Differenz $p'_{-1(k)} - p'_{-1(k-1)}$	0·000	$-$0·001
Mittlerer Fehler	$\pm$0·010	$\pm$0·008	Mittlerer Fehler	$\pm$0·010	$\pm$0·009
Verhältnis: $\frac{\text{Differenz}}{\text{Mittl. Fehler}}$	$\pm$0·7	$\pm$1·4	Verhältnis: $\frac{\text{Differenz}}{\text{Mittl. Fehler}}$	0	$\pm$0·1

Somit haben die Zähler nicht nur von $\rho_{0(k)}$, sondern auch von $\rho_{-1(k)}$ in der 5. und 6. Differenz eine den Anforderungen der Theorie genügende Stabilität erreicht. Die Koexistenz zweier stabiler Korrelationskoeffizienten widerspricht aber vom Grunde aus der Annahme, daß es nur ein einziges $r_{1/1}^{(j)}$ gibt, welches nicht gleich Null ist.

Deshalb müssen jetzt die Korrelationskoeffizienten daraufhin untersucht werden, ob sie nicht als Funktionen *einiger* Koeffizienten $r_{1/1}^{(j)}$ angesehen werden *müssen*, die

TABELLE X.

	(1)	(2) Mittl. Fehler	(3) Dieselben Koeffizienten $\rho_{j(6)}$ in der Annahme $r_{1/1}^{(-1)} = +0,5$, und alle $r_{1/1}^{(-1\pm j)}=0$, bei $j \neq 0$	(4) Differenz zwischen Kolonne 3 und Kolonne 1
$\rho_{3(6)}$	= +0·041	$\pm$0·197	+0·036	$-$0·005
$\rho_{2(6)}$	= $-$0·125	$\pm$0·193	$-$0·119	+0·006
$\rho_{1(6)}$	= +0·280	$\pm$0·180	+0·268	$-$0·012
$\rho_{0(6)}$	= $-$0·417	$\pm$0·160	$-$0·429	$-$0·012
$\rho_{-1(6)}$	= +0·497	$\pm$0·147	+0·500	$-$0·003
$\rho_{-2(6)}$	= $-$0·377	$\pm$0·168	$-$0·429	$-$0·052
$\rho_{-3(6)}$	= +0·155	$\pm$0·192	+0·268	+0·113
$\rho_{-4(6)}$	= +0·051	$\pm$0·198	$-$0·119	$-$0·170
$\rho_{-5(6)}$	= $-$0·164	$\pm$0·195	+0·036	+0·200

eine derartige Gesamtwirkung hervorrufen. Zu diesem Zweck berechnen wir für die 6-te Differenz die Werte der Koeffizienten $\rho_{j\,(6)}$ von $j=3$ bis $j=-5$ zusammen mit ihren nach Formel (27) bestimmten mittleren Fehlern. Um die Rechnungen zu vereinfachen, nehmen wir also an, die entsprechenden Korrelationsflächen seien "normal." Die Ergebnisse sind in den zwei ersten Kolonnen der Tabelle X auf S. 79 enthalten.

Die Glieder der Reihe verteilen sich ungefähr symmetrisch um das Maximum: $\rho_{-1\,(6)}$.

Aus Formel (4) von "Über ein neues Verfahren, etc.," welche leicht auf die unseren Bezeichnungen entsprechende Form:

$$E\rho_{j\,(k)} = \frac{(-1)^k}{C_{2k}^k}\{C_{2k}^{2k} r_{1/1}^{(k+j)} - C_{2k}^{2k-1} r_{1/1}^{(k+j-1)} + C_{2k}^{2k-2} r_{1/1}^{(k+j-2)} - \ldots + C_{2k}^{0} r_{1/1}^{(j-k)}\} \qquad \ldots\ldots(41),$$

gebracht werden kann, ist es ersichtlich, daß wenn alle $r_{1/1}^{(i)}$, ausgenommen ein einziges beliebiges $r_{1/1}^{(j)}$, gleich Null sind, die mathem. Erwartungen der Koeffizienten $\rho_{i\,(k)}$ folgende Werte erhalten würden:

$$E\rho_{j\,(k)} = r_{1/1}^{(j)}; \quad E\rho_{j\pm 1,\,(k)} = -\frac{k}{k+1} r_{1/1}^{(j)};$$

$$E\rho_{j\pm 2,\,(k)} = +\frac{k(k-1)}{(k+1)(k+2)} r_{1/1}^{(j)}, \;\ldots\; E\rho_{j\pm i,\,(k)}$$

$$= (-1)^i \frac{k(k-1)(k-2)\ldots(k-i+1)}{(k+1)(k+2)\ldots(k+i)} r_{1/1}^{(j)}.$$

Nehmen wir an, $r_{1/1}^{(-1)}$ sei $+0{\cdot}50$ und alle übrigen $r_{1/1}^{(i)}$ gleich Null, so würden wir dann für $\rho_{0\,(6)}$ und $\rho_{-2\,(6)}$ den Wert $-\frac{6}{7}\times 0{\cdot}50 = -0{\cdot}429$ erhalten, für $\rho_{1\,(6)}$ und $\rho_{-3\,(6)}$ den Wert $\frac{6\cdot5}{7\cdot8}\times 0{\cdot}50 = +0{\cdot}268$, u.s.w.

Diese Werte sind in der dritten Kolonne unserer Tabelle X wiedergegeben. Vergleichen wir die Differenzen zwischen den wirklich gefundenen Werten der $\rho_{j\,(k)}$ und diesen berechneten (vierte Kolonne), einerseits, mit den mittleren Fehlern der ersteren (zweite Kolonne), andererseits, so ist die gute Uebereinstimmung der Werte von Kol. (3) und Kol. (1) augenfällig. Die Notwendigkeit, auch andere $r_{1/1}^{(j)} \neq 0$ einzuführen, ist nicht ersichtlich. Der Koeffizient $r_{1/1}^{(-1)}$ entspricht dem Korrelationskoeffizienten zwischen New York heute und Berlin morgen.

Wir kommen also zum Schluß, daß unser Zahlenmaterial sich sehr wohl mit der Hypothese verträgt, *der Einfluß der New Yorker Weizennotierungen auf die Berliner könne im Herbst* 1908, *durch einen Korrelationskoeffizienten etwa von der Größe* $+0{\cdot}5$ *mit einem mittleren Fehler von etwa* $\pm 0{\cdot}15$ *gemessen werden, während, umgekehrt, der entsprechende Einfluß des Berliner Kurszettels auf New Yorker Weizengeschäfte durch unser Material nicht nachzuweisen ist und, jedenfalls, recht gering sein muß.*

Betrachten wir jetzt noch kurz die Zusammenhänge zwischen den Weizen-notierungen von Chicago (s) und New York (s').

Wir erhalten hier folgende Werte:

$$\rho_0 = \frac{+12{\cdot}134}{\sqrt{11{\cdot}033 \times 16{\cdot}140}} = +0{\cdot}909; \quad \rho_{0(1)} = \frac{+0{\cdot}757}{\sqrt{0{\cdot}807 \times 1{\cdot}021}} = +0{\cdot}834;$$

$$\rho_{0(2)} = \frac{+0{\cdot}545}{\sqrt{0{\cdot}601 \times 0{\cdot}728}} = 0{\cdot}825; \quad \rho_{0(3)} = \frac{+0{\cdot}498}{\sqrt{0{\cdot}560 \times 0{\cdot}659}} = +0{\cdot}819;$$

$$\rho_{0(4)} = \frac{+0{\cdot}469}{\sqrt{0{\cdot}531 \times 0{\cdot}629}} = +0{\cdot}811; \quad \rho_{0(5)} = \frac{+0{\cdot}444}{\sqrt{0{\cdot}502 \times 0{\cdot}612}} = +0{\cdot}802;$$

$$\rho_{0(6)} = \frac{+0{\cdot}430}{\sqrt{0{\cdot}476 \times 0{\cdot}601}} = +0{\cdot}805.$$

Wie auch zu erwarten stand, sind die Korrelationskoeffizienten viel größer als im vorigen Falle.

Die Differenzen der einzelnen Elemente der Koeffizienten und ihre mittleren Fehler finden sich in folgender Tabelle. (Diejenigen für New York siehe oben.)

	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$
Differenz $p_0'(k) - p_0'(k-1)$ ...	$-11{\cdot}377$	$-0{\cdot}212$	$-0{\cdot}047$	$-0{\cdot}029$	$-0{\cdot}025$	$-0{\cdot}014$
Mittlerer Fehler ...	$\pm 0{\cdot}089$	$\pm 0{\cdot}031$	$\pm 0{\cdot}020$	$\pm 0{\cdot}015$	$\pm 0{\cdot}012$	$\pm 0{\cdot}010$
Verhältnis: $\frac{\text{Differenz}}{\text{Mittl. Fehler}}$	$\pm 127{\cdot}8$	$\pm 6{\cdot}8$	$\pm 2{\cdot}4$	$\pm 1{\cdot}9$	$\pm 2{\cdot}1$	$\pm 1{\cdot}4$

Chicago.

	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$
Differenz $\sigma_k'^2 - \sigma'^2_{k-1}$ ...	$-10{\cdot}225$	$-0{\cdot}206$	$-0{\cdot}041$	$-0{\cdot}029$	$-0{\cdot}029$	$-0{\cdot}026$
Mittlerer Fehler ...	$\pm 0{\cdot}086$	$\pm 0{\cdot}031$	$\pm 0{\cdot}020$	$\pm 0{\cdot}015$	$\pm 0{\cdot}012$	$\pm 0{\cdot}010$
Verhältnis: $\frac{\text{Differenz}}{\text{Mittl. Fehler}}$	$\pm 118{\cdot}9$	$\pm 6{\cdot}7$	$\pm 2{\cdot}1$	$\pm 1{\cdot}9$	$\pm 2{\cdot}4$	$\pm 2{\cdot}6$

Es macht den Eindruck, als ob in der 6-ten Differenz der Chicagoer Notierungen noch ein gewisses evolutorisches Element enthalten sei. Doch da es sich für uns hier nur um ein Exempel handelt und da jedenfalls die Reste der Komponente nicht groß sein können, so werden wir hier von der Berechnung 7-ter und 8-ter Differenzen abstehen.

Berechnet man jetzt wieder die Koeffizienten $\rho_{j(6)}$ und ihre mittleren Fehler, so kommt man zu folgenden Resultaten.

(1) (2) Mittl. Fehler	(3) Dieselben Koeffizienten $\rho_{j(6)}$ in der Annahme $r_{1/1} = +0{\cdot}75$ und alle $r_{1/1}^{\pm j} = 0$, bei $j \neq 0$	(4) Differenz zwischen Kolonne 3 and Kolonne 1
$\rho_{2(6)} = +0{\cdot}251 \pm 0{\cdot}184$	$+0{\cdot}402$	$+0{\cdot}151$
$\rho_{1(6)} = -0{\cdot}614 \pm 0{\cdot}121$	$-0{\cdot}643$	$-0{\cdot}029$
$\rho_{0(6)} = +0{\cdot}805 \pm 0{\cdot}068$	$+0{\cdot}750$	$-0{\cdot}055$
$\rho_{-1(6)} = -0{\cdot}653 \pm 0{\cdot}112$	$-0{\cdot}643$	$+0{\cdot}010$
$\rho_{-2(6)} = +0{\cdot}266 \pm 0{\cdot}182$	$+0{\cdot}402$	$+0{\cdot}136$

Also auch in diesem Falle kann man annehmen, daß nur die Existenz einer beträchtlichen positiven Korrelation zwischen den Notierungen *desselben Tages* in New York und Chicago bewiesen ist, und daß diese Korrelation etwa durch $+0{\cdot}75 \pm 0{\cdot}07$ gemessen werden könnte. Da aber die Reihe der absoluten Größen $\rho_{j(a)}$ eine gewisse Asymmetrie aufweist, so ist es auch nicht ausgeschlossen, daß neben einem großen positiven $r_{1/1}$ noch kleine positive $r_{1/1}^{(+1)}$ und $r_{1/1}^{(-1)}$ bestehen, wobei jedoch möglicherweise $r_{1/1}^{(+1)} > r_{1/1}^{(-1)}$. Das würde, mit anderen Worten, bedeuten, daß der Einfluß der gestrigen Chicagoer Notierung heute in New York bemerkbarer wäre, als die heutige New-Yorker Notierung—morgen in Chicago. Um aber derartiges positiv an Hand unserer Formeln behaupten zu können, müßte man wohl etwa 10-mal längere Zahlenreihen (ca. 3 Jahre) bearbeitet haben, als es in diesem Beispiel getan wurde.

(19) *Further Evidence of Natural Selection in Man.*

Unter diesem Titel veröffentlichten E. M. Elderton und K. Pearson im x. Band der *Biometrika* (May 1915) eine Monographie, in welcher sie mit Hilfe der Differenzenmethode unter andere sehr beträchtliche negative Korrelationen zwischen Sterblichkeiten in verschiedenen einander nachfolgenden Altersklassen nachgewiesen zu haben glaubten. In einer späteren Arbeit, "On the Variate Difference Method" (*Biometrika*, March 1923) hielten sie es aber für angebracht, der W. Person'schen Kritik einige Zugeständnisse zu machen und dieselben Koeffizienten auf Grund der Rhodes'schen und Sheppard'schen Ausgleichungssysteme zu berichtigen. Die Verfasser kamen dabei zu folgenden Resultaten (S. 308): "We think it safe to say that there really does exist a substantial *negative* correlation between deaths of the same group in the first and second years of life. It is not as great as we found it in the previous paper using hypotheses, which, we admit, ought to have been tested; but it is quite adequate to indicate that natural selection is really at work."

Demgegenüber ist folgendes zu bemerken.

Erstens, wie oben aus §§ 1—3 ersichtlich, führt die Rhodes'sche und Sheppard'sche Ausgleichungsmethode ebenfalls nicht zu den wahren Werten der "restlichen Komponente," und insofern an ihnen keine Korrektionen angebracht werden, können die gefundenen Werte mehr oder weniger falsch sein. So ist es, zum Beispiel, *möglich,* daß die Koeffizienten $R_{x_p y_p}$ auf S. 308 der zitierten Publikation nach solchen Korrektionen sich den durch die Differenzenmethode erbrachten Resultaten beträchtlich nähern würden. Für die Sheppard'sche Methode, in einer gewissen Einstellung, ist das sogar ganz gewiß. Andererseits *können* daselbst die Korrelationskoeffizienten für $x_p x_{p+i}$ (S. 303) und für $x_p y_{p-2}$, $x_p y_{p-1}$ u.s.w. (S. 305) sich sehr wohl als einfach "spurious" erweisen.

Zweitens, wenn man an Hand der Formeln der vorliegenden Arbeit die in "Further Evidence" gefundenen Resultate prüft, so stellt es sich heraus, daß die Differenzen $p'_{0(k)} - p'_{0(k-1)}$ und $\sigma_k'^2 - \sigma'^2_{k-1}$, an ihren mittleren Fehlern gemessen, sogar in der 6-ten Differenz recht klein ausfallen. Das von Pearson und Elderton

bearbeitete Material widerspricht also den Grundhypothesen der Differenzenmethode nicht, und letztere kann hier wohl ruhig angewandt werden. Es fragt sich, ob das Zugeständnis an W. Persons notwendig war.

Aus diesen Feststellungen folgt jedoch keinesfalls, daß alle Details und Schlußfolgerungen von "Further Evidence" jetzt noch ohne Weiteres als vollkommen korrekt angesehen werden müssen.

VARNA, *Januar*, 1926.

Postskriptum.

Das in § 5 des ersten Teiles der vorliegenden Arbeit (*Biometrika*, Vol. XVIII. p. 299 ff.) entwickelte Ausgleichungsverfahren kann auch derart verallgemeinert werden, daß es alle Formeln der Sheppard'schen Methode als Spezialfälle ergibt.

Es sei angenommen, daß die statistische Reihe $u_1, u_2, \ldots u_N$, als eine Summe zweier Komponenten dargestellt werden kann: (*a*) einer "glatten" G, welche in der $2n$-ten endlichen Differenz so kleine Werte aufweist, daß sie für die Praxis bedeutungslos werden, und (*b*) einer "restlichen" Komponente H, welche als eine Reihe empirischer Größen gedacht wird, die eine zufällige Variable mit einem beliebigen, aber konstanten Verteilungsgesetz bei N von einander unabhängigen Versuchen ergibt. Es ist also

$$u_i = G_i + H_i.$$

G braucht hier durchaus nicht immer durch eine Parabel $(2n-1)$-ter Ordnung darstellbar zu sein, denn es sind doch, erfahrungsgemäß, die endlichen Differenzen bestimmter höherer Ordnungen *aller gebräuchlichen Tabellen-Funktionen* gleich Null, oder, genauer ausgedrückt, kleiner als die letzten Dezimalen der Tabellen. Hierauf beruhen ja bekanntlich alle Interpolationsformeln*.

Die Annahme einer derartigen Zusammensetzung der statistischen Reihe ist nur dann gerechtfertigt, wenn ungefähr:

$$\sigma'_{2n} = \sigma'_{2n+1} = \sigma'_{2n+2} = \sigma'_{2n+3} = \ldots \dagger.$$

Der zulässige Schwankungsbereich dieser Größen ist aus den Formeln der Tabellen I und II zu bestimmen ‡.

Setzen wir $H_i - E(H) = \eta_i$, wobei das Symbol E, wie gewöhnlich, "mathematische Erwartung" bedeutet, so ist:

$$u_i = G_i + E(H) + \eta_i \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Will man sich von der "restlichen" Komponente η_i befreien (die konstante Größe $E(H)$ ist praktisch unschädlich), so liegt der Gedanke nahe, u_i eine derartige Funktion beizugeben, daß diese *zusammen mit* η_i möglichst verschwinde. Hierbei kommt uns der Umstand zustatten, daß, für ein jedes $m \geqq 2n$, $\delta^m u_i = \delta^m \eta_i$, wenn das Symbol δ^m, wie üblich, eine endliche *zentrale* Differenz m-ter Ordnung bedeutet, und also:

$$\delta^{2n} u_0 = (-1)^n [C^n_{2n} u_0 - C^{n-1}_{2n} (u_{-1} + u_1) + C^{n-2}_{2n} (u_{-2} + u_2) - \ldots + (-1)^n (u_{-n} + u_n)] \ldots\ldots(2),$$

wobei

$$C^n_m = \frac{m!}{n!\,(m-n)!} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

Wählen wir, *zum Beispiel*, den Ausdruck:

$$u_0' = u_0 + x\,\delta^{2n} u_0 + y\,\delta^{2n+2} u_0 + z\,\delta^{2n+4} u_0 + t\,\delta^{2n+6} u_0 + \ldots \quad \ldots\ldots\ldots\ldots(4),$$

so ist er gleich:

$$u_0' = G_0 + E(H) + \epsilon_0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5),$$

wobei

$$\epsilon_0 = \eta_0 + x\,\delta^{2n} \eta_0 + y\,\delta^{2n+2} \eta_0 + z\,\delta^{2n+4} \eta_0 + t\,\delta^{2n+6} \eta_0 + \ldots \quad \ldots\ldots\ldots\ldots(6).$$

Wie leicht ersichtlich, ist die mathematische Erwartung von ϵ_0 gleich Null. Es gilt also, die Koeffizienten $x, y, z, t, \ldots$ so zu wählen, daß die *Streuung* von ϵ_0, das heißt $E(\epsilon_0^2)$, ein Minimum werde.

* Vergl. Whittaker and Robinson, *The Calculus of Observations*, p. 4.

† Vergl. *Biometrika*, Vol. XVIII. p. 303, Formel (11).

‡ *Ibid.*, pp. 305 und 306.

Nach einigen Umformungen gelangen wir, vermöge der Beziehungen: $E(\eta_i\eta_j)=0$, bei $i\neq j$, und $E\eta_i^2=\mu_2=\text{const.}$, bei beliebigem i, zu folgendem Ausdruck:

$$\begin{aligned}E(\epsilon_0^2)=\mu_2\{1&+x^2C^{2n}_{4n}+y^2C^{2n+2}_{4n+4}+z^2C^{2n+4}_{4n+8}+t^2C^{2n+6}_{4n+12}+\dots\\&+2(-1)^n xC^n_{2n}+2(-1)^{n+1}yC^{n+1}_{2n+2}+2(-1)^{n+2}zC^{n+2}_{2n+4}+2(-1)^{n+3}tC^{n+3}_{2n+6}+\dots\\&-2xyC^{2n+1}_{4n+2}+2xzC^{2n+2}_{4n+4}-2xtC^{2n+3}_{4n+6}+\dots\\&-2yzC^{2n+3}_{4n+6}+2ytC^{2n+4}_{4n+8}-\dots\\&-2ztC^{2n+5}_{4n+10}+\dots\}.\end{aligned}$$

Nach den bekannten Regeln der Theorie der Maxima und Minima ergibt sich hieraus ein folgendes Gleichungssystem zur Bestimmung der vorteilhaftesten Werte der Unbekannten $x, y, z, t, \dots$:

$$\left.\begin{array}{l}+xC^{2n}_{4n}-yC^{2n+1}_{4n+2}+zC^{2n+2}_{4n+4}-tC^{2n+3}_{4n+6}+\dots+(-1)^n\,C^n_{2n}=0\\-xC^{2n+1}_{4n+2}+yC^{2n+2}_{4n+4}-zC^{2n+3}_{4n+6}+tC^{2n+4}_{4n+8}-\dots+(-1)^{n+1}C^{n+1}_{2n+2}=0\\+xC^{2n+2}_{4n+4}-yC^{2n+3}_{4n+6}+zC^{2n+4}_{4n+8}-tC^{2n+5}_{4n+10}+\dots+(-1)^{n+2}C^{n+2}_{2n+4}=0\\-xC^{2n+3}_{4n+6}+yC^{2n+4}_{4n+8}-zC^{2n+5}_{4n+10}+tC^{2n+6}_{4n+12}-\dots+(-1)^{n+3}C^{n+3}_{2n+6}=0\\\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\end{array}\right\}\dots\dots\dots\dots(7).$$

Begnügt man sich mit einer Unbekannten (x) und unterdrückt alle anderen, so erhält man aus der ersten Gleichung:

$$x=-(-1)^n\frac{C^n_{2n}}{C^{2n}_{4n}}\,^*.$$

Behält man 2 Unbekannte, x und y, so erhält man aus den zwei ersten Gleichungen:

$$x=(-1)^{n+1}(2n+1)\frac{C^n_{2n}}{C^{2n}_{4n}};\quad y=(-1)^{n+1}2n\frac{C^n_{2n}}{C^{2n+1}_{4n+2}},\ \text{u.s.w.}$$

Setzt man alle diese Größen $x, y, z, t, \dots$ sukzessive in die Formel (4) ein, so erhält man für u_0', den "ausgeglichenen" Wert von u_0, folgende Ausdrücke:

1te Annäherung:

$$u_0'=u_0+(-1)^{n+1}\frac{C^n_{2n}}{C^{2n}_{4n}}\delta^{2n}u_0;$$

2te Annäherung:

$$u_0'=u_0+(-1)^{n+1}(2n+1)\frac{C^n_{2n}}{C^{2n}_{4n}}\delta^{2n}u_0+(-1)^{n+1}2n\frac{C^n_{2n}}{C^{2n+1}_{4n+2}}\delta^{2n+2}u_0;$$

3tte Annäherung:

$$\begin{aligned}u_0'=u_0&+(-1)^{n+1}(2n+1)(n+1)\frac{C^n_{2n}}{C^{2n}_{4n}}\delta^{2n}u_0+(-1)^{n+1}2n(2n+3)\frac{C^n_{2n}}{C^{2n+1}_{4n+2}}\delta^{2n+2}u_0\\&+(-1)^{n+1}n(2n+3)\frac{C^n_{2n}}{C^{2n+2}_{4n+4}}\delta^{2n+4}u_0;\end{aligned}$$

4te Annäherung:

$$\begin{aligned}u_0'=u_0&+(-1)^{n+1}\frac{(2n+1)(n+1)(2n+3)}{2}\frac{C^n_{2n}}{C^{2n}_{4n}}\delta^{2n}u_0+(-1)^{n+1}2n(2n+3)(n+2)\frac{C^n_{2n}}{C^{2n+1}_{4n+2}}\delta^{2n+2}u_0\\&+(-1)^{n+1}n(2n+3)(2n+5)\frac{C^n_{2n}}{C^{2n+2}_{4n+4}}\delta^{2n+4}u_0+(-1)^{n+1}\frac{n(2n+4)(2n+5)}{3}\frac{C^n_{2n}}{C^{2n+3}_{4n+6}}\delta^{2n+6}u_0;\ \text{u.s.w.}\end{aligned}$$

$$\dots\dots\dots(8).$$

* Vergleiche *Biometrika*, Vol. XVIII. p. 301, Formel (5), wo der Multiplikator $(-1)^k$ durch ein Versehen ausgefallen ist.

Entwickelt man hier, nach (2), die zentralen Differenzen $\delta^{2n}u_0$, $\delta^{2n+2}u_0$, $\delta^{2n+4}u_0$, etc. so erhält man folgendes:

(1) im Falle $n=1$, d.h. bei Annahme, daß schon in der ersten *oder* zweiten endlichen Differenz keine Spuren von G mehr vorhanden sind:

$$\left.\begin{array}{lll}
1^{\text{te}}\text{ Annäherung:} & u_0'=\frac{1}{3}\{u_0+(u_{-1}+u_1)\}, \ldots\ldots & [\frac{1}{3}\mu_2] \\
2^{\text{te}}\text{ Annäherung:} & u_0'=\frac{1}{5}\{u_0+(u_{-1}+u_1)+(u_{-2}+u_2)\}, \ldots\ldots & [\frac{1}{5}\mu_2] \\
3^{\text{tte}}\text{ Annäherung:} & u_0'=\frac{1}{7}\{u_0+(u_{-1}+u_1)+(u_{-2}+u_2)+(u_{-3}+u_3)\}, \ldots & [\frac{1}{7}\mu_2] \\
 & \text{u.s.w.} &
\end{array}\right\} \ldots\ldots (9\,a).$$

(2) im Falle $n=2$, d.h. bei Annahme, daß schon in der dritten *oder* vierten endlichen Differenz keine Spuren von G mehr vorhanden sind:

$$\left.\begin{array}{ll}
1^{\text{te}}\text{ Annäherung:} & \\
u_0'=\frac{1}{35}\{17u_0+12(u_{-1}+u_1)-3(u_{-2}+u_2)\}, \ldots\ldots & [0{\cdot}486\mu_2] \\
2^{\text{te}}\text{ Annäherung:} & \\
u_0'=\frac{1}{21}\{7u_0+6(u_{-1}+u_1)+3(u_{-2}+u_2)-2(u_{-3}+u_3)\}, \ldots\ldots & [\frac{1}{3}\mu_2] \\
3^{\text{tte}}\text{ Annäherung:} & \\
u_0'=\frac{1}{231}\{59u_0+54(u_{-1}+u_1)+39(u_{-2}+u_2)+14(u_{-3}+u_3)-21(u_{-4}+u_4)\}, \ldots & [0{\cdot}255\mu_2] \\
\text{u.s.w.} &
\end{array}\right\} \ldots (9\,b).$$

(3) im Falle $n=3$, d.h. bei Annahme, daß schon in der fünften *oder* sechsten endlichen Differenz keine Spuren von G mehr vorhanden sind:

$$\begin{array}{ll}
1^{\text{te}}\text{ Annäherung:} & \\
u_0'=\frac{1}{231}\{131u_0+75(u_{-1}+u_1)-30(u_{-2}+u_2)+5(u_{-3}+u_3)\}, \ldots\ldots & [0{\cdot}567\mu_2] \\
2^{\text{te}}\text{ Annäherung:} & \\
u_0'=\frac{1}{429}\{179u_0+135(u_{-1}+u_1)+30(u_{-2}+u_2)-55(u_{-3}+u_3)+15(u_{-4}+u_4)\}, \ldots & [0{\cdot}417\mu_2] \\
3^{\text{tte}}\text{ Annäherung:} & \\
u_0'=\frac{1}{429}\{143u_0+120(u_{-1}+u_1)+60(u_{-2}+u_2)-10(u_{-3}+u_3)-45(u_{-4}+u_4) & \\
\qquad +18(u_{-5}+u_5)\}, \ldots & [\frac{1}{3}\mu_2] \\
\text{u.s.w.} &
\end{array} \ldots\ldots (9\,c).$$

Das sind ganz genau die Sheppard'schen Ausgleichungsformeln*, welche sich alle der Reihe nach aus den Gleichungen der Systeme (7) und (8) ableiten lassen.

Wie schon oben angedeutet, wird eine Ausgleichungsformel als desto besser gelten können, je geringer die *Streuung* von ϵ in (5) ausfällt. Es kann nun aus (6) und (8) ohne besondere Mühe nachgewiesen werden, daß die Größe $E(\epsilon_0^2)$ für jedes n und jede Annäherung sich ganz leicht aus den Formeln (9 *a*), (9 *b*), (9 *c*), u.s.w., berechnen läßt: man braucht nur μ_2 mit der Summe der Quadrate der Koeffizienten, welche dort vor einem jedem u_i stehen müssen, zu multiplizieren. Es ist also, zum Beispiel, für die erste Annäherung in (9 *b*):

$$E(\epsilon_0^2)=\mu_2\{(\tfrac{17}{35})^2+(\tfrac{12}{35})^2+(\tfrac{12}{35})^2+(\tfrac{3}{35})^2+(\tfrac{3}{35})^2\}=0{\cdot}486\mu_2.$$

Bei den Formeln (9 *a*), (9 *b*), (9 *c*) sind die einer jeden Annäherung entsprechenden Werte von $E(\epsilon_0)^2$ rechts in eckigen Klammern angeführt, und wir ersehen sofort, daß mit jeder neuen Annäherung diese Größen sich verringern und, folglich, das Resultat sich verbessert. In der *achten* Annäherung, zum Beispiel, würden sich für $E(\epsilon_0^2)$ schon folgende Werte ergeben: in (9 *a*): $0{\cdot}059\mu_2$; in (9 *b*): $0{\cdot}119\mu_2$; in (9 *c*): $0{\cdot}169\mu_2$, u.s.w.

Das ganze Sheppard'sche Ausgleichungsverfahren sollte wie folgt konstruiert werden. Man berechnet zuerst die Größen $\sigma_1'^2$, $\sigma_2'^2$, $\sigma_3'^2$, ... und bestimmt, nach den Regeln der Differenzenmethode, die Ordnungszahl der Differenz, $2n$ oder $2n-1$, für welche man schon die Anwesenheit eines G-Elements nicht mehr anzunehmen braucht. Ist $2n=2$, so wählt man hierauf eine der

* Vergl., zum B., Whittaker and Robinson, *loc. cit.*, pp. 295, 296.

Ausgleichungsformeln in (9 *a*). Ist $2n=4$, so benutzt man die Formeln (9 *b*); ist $2n=6$, so bedient man sich der Formeln (9 *c*), u.s.w. Je höher die Ordnungszahl der Annäherung und je größer die Zahl der in die Formel eintretenden Reihenglieder, desto genauer wird das Resultat und desto näher kommt u_0' zu $G_0+E(H)$; desto größer wird aber auch die Rechnungsarbeit und desto kürzer die ausgeglichene Reihe, da jede neue Annäherung dieselbe um je 2 Endglieder verringert*. Die Bestimmung der die Komponente η kennzeichenden Mittelwerte erfolgt nach den Regeln der Differenzenmethode aus den endlichen Differenzen von der Ordnung $2n$ oder $2n-1$.

Es sei jedoch zu bemerken, daß die hier entwickelte Modifikation des Sheppard'schen Verfahrens noch einer durchgreifenden Nachprüfung an Hand von verschiedenem Tatsachenmaterial und künstlich konstruierten Reihen bedarf, ehe man über seine praktische Anwendbarkeit ein endgiltiges Urteil fällen könnte.

Es ist nicht ausgeschlossen, daß die Wahl einer anderen Grundformel als (4) zu einem schnelleren Schwund der Komponente ϵ_0 führen könnte. In dieser Hinsicht scheint, zum Beispiel, eine genauere Untersuchung der Methoden von Woolhouse, Spencer, Rhodes, u.s.w. noch gewisse Aussichten zu eröffnen.

VARNA, *March*, 1927.

* Wenn die Differenzen von der Ordnung $2n$ aufwärts schon alle berechnet sind, so dürfte der Gebrauch der Formeln (8) geringere Rechenarbeit erfordern, als die Benutzung der ursprünglichen Sheppardschen (9). Zieht man in Betracht, daß jede endliche Differenz durch eine lineare Funktion entsprechender Differenzen unterer Ordnungen ausgedrückt werden kann, so vermögen die Formeln (8) auch mühelos auf eine Form gebracht werden, wo nur Differenzen einer und derselben Ordnung auftreten.

BERICHTIGUNGEN ZUM ERSTEN TEIL (*Biometrika*, Vol. XVIII. Nos. 3 und 4).

Seite 300 3[tte] Zeile von unten: statt

$$Z=\frac{C_{2k}^{k}}{C_{4k}^{2k}}=\frac{[(2k)!]^3}{(4k)!\,k!\,k!}\ \text{lies}: Z=(-1)^k\frac{C_{2k}^{k}}{C_{4k}^{2k}}=(-1)^k\frac{[(2k)!]^3}{(4k)!\,k!\,k!}.$$

Seite 301 6[te] Zeile von oben: statt

$$u_i-\frac{[(2k)!]^3}{(4k)!\,k!\,k!}\delta^{2k}u_i\ \text{lies}\cdot\ u_i-(-1)^k\frac{[(2k)!]^3}{(4k)!\,k!\,k!}\delta^{2k}u_i.$$

NOTES ON CERTAIN EXPANSIONS IN ORTHOGONAL AND SEMI-ORTHOGONAL FUNCTIONS.

THESE Notes are published in *Biometrika*, because that Journal has been accused of not doing justice to Russian investigators. We would repeat for the benefit of any readers especially interested in semi-orthogonal functions, that the most adequate treatment that has come to our notice of the subject is that of J. P. Gram.

I. Note on Chebysheff's Interpolation Formula.

BY L. ISSERLIS, M.A., D.Sc.

§ 1. Between 1854 and 1875 the Russian mathematician P. L. Chebysheff* wrote a series of memoirs dealing with interpolation by the method of least squares. Although many of his memoirs on this and other subjects were published in French journals as well as in the proceedings of Russian Academies and although a French Edition of his collected works has been available since 1907, his work is not as well known as it should be and the ground he has thoroughly covered is from time to time re-traversed by workers unfamiliar with the whole of his investigations. One example of this is furnished by a short paper by Professor Pearson in Vol. XIII. of *Biometrika* entitled "On a general method of determining the successive terms in a skew regression line." In this paper Pearson obtains the values of the first four terms of Chebysheff's series in the special case when both variables are referred to their means, by repeated applications of Chebysheff's orthogonality conditions. He states that the general principle involved has been discussed by Chebysheff and "more adequately" by J. P. Gram, and is clearly under the impression that Chebysheff assumed equal weights and equidistant ordinates. The contrary is the case, although Chebysheff devoted much attention to this particular case, so important in practical applications.

§ 2. As Chebysheff's formula and method are to-day of great practical as well as theoretical importance the present summary of his results may be of interest. There is no novelty in any part of what follows. I have adopted as far as possible Chebysheff's own exposition and notation. Other accounts of his work, in addition to a rather inadequate one by Bauschinger in the *Encyclopédie des Sciences mathématiques*†, are to be found in the fourth (Russian) edition of Markoff's *Calculus of Probabilities* and in V. Khotimsky's *Smoothing of Statistical Series by Least Squares (Chebysheff's Method)*, published (in Russian) by the Soviet Press in Moscow in 1925.

The last-named is a very complete account and contains tables intended to facilitate computation when the argument proceeds by equal increments.

The following is a list of Chebysheff's memoirs summarised below.

(1) *On a Formula in Analysis* (Russian). 20 October (1 November) 1854. (Bull. phys.-mathém. T. XIII. p. 210.) Collected Works, Vol. I. pp. 701—702.

This is a statement (without proof) of the equivalence of Lagrange's interpolation formula and Chebysheff's expansion.

* The spelling *Chebysheff* gives English readers a very fair idea of the pronunciation of the name in Russian. The simpler form *Čebyšev* has its advantages and is applicable to many similar transliterations of Slavonic names. The combinations "tsch" and "shtch" for the initial consonant occasionally met with are not only ugly but actually misrepresent the sound.

† T. I. IV. I. Interpolation.

(2) *On continued Fractions* (Russian). Trans. Imp. Acad. of Science, Vol. III. pp. 636—664, St Petersburg, 1855, read 22 January 1855. Collected Works, Vol. I. pp. 203—230.

Sur les Fractions continues. Journal de mathématiques pures et appliquées, II Série, T. III. 1858, pp. 289—323.

This is primarily an article on the properties of continued fractions illustrated by the theorem of (1) and incidentally contains a proof of the theorem.

(3) *Sur une nouvelle Série.* (Bull. phys.-mathém., T. XVII. pp. 257—261), read 8 October 1858. Collected Works, Vol. I. pp. 381—384.

This is a statement of the simplified form of Chebysheff's expansion for equidistant intervals and equal weights. If there are n ordinates and the distance between consecutive ordinates equals h then if $h=\frac{1}{n}$ the series becomes an expansion in Legendre Polynomials when n tends to infinity, and if $h=\frac{1}{n^2}$ we get Maclaurin's Series when n tends to infinity.

(4) *On Interpolation by the Method of Least Squares.* Memoirs of the Imp. Acad. of Science, VIIth Series, Vol. I. 1859, No. 15, pp. 1—24, read 29 April 1859. Collected Works, Vol. I. pp. 473—498.

The results of the previous papers are here put in a form suitable for rapid numerical computation in the general case and formulae for the sums of the squares of the residuals are obtained.

(5) *On Interpolation* (Russian). Appendix to Vol. IV. of Proc. Imp. Acad. Sci. No. 5, 1864. Collected Works, Vol. I. pp. 541—560.

The special case of equal intervals and equal weights is dealt with in detail and formulae for systematic computation and the evaluation of the sum of the residuals are obtained.

(6) *On the Interpolation of Quantities for Equidistant Values of the Argument* (Russian). Appendix to Vol. XXV. of the Proceedings of the Imp. Acad. of Sci. No. 5, 1875. Collected Works, Vol. II. pp. 219—242.

In this memoir the formulae for the equidistant case dealt with in memoir (5) are again taken up and closed expressions for the coefficients as differences of a simple factorial expression are obtained.

§ 3. Let $y_1, y_2, \ldots, y_n$ be the values of y corresponding, with weights $p_1, p_2, \ldots, p_n$, to the values $x_1, x_2, \ldots, x_n$ of x.

Let
$$P_\lambda(x)=k_0\psi_0(x)+k_1\psi_1(x)+\ldots+k_\lambda\psi_\lambda(x),$$
where $\psi_\mu(x)$ is a polynomial of degree μ and the k's are numerical coefficients, be the parabola of order λ of closest fit as determined by the method of least squares, for $\lambda=0, 1, 2, \ldots, n-1$.

Then if the summation sign Σ refers to summation for i from $i=1$ to $i=n$, we must have
$$\Sigma p_i[y_i-P_\lambda(x_i)]^2=\text{minimum, for } \lambda=0, 1, 2, \ldots, n-2,$$
and equals zero, for $\lambda=n-1$.

In the latter case $y_i=P_{n-1}(x_i)$. P_λ can be written in the form
$$P_\lambda=a_0+a_1x+a_2x^2+\ldots+a_\lambda x^\lambda.$$

Substituting in the minimum condition and differentiating with respect to a_μ we get
$$\Sigma p_i[y_i-P_\lambda(x_i)]\,x_i^\mu=0, \text{ for } \mu=0, 1, 2, \ldots, \lambda, \text{ and } \lambda=0, 1, 2, \ldots, n-1,$$
or since $y_i=P_{n-1}(x_i)$,
$$\Sigma\,[P_{n-1}(x_i)-P_\lambda(x_i)]\,p_ix_i^\mu=0.$$

Changing λ into $\lambda+1$ and subtracting we find
$$\Sigma\,[P_{\lambda+1}(x_i)-P_\lambda(x_i)]\,p_ix_i^\mu=0,$$
or
$$\Sigma\psi_{\lambda+1}(x_i)\,p_ix_i^\mu=0 \quad\ldots\ldots\ldots\ldots(1).$$

It follows that if $\Phi_\mu(x)$ be any polynomial of degree μ in x then if $\mu < \lambda + 1$

$$\Sigma p_i \psi_{\lambda+1}(x_i)\Phi_\mu(x_i) \equiv 0 \quad \text{.......................(2)}.$$

Assuming that the ψ's are known, the k's are determined as follows:

$$\Sigma p_i [y_i - P_\lambda(x_i)]^2 = \text{minimum},$$

or

$$\Sigma p_i [y_i - k_0\psi_0(x_i) - k_1\psi_1(x_i) - \dots - k_\mu \psi_\mu(x_i) - \dots - k_\lambda \psi_\lambda(x_i)]^2 = \text{minimum}.$$

Differentiating with respect to k_μ,

$$\Sigma p_i \psi_\mu(x_i)[y_i - k_0\psi_0(x_i) - \dots - k_\mu\psi_\mu(x_i) - \dots - k_\lambda\psi_\lambda(x_i)] = 0.$$

But

$$\Sigma p_i \psi_\mu(x_i)\psi_\nu(x_i) \equiv 0 \text{ if } \mu \neq \nu,$$

therefore

$$k_\mu = \frac{\Sigma p_i y_i \psi_\mu(x_i)}{\Sigma p_i \psi_\mu^2(x_i)} \quad \text{.......................(3)}.$$

§ 4. To determine the ψ's we note that

$$\frac{\psi_{\lambda+1}(x_i)}{x - x_i} = \psi_{\lambda+1}(x_i)\left[\frac{1}{x} + \frac{x_i}{x^2} + \dots + \frac{x_i^{\mu-1}}{x^\mu} + \dots\right],$$

and remembering that

$$\Sigma \psi_\lambda(x_i) p_i x_i^\mu = 0, \ \mu < \nu, \ \lambda$$

we have

$$\Sigma p_i \frac{\psi_{\lambda+1}(x_i)}{x - x_i} = \Sigma p_i \psi_{\lambda+1}(x_i)\left[\frac{x_i^{\lambda+1}}{x^{\lambda+2}} + \frac{x_i^{\lambda+2}}{x^{\lambda+3}} + \dots\right].$$

But

$$\psi_{\lambda+1}(x)\,\Sigma \frac{p_i}{x - x_i} = \Sigma p_i \frac{[\psi_{\lambda+1}(x) - \psi_{\lambda+1}(x_i)]}{x - x_i} + \Sigma p_i \frac{\psi_{\lambda+1}(x_i)}{x - x_i}$$

$$= \Sigma p_i \frac{\psi_{\lambda+1}(x) - \psi_{\lambda+1}(x_i)}{x - x_i} + \Sigma p_i \psi_{\lambda+1}(x_i)\left[\frac{x_i^{\lambda+1}}{x^{\lambda+2}} + \frac{x_i^{\lambda+2}}{x^{\lambda+3}} + \dots\right].$$

The first term is clearly integral and of degree λ for the numerator of each term under the summation sign vanishes when $x = x_i$.

Hence

$$\Sigma \frac{p_i}{x - x_i} = \frac{\Sigma p_i \left\{\dfrac{\psi_{\lambda+1}(x) - \psi_{\lambda+1}(x_i)}{x - x_i}\right\}}{\psi_{\lambda+1}(x)},$$

plus terms in $\frac{1}{x}$ of higher order than $\frac{1}{[\psi_{\lambda+1}(x)]^2}$.

Hence

$$\frac{\Sigma p_i \left\{\dfrac{\psi_{\lambda+1}(x) - \psi_{\lambda+1}(x_i)}{x - x_i}\right\}}{\psi_{\lambda+1}(x)}$$

is the $(\lambda+1)$th convergent to $\Sigma \frac{p_i}{x - x_i}$ when expressed in the form of a continued fraction

$$\frac{a_1}{(x - b_1) -}\ \frac{a_2}{(x - b_2) -}\ \cdots \frac{a_{\lambda+1}}{(x - b_{\lambda+1}) -}\ \cdots.$$

§ 5. To determine the a's and b's we note first that

$$\frac{a_1}{x - b_1} = \frac{\Sigma p_i \dfrac{\psi_1(x) - \psi_1(x_i)}{x - x_i}}{\psi_1(x)},$$

so that

$$a_1 = \Sigma p_i \left\{\frac{\psi_1(x) - \psi_1(x_i)}{x - x_i}\right\},$$

and

$$x - b_1 = \psi_1(x).$$

Hence

$$\Sigma p_i (x_i - b_1) = \Sigma p_i \psi_1(x_i) = 0$$

by (1), and therefore

$$b_1 = \frac{\Sigma p_i x_i}{\Sigma p_i} \quad \text{.......................(4)},$$

while

$$a_1 = \Sigma p_i \frac{[x - b_1 - (x_i - b_1)]}{x - x_i} = \Sigma p_i \quad \text{.......................(5)}.$$

§ 6. In general, by known properties of continued fractions

$$\psi_{\lambda+1}(x)=(x-b_{\lambda+1})\psi_\lambda(x)-a_{\lambda+1}\psi_{\lambda-1}(x) \quad\text{.........................(6).}$$

Multiply (6) by $p_i x_i^{\lambda-1}$ and sum, using (1). We get

$$0=\Sigma p_i x_i^\lambda \psi_\lambda(x_i)-a_{\lambda+1}\Sigma p_i\psi_{\lambda-1}(x_i)x_i^{\lambda-1},$$

or

$$a_{\lambda+1}=\frac{(\lambda,\lambda)}{(\lambda-1,\lambda-1)} \quad\text{...(7),}$$

introducing Chebysheff's notation, viz.:

$$(a,\beta)=\Sigma p_i\psi_a(x_i)x_i^\beta \quad\text{..(8).}$$

Multiply (6) by $p_i x_i^\lambda$ and sum; this gives

$$0=(\lambda,\lambda+1)-b_{\lambda+1}(\lambda,\lambda)-a_{\lambda+1}(\lambda-1,\lambda),$$

therefore

$$b_{\lambda+1}=\frac{(\lambda,\lambda+1)}{(\lambda,\lambda)}-\frac{(\lambda-1,\lambda)}{(\lambda-1,\lambda-1)} \quad\text{...................................(9).}$$

§ 7. When any number of terms of Chebysheff's series have been obtained, the accuracy of the approximation reached at this stage is measured by the sum of the weighted squares of the differences between the actual values of the terms y_i and the calculated values $P_\lambda(x_i)$. Denoting this by R_λ, we have

$$\begin{aligned}R_\lambda&=\Sigma p_i[y_i-P_\lambda(x_i)]^2\\&=\Sigma p_i y_i^2-2\Sigma p_i[k_0\psi_0(x_i)+\ldots+k_\lambda\psi_\lambda(x_i)+\ldots+k_{n-1}\psi_{n-1}(x_i)]\times[k_0\psi_0(x_i)+\ldots+k_\lambda\psi_\lambda(x_i)]\\&\qquad+\Sigma p_i[k_0\psi_0(x_i)+k_1\psi_1(x_i)+\ldots+k_\lambda\psi_\lambda(x_i)]^2\\&=\Sigma p_i y_i^2-\Sigma p_i[k_0\psi_0(x_i)+\ldots+k_\lambda\psi_\lambda(x_i)]^2.\end{aligned}$$

Similarly,

$$\begin{aligned}R_{\lambda-1}&=\Sigma p_i y_i^2-\Sigma p_i[k_0\psi_0(x_i)+\ldots+k_{\lambda-1}\psi_{\lambda-1}(x_i)]^2\\&=R_{\lambda-1}-\Sigma p_i k_\lambda^2\psi_\lambda^2(x_i),\end{aligned}$$

therefore

$$R_\lambda=R_{\lambda-1}-\Sigma p_i k_\lambda^2\psi_\lambda^2(x_i).$$

But

$$\Sigma p_i\psi_\lambda^2(x_i)=\Sigma p_i\psi_\lambda(x_i)[x_i^\lambda+\text{terms of lower order}]^*.$$

Hence by (1)

$$R_\lambda=R_{\lambda-1}-k_\lambda^2(\lambda,\lambda) \quad\text{.......................................(10).}$$

§ 8. Multiplying equation (6) by $p_i x_i^\mu$ and summing we get the general recurrence formula for (a,β),

$$(\lambda+1,\mu)=(\lambda,\mu+1)-b_{\lambda+1}(\lambda,\mu)-a_{\lambda+1}(\lambda-1,\mu) \quad\text{....................(11).}$$

If we denote the data by

$$x_1,\ x_2,\ \ldots,\ x_n,$$
$$y_1,\ y_2,\ \ldots,\ y_n,$$
$$p_1,\ p_2,\ \ldots,\ p_n,$$

Chebysheff's series by

$$y=k_0\psi_0(x)+k_1\psi_1(x)+\ldots+k_\lambda\psi_\lambda(x),$$

and the successive terms of this series by T_0, T_1, T_2, ..., T_λ, we may arrange the formulae for the purposes of computation as follows:

T_0. $(0,0)=\Sigma p$; $k_0=\Sigma py/(0,0)$,

$\psi_0(x)=1$, $T_0=k_0\psi_0(x)$, $R_0=\Sigma py^2-(0,0)\,k_0^2$.

T_1. $(0,1)=\Sigma px$; $(0,2)=\Sigma px^2$; $a_1=(0,0)$; $b_1=\dfrac{(0,1)}{(0,0)}$,

$(1,1)=(0,2)-b_1(0,1)$; $k_1=[\Sigma pxy-(0,1)\,k_0]/(1,1)$,

$\psi_1(x)=x-b_1$; $T_1=k_1\psi_1(x)$; $R_1=R_0-k_1^2(1,1)$.

* For $\psi_1(x)=x-b_1$, and $\psi_\lambda(x)=(x-b_\lambda)\psi_{\lambda-1}(x)-a_\lambda\psi_{\lambda-2}(x)$; it is therefore clear that the coefficient of x^λ in $\psi_\lambda(x)$ is 1.

T_2. $(0, 3) = \Sigma px^3$; $(0, 4) = \Sigma px^4$; $(1, 2) = (0, 3) - b_1 (0, 2)$,

$(1, 3) = (0, 4) - b_1 (0, 3)$; $a_2 = (1, 1)/(0, 0)$; $b_2 = \dfrac{(1, 2)}{(1, 1)} - \dfrac{(0, 1)}{(0, 0)}$,

$(2, 2) = (1, 3) - b_2 (1, 2) - a_2 (0, 2)$,

$k_2 = [\Sigma pyx^2 - k_0 (0, 2) - k_1 (1, 2)]/(2, 2)$,

$\psi_2 (x) = (x - b_2) \psi_1 (x) - a_2$; $T_2 = k_2 \psi_2 (x)$; $R_2 = R_1 - k_2^2 (2, 2)$.

T_λ. When $T_0, T_1, \ldots T_{\lambda-1}$ have been computed,

$(0, 2\lambda - 1) = \Sigma px^{2\lambda-1}$; $(0, 2\lambda) = \Sigma px^{2\lambda}$,

$(1, 2\lambda - 2) = (0, 2\lambda - 1) - b_1 (0, 2\lambda - 2)$,

$(1, 2\lambda - 1) = (0, 2\lambda) - b_1 (0, 2\lambda - 1)$,

$(2, 2\lambda - 3) = (1, 2\lambda - 2) - b_2 (1, 2\lambda - 3) - a_2 (0, 2\lambda - 3)$,

$(2, 2\lambda - 2) = (1, 2\lambda - 1) - b_2 (1, 2\lambda - 2) - a_2 (0, 2\lambda - 2)$,

....

$(\lambda - 1, \lambda) = (\lambda - 2, \lambda + 1) - b_{\lambda-1} (\lambda - 2, \lambda) - a_{\lambda-1} (\lambda - 3, \lambda)$,

$(\lambda - 1, \lambda + 1) = (\lambda - 2, \lambda + 2) - b_{\lambda-1} (\lambda - 2, \lambda + 1) - a_{\lambda-1} (\lambda - 3, \lambda + 1)$,

....

$k_\lambda = [\Sigma pyx^\lambda - (0, \lambda) k_0 - (1, \lambda) k_1 - \ldots - (\lambda - 1, \lambda) k_{\lambda-1}]/(\lambda, \lambda)$,

$\psi_\lambda (x) = (x - b_\lambda) \psi_{\lambda-1} (x) - a_\lambda \psi_{\lambda-2} (x)$,

$T_\lambda = k_\lambda \psi_\lambda (x)$; $R_\lambda = R_{\lambda-1} - (\lambda, \lambda) k_\lambda^2$,

$a_\lambda = (\lambda - 1, \lambda - 1)/(\lambda - 2, \lambda - 2)$,

$b_\lambda = (\lambda - 1, \lambda)/(\lambda - 1, \lambda - 1) - (\lambda - 2, \lambda - 1)/(\lambda - 2, \lambda - 2)$.

In the summations p, x, y have suffixes i and the summation is from $i = 1$ to $i = n$.

The formula becomes the regression equation of y on x if p_i is the population of the x_i array and y_i is the mean of that array.

Numerical Example.

Data			T_0		T_1			T_2			R_2 by direct calculation as check.		
p	x	y	py	py^2	px	px^2	pxy	px^3	px^4	px^2y	$2418 Y_2$	$2418y$	$\{2418 (Y_2 - y)\}^2 p$
1	0	0	0	0	0	0	0	0	0	0	132	0	17,424
2	1	1	2	2	2	2	2	2	2	2	2478	2418	7,200
2	2	2	4	8	4	8	8	16	32	16	4346	4836	480,200
1	3	2	2	4	3	9	6	27	81	18	5736	4836	810,000
1	5	3	3	9	5	25	15	125	625	75	7082	7254	29,584
7	11	8	11	23	14	44	31	170	740	111			1,344,408
(0, 0)					(0, 1)	(0, 2)		(0, 3)	(0, 4)		$R_2 = \dfrac{1,344,408}{2418} = \dfrac{278}{1209}$		

T_0. $\overset{k_0}{\cancel{R_0}} = \Sigma py/(0, 0) = 11/7$; $\psi_0 (x) = 1$; $T_0 = k_0 \psi_0 (x) = 11/7$,

$R_0 = \Sigma py^2 - (\Sigma py)^2/(0, 0) = 23 - 121/7 = 5\frac{5}{7} = 5{\cdot}714.$

T_1. $b_1=(0,1)/(0,0)=2$; $(1,1)=(0,2)-b_1(0,1)=44-28=16$,

$$k_1=[\Sigma pxy-k_0(0,1)]/(1,1)=(31-22)/16=9/16\,;\quad \psi_1(x)=x-b_1=x-2,$$

$$T_1=k_1\psi_1(x)=(9/16)(x-2)\,;\qquad T_0+T_1=\frac{25}{56}+\frac{9x}{16},$$

$$R_1=R_0-k_1^2(1,1)=\frac{40}{7}-\frac{81}{6}=\frac{73}{112}=0{\cdot}552.$$

T_2. $(1,2)=(0,3)-b_1(0,2)=170-88=82$,

$$(1,3)=(0,4)-b_1(0,3)=740-2(170)=400,$$

$$a_2=(1,1)/(0,0)=16/7\,;\quad b_2=\frac{(1,2)}{(1,1)}-\frac{(0,1)}{(0,0)}=\frac{82}{16}-\frac{14}{7}=3\tfrac{1}{8},$$

$$\begin{aligned}(2,2)&=(1,3)-b_2(1,2)-a_2(0,2)\\&=400-3\tfrac{1}{8}(82)-(16/7)\,44=43\tfrac{5}{28}.\end{aligned}$$

$$\begin{aligned}k_2&=[\Sigma pyx^2-k_0(0,2)-k_1(1,2)]/(2,2)\\&=[111-(11/7)\,44-(9/16)\,82]/43\tfrac{5}{28}\\&=-239/2418,\end{aligned}$$

$$\begin{aligned}\psi_2(x)&=(x-b_2)\,\psi_1(x)-a_2\psi_0(x)\\&=(x-3\tfrac{1}{8})\,(x-2)-16/7.\end{aligned}$$

$$T_2=k_2\psi_2(x)=-(239/2418)\,(x^2-41x/8+111/28),$$

$$T_0+T_1+T_2=(-239x^2+2585x+132)/2418.$$

$$R_2=R_1-k_2^2\,(2,2)=73/112-(4\tfrac{15}{56})^2/43\tfrac{5}{28}=278/1209=0{\cdot}23,$$

which agrees with R_2 obtained by direct calculation.

§ 9. The particular case in which the ordinates are equidistant and the weights are equal is reduced by Chebysheff to the following simple form:

$$\begin{aligned}y&=\frac{1}{n}\,\Sigma y_i\phi_0(t)\\&+\frac{3}{n(n^2-1^2)}\,\Sigma\,\frac{i}{1}\cdot\frac{n-i}{1}\,\Delta y_i\phi_1(t)\\&+\frac{}{n(n^2-1^2)(n^2-2^2)}\,\Sigma\,\frac{i(i+1)(n-i)(n-i-1)}{1\,.\,2\,.\,1\,.\,2}\,\Delta^2 y_i\phi_2(t)\\&+\frac{7}{n(n^2-1^2)(n^2-2^2)(n^2-3^2)}\,\Sigma\,\frac{i(i+1)(i+2)}{1\,.\,2\,.\,3}\,\frac{(n-i)(n-i-1)(n-i-2)}{1\,.\,2\,.\,3}\times\Delta^3 y_i\phi_3(t)\\&+\ldots\ldots,\end{aligned}$$

where

$$\begin{aligned}t&=\frac{2x-x_1-x_n}{x_2-x_1}\\&=2x-n-1\ \text{ if }\ x_i=i,\end{aligned}$$

$$2^{2\lambda}\phi_\lambda(t)=\Delta^\lambda\,(t+n-1)(t+n-3)\ldots(t+n-2\lambda+1)(t-n+1)(t-n+2)\ldots(t-n+2\lambda-1)$$

and

$$\phi_\lambda(t)=(2\lambda-1)\,t\phi_{\lambda-1}(t)-(\lambda-1)^2[n^2-(\lambda-1)^2]\,\phi_{\lambda-2}(t).$$

The first six ϕ's have the following values*:

$$\begin{aligned}&\phi_0(t)=1,\\&\phi_1(t)=t,\\&\phi_2(t)=3t^2-(n^2-1),\\&\phi_3(t)=15t^3-3(3n^2-7)\,t,\\&\phi_4(t)=105t^4-30(3n^2-13)\,t^2+9(n^2-1)(n^2-9),\\&\phi_5(t)=945t^5-1050(n^2-7)\,t^3+15(15n^4-230n^2+407)\,t,\\&\phi_6(t)=10{,}395t^6-4725(3n^2-31)\,t^4+945(5n^4-110n^2+329)\,t^2-225(n^2-1)(n^2-9)(n^2-25).\end{aligned}$$

* The first five are given by Chebysheff and reproduced by Markoff, Khotimsky and the *Encyc. des Sci. Math.* loc. cit.

The following numerical example illustrates the application of the formulae in this case.

Numerical Example of Equidistant Ordinates and Equal Weights.

(1) x_i	(2) y_i	(3) Δy_i	(4) $i(n-i)$	(5) (3) × (4)	(6) $\Delta^2 y_i$	(7) $i(n-i) \times (i+1)(n-i-1)$	(8) (6) × (7)	(9) $\Delta^3 y_i$	(10) $i(n-i) \times (i+1)(n-i-1) \times (i+2)(n-i-2)$	(11) (9) × (10)	(12) y_i^2
1	1	—	—	—	—	—	—	—	—	—	1
2	8	7	4	28	—	—	—	—	—	—	64
3	27	19	6	114	12	24	288	—	—	—	729
4	64	37	6	222	18	36	648	6	144	864	4096
5	125	61	4	244	24	24	576	6	144	864	15625
	225	—	—	608	—	—	1512	—	—	1728	20515

$$n=5\,;\quad z=\tfrac{1}{2}t=\frac{x-\frac{1}{2}(x_1+x_5)}{x_2-x_1}=x-3,$$

$$y=T_0+T_1+T_2+T_3+\ldots,$$

$$\phi_0(2z)=1,\quad \phi_1(2z)=2z,\quad \phi_2(2z)=12z^2-24,$$

$$\phi_3(2z)=120z^3-6z(68)=24(5z^3-17z),$$

$$T_0=\frac{225}{5}=45,\qquad R_0=20{,}515-\frac{(225)^2}{5}=20{,}515-10{,}125=10{,}390,$$

$$T_1=\frac{3\,.\,608\,.\,2z}{5\,.\,24}=30{\cdot}4x-91{\cdot}2,\qquad R_1=10{,}390-\frac{(15{\cdot}2)^2\,.\,5\,(24)}{3}=10{,}390-9241{\cdot}6=1148{\cdot}4,$$

$$T_2=\frac{5\,.\,1512\,(12z^2-24)}{5\,.\,24\,.\,21\,.\,4}=9\,(z^2-2)=9\,(x^2-6x+7),\qquad R_2=1148{\cdot}4-(0{\cdot}75)^2\,\frac{4\,.\,5\,.\,24\,.\,21}{5}$$

$$=1148{\cdot}4-1134=14{\cdot}4,$$

$$T_3=\frac{7\,.\,1728\,(120z^3-408z)}{5\,.\,24\,.\,21\,.\,16\,.\,36}=z^3-3{\cdot}4z=x^3-9x^2+23{\cdot}6x-16{\cdot}8,\qquad R_3=14{\cdot}4-\frac{2^2\,.\,3^2\,.\,5\,.\,24\,.\,21\,.\,1}{3^2\,.\,5^2\,.\,2^6\,.\,7}$$

$$=14{\cdot}4-14{\cdot}4=0,$$

$$y=45+30{\cdot}4\,(x-3)+9\,(x^2-6x+7)+x^3-9x^2+23{\cdot}6x-16{\cdot}8=x^3.$$

Khotimsky's book previously referred to contains several numerical examples as well as tables to facilitate computation for values of n up to 50.

II. Note on Orthogonalising Series of Functions and Interpolation.

BY PROFESSOR V. ROMANOVSKY (The University Tashkend).

I. *General Method of Interpolation with the aid of a given System of Functions.*

We shall consider an independent variable x and its function y. Let $y_1, y_2, \ldots, y_n$ be the values of y observed for the values $x_1, x_2, \ldots, x_n$ of x, and let $p_1, p_2, \ldots, p_n$ be the weights of the observed values of y. The system of values

$$x_i,\ y_i,\ p_i\ (i=\overline{1,n})\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

($i=\overline{1,n}$ is equivalent to $i=1, 2, \ldots, n$) we shall name the *base of interpolation*.

Now let
$$u_0(x),\ u_1(x),\ u_2(x),\ \ldots,\ u_s(x) \quad \ldots\ldots(2)$$
be a system of some functions well defined and one valued for the values of x considered. We shall treat the following problem of interpolation by the method of least squares with the aid of the system (2).

It is required to find an expression
$$y=\sum_{h=0}^{s} a_h u_h(x) \quad \ldots\ldots(3)$$
such that the sum
$$S=\sum_{i=1}^{n} p_i\left[y_i-\sum_{h=0}^{s} a_h u_h(x_i)\right]^2 \quad \ldots\ldots(4)$$
shall be a minimum.

The values of the coefficients a_h giving the solution of this problem are, as it is well known, to be found from the normal equations
$$-\frac{1}{2}\frac{\partial S}{\partial a_k}=(yu_k)-\sum_{h=0}^{s} a_h(u_h u_k)=0 \qquad (k=\overline{0,s}) \quad \ldots\ldots(5),$$
where we use the notations
$$(yu_k)=\sum_{i=1}^{n} p_i y_i u_k(x_i), \quad (u_h u_k)=\sum_{i=1}^{n} p_i u_h(x_i)\, u_k(x_i) \quad \ldots\ldots(6).$$

The coefficients a_h being found from (5) the degree of approximation of the expression (3) to the interpolated function y can be judged from the quantity
$$\delta_s^2=\operatorname{Min} S \quad \ldots\ldots(7),$$
which we obtain by introducing in (4) the values of a_h from (5).

Now the great inconvenience of the method described consists in the necessity of remaking all calculations if the approximation gained is insufficient and if we desire, in order to receive a greater approximation, to add to the system (2) some additional interpolating functions $u_{s+1}(x)$, $u_{s+2}(x)$, ... We shall have then a new normal system for the solution of which the solution of (5) is of no use, and a new minimum of the sum analogous to the sum S, the evaluation of which is quite independent of the evaluation of δ_s^2.

But this inconvenience can be avoided by the aid of the following device which can be named *orthogonalisation of the system* (2) *on the base* (x, p) as we shall denote the system of numbers x_i, p_i $(i=\overline{1,n})$.

We shall say that the system of functions (2) is orthogonalised on the base (x, p) if we can find a new system of functions
$$\phi_0(x), \quad \phi_1(x), \quad \phi_2(x), \ \ldots,\ \phi_s(x) \quad \ldots\ldots(8)$$
possessing the two properties:

1°—we have
$$\phi_h(x)=\sum_{g=0}^{s} a_{gh} u_g(x) \qquad (h=\overline{0,s}) \quad \ldots\ldots(9),$$
a_{gh} being some constant coefficients;

2°—we have
$$(\phi_h\phi_k)=\sum_{i=1}^{n} p_i\phi_h(x_i)\,\phi_k(x_i) \qquad \begin{cases}=0 \text{ if } h\neq k\\ \neq 0 \text{ if } h=k\end{cases} \quad \ldots\ldots(10),$$
i.e. the functions $\phi_h(x)$ represent orthogonal functions (on the set of values of x considered).

Now it is very easy to find the functions (8) with the required properties.

We shall put, to begin with,
$$\phi_0(x)=u_0(x).$$
Then we write $\phi_1(x)$ in the form
$$\phi_1(x)=a\phi_0(x)+bu_1(x),$$
and find a and b from the condition that $\phi_1(x)$ must be orthogonal to $\phi_0(x)$, that is from the relation
$$(\phi_0\phi_1)=a(\phi_0\phi_0)+b(u_1\phi_0)=0.$$

Hence, putting $b=1$, $a=-(u_1\phi_0)/(\phi_0\phi_0)$ and

$$\phi_1(x)=-\frac{(u_1\phi_0)}{(\phi_0\phi_0)}\phi_0(x)+u_1(x).$$

Similarly, writing $$\phi_2(x)=a'\phi_0(x)+b'\phi_1(x)+c'u_2(x)$$

we find $$\phi_2(x)=-\frac{(u_2\phi_0)}{(\phi_0\phi_0)}\phi_0(x)-\frac{(u_2\phi_1)}{(\phi_1\phi_1)}\phi_1(x)+u_2(x)$$

from the conditions of orthogonality of $\phi_2(x)$ with $\phi_0(x)$ and $\phi_1(x)$.

Continuing in this manner we find without difficulty that the problem of the orthogonalisation of the system (2) is solved by the system

$$\left.\begin{array}{c}\phi_0(x)=u_0(x),\\ \phi_h(x)=-\sum_{k=0}^{h-1}\frac{(u_h\phi_k)}{(\phi_k\phi_k)}\phi_k(x)+u_h(x)\\ (h=\overline{1,s})\end{array}\right\}\quad\text{.....(11)},$$

if none of the sums $(\phi_h\phi_h)$ $(h=\overline{0,s})$ vanish for the given values of x and p. Evidently this latter condition is satisfied if none of the functions $\phi_h(x)$ vanish for all values $x_i\,(i=\overline{1,n})$. We shall suppose that the given functions $u_h(x)$ are of such a nature that $\phi_h(x)$ will not vanish for all values of x mentioned. For example, $u_h=x^h\,(h=\overline{0,s})$ when $n>s$ are such functions.

Let us note one result of the orthogonality of $\phi_h(x)$ with $\phi_0(x)$, $\phi_1(x)$, ..., $\phi_{h-1}(x)$. We have

$$(\phi_h\phi_h)=-\sum_{k=0}^{h-1}\frac{(u_h\phi_k)}{(\phi_k\phi_k)}(\phi_h\phi_k)+(u_h\phi_h).$$

But $(\phi_h\phi_k)=0$ for $k=\overline{0,h-1}$, therefore

$$(\phi_h\phi_h)=(u_h\phi_h)\qquad(h=0,s)\quad\text{.....(12)}.$$

These relations simplify very much the evaluation of the sums $(\phi_h\phi_h)$.

We shall consider now the application of the functions $\phi_h(x)$ in our problem of interpolation.

It is clear that the relation (3) $$y=\sum_{h=0}^{s}a_hu_h(x)$$

can be transformed identically into $$y=\sum_{h=0}^{s}A_h\phi_h(x)\quad\text{.....(13)}.$$

Therefore the minimum of the sum S is equal to the minimum of the sum

$$S_1=\sum_{i=1}^{n}p_i\left[y_i-\sum_{h=0}^{s}A_h\phi_h(x_i)\right]^2\quad\text{.....(14)},$$

and is attained at the same time as the minimum of S_1. Thus our problem is reduced, by the well-known rules, to that of obtaining A_h from the equations

$$\frac{1}{2}\frac{\partial S_1}{\partial A_h}=-\sum_{i=1}^{n}p_i\phi_h(x_i)\left[y_i-\sum_{k=0}^{s}A_k\phi_k(x_i)\right]=0,$$

or $$(y\phi_h)-\sum_{k=0}^{s}A_k(\phi_h\phi_k)=0\qquad(h=\overline{0,s}).$$

But $(\phi_h\phi_k)=0$ for $k\neq h$ and $\neq 0$ for $k=h$ and we find

$$A_h=\frac{(y\phi_h)}{(\phi_h\phi_h)}\qquad(h=\overline{0,s})\quad\text{.....(15)}.$$

In this manner we reach the expression

$$y=\sum_{h=0}^{s}\frac{(y\phi_h)}{(\phi_h\phi_h)}\phi_h(x)\quad\text{.....(16)}$$

for the interpolation formula reducing S_1 to a minimum. It is easy to reduce this to the form (3).

Let us find further the minimum δ_s^2 of the sum S_1. From

$$\delta_s^2=\sum_{i=1}^{n}p_i\left[y_i-\sum_{h=0}^{s}\frac{(y\phi_h)}{(\phi_h\phi_h)}\phi_h(x_i)\right]^2$$

we find without difficulty $$\delta_s^2 = (yy) - \sum_{h=0}^{s} \frac{(y\phi_h)^2}{(\phi_h\phi_h)} \quad \text{.........(17)},$$

or $$\delta_s^2 = (yy) - \sum_{h=0}^{s} A_h (y\phi_h) \quad \text{.........(17 \textit{bis})},$$

which is more appropriate for numerical calculations.

It is clear that $$\delta_s^2 = \delta^2_{s-1} - A_s (y\phi_s) \quad \text{.........(18)}.$$

The relations (15) and (18) show at once that the method just described has not the inconvenience mentioned above of the method of least squares in its habitual form.

I may make some remarks on the convergence of the interpolation formula (13). If n be constant and finite then taking $s=n-1$ we can obtain a function

$$\sum_{h=0}^{n-1} B_h \phi_h (x),$$

which for $x=x_i \ (i=\overline{1, n})$ will take just the values $y_i \ (i=\overline{1, n})$ provided that the determinant

$$\begin{matrix} \phi_0(x_1), & \phi_0(x_2), & \ldots, & \phi_0(x_n) \\ \phi_1(x_1), & \phi_1(x_2), & \ldots, & \phi_1(x_n) \\ & & & \\ \phi_{n-1}(x_1), & \phi_{n-1}(x_2), & \ldots, & \phi_{n-1}(x_n) \end{matrix}$$

or, what is the same,

$$\begin{matrix} u_0(x_1), & u_0(x_2), & \ldots, & u_0(x_n) \\ u_1(x_1), & u_1(x_2), & \ldots, & u_1(x_n) \\ & & & \\ u_{n-1}(x_1), & u_{n-1}(x_2), & \ldots, & u_{n-1}(x_n) \end{matrix}$$

be different from zero. Then the coefficients will coincide with $A_h=(y\phi_h)/(\phi_h\phi_h)$ and we shall find $\delta^2_{n-1}=0$. Adding the functions $\phi_n(x)$, $\phi_{n+1}(x)$, ... we shall find $A_n=A_{n+1}=\ldots=0$ and we see that the interpolation formula (13) converges to full coincidence with the function *y at given points* $x_i \ (i=\overline{1, n})$ when s becomes equal to $n-1$.

The question will also arise of the behaviour of the interpolation function $\sum_{h=0}^{n-1} A_h \phi_h (x)$ at other points. But this is a delicate and difficult problem which cannot be considered here. The reader may consult on this point the excellent monograph by E. Borel, "Leçons sur les fonctions de variables réelles et développements en séries de polynômes."

When $n=\infty$ the question on the convergence of the series $\sum_{h=0}^{\infty} \frac{(y\phi_h)}{(\phi_h\phi_h)} \phi_h (x)$ is also very difficult and belongs to the theory of expansions of arbitrary functions in series of orthogonal functions, much discussed and highly developed in the modern analysis. We shall omit it as too special for this note.

II. *Illustrations and extensions.*

1. Choosing for $u_h(x)$ any particular function we shall obtain from the preceding analysis the corresponding particular form of interpolation. For example, if we choose

$$u_h(x) = x^h \qquad (h=\overline{0, s}),$$

we shall obtain the parabolic interpolation described in the form considered above by the great Russian mathematician P. Tchebysheff, who deduced it from the theory of continuous fractions*.

* P. Tchebysheff, "Sur les fractions continues," *Journal de Mathématiques pures et appliquées*, t. III, pp. 289—323; "Sur l'interpolation par la méthode des moindres carrés," *Mémoires de l'Ac. des Sc. de Pétersbourg*, VII série, t. I. 1859, No. 15. There are some other papers of Tchebysheff on interpolation which can be found in *Oeuvres de P. L. Tchebysheff*, Pétersbourg, 1899. [We must leave to the Russians themselves the choice of spelling Tchebysheff's name (see footnote, p. 87). There is no doubt of the way its owner himself spelt it when writing for Western Europe. EDITORS.]

Continuous fractions are applied by P. Tchebysheff for the deduction of the functions orthogonalising the functions $1, x, x^2, \ldots, x^s$ on the base (x, p). The deduction is long and complicated, and this fact, without doubt, prevented the wide application of Tchebysheff's ingenious method. The shortest and simplest way in which this method can be expounded is that considered above with the due specification of the functions $u_h(x)$. I have published a short note on such a deduction of the Tchebysheff's parabolic method in the *Comptes Rendus de l'Ac. des Sc. de Paris*, t. CLXXXI, p. 595, and a larger paper—in Russian—in the journal Вестник Ирригации (Messenger of Irrigation), 1926, No. 1.

I shall note here a result from the latter paper which is new and of importance in the applications of parabolic interpolation. For $u_h(x) = x^h$ $(h=\overline{0, s})$ we shall have

$$\phi_h(x) = -\sum_{k=0}^{h-1} \frac{(x^h \phi_k)}{(\phi_k \phi_k)} \phi_k(x) + x^h \qquad (\phi_0 = 1;\ h = \overline{1, s}).$$

But clearly we can write

$$\phi_h(x) = a_{h_0} + a_{h_1} x + a_{h_2} x^2 + \ldots + a_{h, h-1} x^{h-1} + x^h,$$

and then the interpolation formula

$$y = \sum_{h=0}^{s} A_h \phi_h(x)$$

can be written in the form
$$y = \sum_{h=0}^{s} a_h x^h,$$

where
$$a_h = A_h + A_{h+1} a_{h+1, h} + \ldots + A_s a_{s, h},$$

$$A_h = \frac{(y \phi_h(x))}{(\phi_h \phi_h)} \qquad (h = \overline{0, s}).$$

Let now P_h be the weight of the coefficient a_h. Then, if the errors in a_h are independent for $h = \overline{0, s}$ and normally distributed as it is supposed in the classical method of least squares, we shall have

$$\frac{1}{P_h} = \frac{1}{(\phi_h \phi_h)} + \frac{a^2_{h+1, h}}{(\phi_{h+1} \phi_{h+1})} + \ldots + \frac{a^2_{s, h}}{(\phi_s \phi_s)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(19),$$

and

$$\sigma_{a_h} = \sqrt{\frac{\delta_s^2}{(n-s-1) P_h}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(20),$$

where σ_{a_h} denotes the standard error of a_h and

$$\delta_s^2 = (yy) - \sum_{h=0}^{s} A_h (y \phi_h).$$

The formulae (19) and (20) contain the result we have mentioned above.

2. For the second illustration we shall consider the following problem.

Suppose that the data $y_1, y_2, \ldots, y_n$ of our base of interpolation (x_i, y_i, p_i) $(i = \overline{1, n})$ show a secular trend and two periodic variations superposed on this trend. We shall suppose that the secular trend can be described by a parabola of the second order and that the periodic variations have the known periods $2\pi/\alpha$ and $2\pi/\beta$. Then our data may be interpolated by means of a relation of the form

$$y = a_0 + a_1 x + a_2 x^2 + b_1 \sin \alpha x + b_2 \cos \alpha x + c_1 \sin \beta x + c_2 \cos \beta x \quad \ldots\ldots\ldots\ldots(21),$$

where the coefficients $a_0, a_1, \ldots, c_2$ are to be found.

Now we have

$$u_0 = 1, \quad u_1 = x, \quad u_2 = x^2, \quad u_3 = \sin \alpha x, \quad u_4 = \cos \alpha x, \quad u_5 = \sin \beta x, \quad u_6 = \cos \beta x \quad \ldots\ldots(22),$$

and, applying the general formulae (11) to this case, we easily find the following system of functions $\phi_k(x)$ orthogonalising the functions (22) on the base (x, p):

$$\left.\begin{aligned}
\phi_0(x)&=1\\
\phi_1(x)&=-\frac{(x\phi_0)}{(\phi_0\phi_0)}\phi_0(x)+x\\
\phi_2(x)&=-\frac{(x^2\phi_0)}{(\phi_0\phi_0)}\phi_0(x)-\frac{(x^2\phi_1)}{(\phi_1\phi_1)}\phi_1(x)+x^2\\
\phi_3(x)&=-\sum_{h=0}^{2}\frac{(\sin ax\,\phi_h)}{(\phi_h\phi_h)}\phi_h(x)+\sin ax\\
\phi_4(x)&=-\sum_{h=0}^{3}\frac{(\cos ax\,\phi_h)}{(\phi_h\phi_h)}\phi_h(x)+\cos ax\\
\phi_5(x)&=-\sum_{h=0}^{4}\frac{(\sin \beta x\,\phi_h)}{(\phi_h\phi_h)}\phi_h(x)+\sin \beta x\\
\phi_6(x)&=-\sum_{h=0}^{5}\frac{(\cos \beta x\,\phi_h)}{(\phi_h\phi_h)}\phi_h(x)+\cos \beta x
\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots(23).$$

The functions (23) being evaluated we can write down the interpolation formula

$$y=\sum_{h=0}^{6}A_h\phi_h(x)\ldots\ldots\ldots\ldots(24),$$

where
$$A_h=\frac{(y\phi_h)}{(\phi_h\phi_h)}\qquad (h=\overline{0,6})\ \ldots\ldots\ldots\ldots(25).$$

The goodness of fit of the formula (24) can be estimated by means of the quantity

$$\delta_6^2=(yy)-\sum_{h=0}^{6}A_h(y\phi_h)\ \ldots\ldots\ldots\ldots(26).$$

When the formula (24) is obtained it is easy to bring it to the usual form

$$y=a_0+a_1x+a_2x^2+A\sin(ax+\gamma)+B\sin(\beta x+\delta),$$

and thus find the amplitudes A and B and the phases γ and δ of the periodic variations of our data.

3. The method of interpolation described above for functions of a single variable can be extended without difficulty to the functions of many variables.

Thus let
$$(x_i,\ y_h,\ z_{ih},\ p_{ih})\qquad (i=\overline{1,m},\ \ h=\overline{1,n})\ \ldots\ldots\ldots\ldots(27)$$

be a base of interpolation for a function z of two variables x, y, where z_{ih} is a value of z for $x=x_i$ and $y=y_h$ and p_{ih} is the weight of z_{ih}. Then, given the functions

$$u_0(x,y),\ u_1(x,y),\ \ldots,\ u_s(x,y),$$

by means of which we desire to interpolate z, we can write down the interpolation formula

$$z=\sum_{h=0}^{s}A_h\phi_h(x,y),$$

where
$$\phi_h(x,y)=-\sum_{k=0}^{h-1}\frac{(u_h\phi_k)}{(\phi_k\phi_k)}\phi_k(x,y)+u_h(x,y),$$

$$A_h=\frac{(z\phi_h)}{(\phi_h\phi_h)},$$

and
$$(u_k\phi_k)=\sum_{i,h}p_{ih}u_k(x_i,y_h)\phi_k(x_i,y_h),$$

$$(z\phi_k)=\sum_{i,h}p_{ih}z_{ih}\phi_k(x_i,y_h),$$

$$(\phi_k\phi_k)=\sum_{i,h}p_{ih}\phi_k(x_i,y_h)\phi_k(x_i,y_h),$$

$$(i=\overline{1,m};\ h=\overline{1,n}).$$

We remark that
$$(\phi_k\phi_k)=(u_k\phi_k),$$
and that
$$\delta_s^2=(zz)-\sum_{h=0}^{s}A_h(z\phi_h),$$
δ_s^2 being the value of the sum
$$\sum_{i,h}p_{ih}\left[z_{ih}-\sum_{h=0}^{s}A_h\phi_h(x_i,y_h)\right]^2.$$

I will make now some remarks on the case of the parabolic interpolation for functions of two variables, the base of interpolation being the same, (27).

In this case we shall put
$$\left.\begin{matrix} u_{00}=1, \quad u_{10}=x, \quad u_{01}=y, \quad u_{20}=x^2, \quad u_{11}=xy, \quad u_{02}=y^2 \\ \text{etc.} \\ u_{s0}=x^s, \quad u_{s-1,1}=x^{s-1}y, \quad u_{s-2,2}=x^{s-2}y^2, \ \ldots, \ u_{0s}=y^s \end{matrix}\right\} \quad \ldots\ldots\ldots\ldots(28).$$
Then the functions $\phi_{hk}(x, y)$ orthogonalising these functions u_{hk} will be
$$u_{00}=\phi_{00}, \quad \phi_{10}=-\frac{(x\phi_{00})}{(\phi_{00}\phi_{00})}\phi_{00}+x,$$
$$\phi_{01}=-\frac{(y\phi_{00})}{(\phi_{00}\phi_{00})}\phi_{00}-\frac{(y\phi_{10})}{(\phi_{10}\phi_{10})}\phi_{10}+y,$$
$$\phi_{20}=-\frac{(x^2\phi_{00})}{(\phi_{00}\phi_{00})}\phi_{00}-\frac{(x^2\phi_{10})}{(\phi_{10}\phi_{10})}\phi_{10}-\frac{(x^2\phi_{01})}{(\phi_{01}\phi_{01})}\phi_{01}+x^2,$$
and so on. Generally for $h=\overline{0, s}$, $k=\overline{0, s}$, $h+k\leqq s$,
$$\phi_{hk}=-\sum_{i,j}\frac{(x^hy^k\phi_{ij})}{(\phi_{ij}\phi_{ij})}\phi_{ij}(x,y)+x^hy^k \quad \ldots\ldots\ldots\ldots(29)$$
($i=\overline{0, g-1}$ and $j=\overline{0, g-1}$ if $i+j\leqq g-1$, where $g=h+k$, and $j=\overline{0, k-1}$ if $i+j=g$).
Here
$$(x^hy^k\phi_{ij})=\sum_{\lambda,\mu}p_{\lambda\mu}x_\lambda{}^hy_\mu{}^k\phi_{ij}(x_\lambda, y_\mu) \qquad (\lambda=\overline{1, m};\ \mu=\overline{1, n}).$$
The functions (29) are orthogonal:
$$(\phi_{hk}\phi_{h_1k_1})\begin{cases}=0 \text{ if } h\neq h_1 \text{ or } k\neq k_1 \\ \neq 0 \text{ if } h=h_1 \text{ and } k=k_1\end{cases} \quad \ldots\ldots\ldots\ldots(30),$$
where
$$(\phi_{hk}\phi_{h_1k_1})=\sum_{\lambda,\mu}p_{\lambda\mu}\phi_{hk}(x_\lambda, y_\mu)\,\phi_{h_1k_1}(x_\lambda, y_\mu) \qquad (\lambda=\overline{1, m};\ \mu=\overline{1, n}).$$

From (30) it follows that
$$(\phi_{hk}\phi_{hk})=(x^hy^k\phi_{hk}).$$

4. We shall conclude this paper by indicating a very remarkable property of the functions (29) when for all integral positive values of λ and μ we have
$$p_{\lambda\mu}=p_\lambda q_\mu \quad \ldots\ldots\ldots\ldots(31),$$
which is always the case when $p_{\lambda\mu}=1$ ($\lambda=\overline{1, m}$; $\mu=\overline{1, n}$).

This property consists in the relations
$$\left.\begin{aligned} \phi_{hk}&=\phi_{h0}\phi_{0k} \\ \phi_{h0}&=-\sum_{g=0}^{h-1}\frac{(x^h\phi_{g0})}{(\phi_{g0}\phi_{g0})}\phi_{g0}+x^h \\ \phi_{0k}&=\sum_{l=0}^{k-1}\frac{(y^k\phi_{0l})}{(\phi_{0l}\phi_{0l})}\phi_{0l}+y^k \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(32),$$
which show at once that ϕ_{h0} and ϕ_{0k} are respectively functions of x and y only, so that ϕ_{hk} is a product of two functions depending only on x and y respectively. The functions ϕ_{h0} and ϕ_{0k} give two sets of orthogonal functions, orthogonalising respectively functions $1, x, x^2, \ldots, x^s$ on the base (x_λ, p_λ) ($\lambda=\overline{1, m}$) and functions $1, y, y^2, \ldots, y^s$ on the base (y_μ, q_μ) ($\mu=\overline{1, n}$).

It is evident what simplifications can be attained in the parabolic interpolation of functions of two variables when the condition (31) is satisfied and we shall not insist on them.

The relations (32) may be easily proved by the method of complete induction.

MULTIPLE AND PARTIAL CORRELATION COEFFICIENTS IN THE CASE OF AN n-FOLD VARIATE SYSTEM.

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THE problem which we are to discuss is that of the correlation coefficients of samples of size N taken from an indefinitely large population. The population is supposed to be characterised by an arbitrary number n of variates, and our object will be to obtain formulae for the means and standard deviations of the multiple and partial correlation coefficients of these n-variates with respect to one another for a large number of samples, terms of higher order than $1/N$ being neglected. So far as we are aware, these topics have only been treated in two papers, the well-known one by G. U. Yule* who uses a method largely based on analogy, and the other by L. Isserlis† dealing with the case of three variates.

Notation. Let r_{st} be the correlation coefficient of the sth with the tth variate for a typical sample, and $\bar{r}_{st}$ the mean of the r_{st} for a large number of samples. We write $r_{st} = \bar{r}_{st} + \delta r_{st}$ and in general δ will refer to variations from the $\bar{r}$-values of the functions concerned. If, further, ρ_{st} is the same coefficient for the sampled population, then‡ to the approximation mentioned

$$= \rho_{st}\left(1 - \frac{1-\rho^2_{st}}{2N}\right) \quad \text{.............................(1).}$$

We shall use curled brackets throughout to signify that the mean for a large number of samples is to be taken, so that e.g. $\bar{r}_{st} = \{r_{st}\}$. Then we have the Pearson-Filon formula§

$$\{\delta r_{st}\, \delta r_{\sigma\tau}\} = \frac{1}{2N}[P_{st,\sigma\tau} + P_{st,\tau\sigma} + P_{ts,\sigma\tau} + P_{ts,\tau\sigma}] \quad \text{...........(2),}$$

where
$$P_{st,\sigma\tau} = (\rho_{s\sigma} - \rho_{st}\,\rho_{t\sigma})(\rho_{t\tau} - \rho_{t\sigma}\,\rho_{\sigma\tau}) \quad \text{........................(3),}$$
this formula remaining true even when $st = \sigma\tau$ or $\tau\sigma$.

We shall denote by Δ the determinant $|r_{st}|\ s, t = 1, 2, \ldots, n$ and by $\bar{D}$ and D respectively its $\bar{r}$ and ρ values. Δ_{st} denotes the minor of r_{st}, and $\Delta^{t\tau}_{s\sigma}$ the second minor complementary to the minor $\rho_{st}\,\rho_{\sigma\tau} - \rho_{s\tau}\,\rho_{\sigma t}$ so that e.g. $\Delta^{t\tau}_{s\sigma} + \Delta^{\tau t}_{s\sigma} = 0$; and similarly $\Delta^{\alpha_1 \ldots \alpha_\mu}_{\beta_1 \ldots \beta_\mu}$ will denote the minor obtained from Δ by suppressing the α_1th, ... α_μth rows and the β_1th, ... β_μth columns.

* *Proc. Roy. Soc.* A. Vol. LXXIX. p. 182.

† *Phil. Mag.* Vol. XXXIV. 1917.

‡ H. E. Soper, *Biometrika*, Vol. IX. p. 105.

§ *Phil. Trans.* A. 1898, Vol. 191, p. 262.

We have $r_{ss} = 1$; and $r_{st} = r_{ts}$, so that $\Delta_{st} = \Delta_{ts}$ &c. Finally, ϵ_{st} is Kronecker's symbol, $= 1$ when $s = t$ and $= 0$ otherwise.

To find $$\Sigma D^{\alpha_1 \ldots \alpha_\mu s\sigma}_{\beta_1 \ldots \beta_\mu t\tau} \{\delta r_{st} \delta r_{\sigma\tau}\} = \Sigma D^{(\alpha) s\sigma}_{(\beta) t\tau} \{\delta r_{st} \delta r_{\sigma\tau}\}:$$

we have $$\Sigma D^{(\alpha) s\sigma}_{(\beta) t\tau} P_{st,\sigma\tau}$$

$$= \Sigma \left[(\rho_{s\sigma} - \rho_{st}\rho_{t\sigma}) \left\{(\epsilon_{\sigma t} - \rho_{\sigma t}) D^{(\alpha) s}_{(\beta) t} - \epsilon_{st} D^{(\alpha) \sigma}_{(\beta) t} - \underset{i}{S} (\epsilon_{t\alpha_i} D^{\alpha_1 \ldots \sigma \ldots \alpha_\mu s}_{\beta_1 \ldots \beta_i \ldots \beta_\mu t})\right\}\right]$$

$$= \Sigma \left[D^{(\alpha) s}_{(\beta) t} (\rho^2_{\sigma t} \rho_{st} - \rho_{\sigma t} \rho_{\sigma s})\right] - \Sigma \left[\underset{i}{S} D^{\alpha_1 \ \sigma \ldots \alpha_\mu s}_{\beta_1 \ldots \beta_i \ldots \beta_\mu \alpha_i} (\rho_{s\sigma} - \rho_{s\alpha_i} \rho_{\alpha_i \sigma})\right]$$

$$= D^{(\alpha)}_{(\beta)} \Sigma^{\sigma \neq (\alpha)}_{t \neq (\beta)} \text{ of } (\rho^2_{\sigma t}) - \Sigma (D^{(\alpha) \sigma}_{(\beta) t} \rho^3_{\sigma t}) + \Sigma \left[\underset{i,j}{S} (\alpha_i \neq \beta_j) \epsilon_{s\beta_j} D^{\alpha_1 \ldots \sigma \ldots \alpha_j \ldots \alpha_\mu}_{\beta_1 \ldots \beta_i \ldots \alpha_i \ldots \beta_\mu}\right]$$

$$+ \Sigma \left[\underset{i}{S} (\alpha_i \text{ not a } \beta) \rho_{\alpha_i \sigma} D^{\alpha_1 \ldots \sigma \ldots \alpha_\mu}_{\beta_1 \ldots \beta_i \ldots \beta_\mu}\right]$$

$$= D^{(\alpha)}_{(\beta)} \left[\sum_{s,t} (\rho^2_{st}) - \sum_\lambda \{\underset{i}{S} (\rho^2_{\lambda\alpha_i} + \rho^2_{\lambda\beta_i})\} + \underset{i,j}{S} \rho^2_{\alpha_i \beta_j} - \text{number of } \alpha\text{'s which are not } \beta\text{'s}\right]$$

$$+ \underset{i,j}{S} (\alpha_i \neq \beta_j) \text{ of } D^{\alpha_1 \ldots \beta_j \ \ldots \alpha_j \ldots \alpha_\mu}_{\beta_1 \ldots \beta_i \ldots \alpha_i \ldots \beta_\mu} - \Sigma (D^{(\alpha) s}_{(\beta) t} \rho^3_{st}).$$

Interchanging each α with the corresponding β gives $\Sigma D^{(\alpha) s\sigma}_{(\beta) t\tau} P_{ts,\tau\sigma}$. While $\Sigma D^{(\alpha) s\sigma}_{(\beta) t\tau} P_{ts,\sigma\tau} = \Sigma \left[(\rho_{t\sigma} - \rho_{st}\rho_{s\sigma})(-D^{(\alpha) \sigma}_{(\beta) t} - \rho_{s\sigma} D^{(\alpha) s}_{(\beta) t})\right] = -D^{(\alpha)}_{(\beta)} \Sigma^{s \neq \sigma}_{s, \sigma \neq (\alpha)} \text{ of } (1 - \rho^2_{s\sigma})$,
and therefore $$\Sigma D^{(\alpha) s\sigma}_{(\beta) t\tau} P_{st,\tau\sigma} = -D^{(\alpha)}_{(\beta)} \Sigma^{t \neq \tau}_{t, \tau \neq (\beta)} \text{ of } (1 - \rho^2_{t\tau}).$$

Adding, we get

$$\Sigma D^{(\alpha) s\sigma}_{(\beta) t\tau} \{\delta r_{st} \delta r_{\sigma\tau}\} = \frac{1}{N} \left[\Sigma \{D^{(\alpha) s}_{(\beta) t} \rho_{st} (1 - \rho^2_{st})\} + \underset{i,j}{S} (\alpha_i \neq \beta_j) D^{\alpha_1 \ldots \beta_j \ldots \alpha_j \ldots \alpha_\mu}_{\beta_1 \ldots \beta_i \ldots \alpha_i \ldots \beta_\mu}\right]$$

$$+ \frac{2D^{(\alpha)}_{(\beta)}}{N} \left[\sum_{\kappa,\lambda} (\rho^2_{\kappa\lambda}) - \sum_\lambda \underset{i}{S} (\rho^2_{\lambda\alpha_i} + \rho^2_{\lambda\beta_i}) + \tfrac{1}{4} \underset{i,j}{S} (\rho^2_{\alpha_i \alpha_j} + 2\rho^2_{\alpha_i \beta_j} + \rho^2_{\beta_i \beta_j})\right.$$
$$\left. - \tfrac{1}{2} (n - \mu)(n - \mu + 1) - \tfrac{1}{2} (\text{number of } \alpha\text{'s which are not } \beta\text{'s})\right] \ldots(4).$$

Or, if by Σ', S' we denote that the summation is not to include those ρ's which have equal suffixes, the coefficient of $\dfrac{2D^{(\alpha)}_{(\beta)}}{N}$ may be written in the form

$$\Sigma' (\rho^2_{\kappa\lambda}) - \sum_\lambda{}' \underset{i}{S} (\rho^2_{\lambda\alpha_i} + \rho^2_{\lambda\beta_i}) + \tfrac{1}{4} \underset{i,j}{S'} (\rho^2_{\alpha_i \alpha_j} + 2\rho^2_{\alpha_i \beta_j} + \rho^2_{\beta_i \beta_j})$$
$$- \tfrac{1}{2} (n - \mu)(n - \mu - 1) - (\text{number of } \alpha\text{'s not equal to } \beta\text{'s}) \ldots(5).$$

Now

$$\bar{D}^{(\alpha)}_{(\beta)} - D^{(\alpha)}_{(\beta)} = -\frac{1}{2N} \Sigma D^{(\alpha) s}_{(\beta) t} \rho_{st} (1 - \rho^2_{st}) + O\left(\frac{1}{N^2}\right)^* \text{ by equation (1)} \quad \ldots(6),$$

while $$\Delta^{(\alpha)}_{(\beta)} - \bar{D}^{(\alpha)}_{(\beta)} = \Sigma \bar{D}^{(\alpha) s}_{(\beta) t} \delta r_{st} + \tfrac{1}{2} \Sigma \bar{D}^{(\alpha) s\sigma}_{(\beta) t\tau} \delta r_{st} \delta r_{\sigma\tau} + \ldots \quad \ldots\ldots\ldots\ldots(7).$$

Hence $$\Sigma \bar{D}^{(\alpha) s\sigma}_{(\beta) t\tau} \{\delta r_{st} \delta r_{\sigma\tau}\} = \Sigma D^{(\alpha) s\sigma}_{(\beta) t\tau} \{\delta r_{st} \delta r_{\sigma\tau}\} + O\left(\frac{1}{N^2}\right),$$

* $O\left(\frac{1}{N^2}\right)$ means of the order of $\frac{1}{N^2}$ at most.

and therefore since $\{\delta r_{st}\} = 0$, we have to our approximation

$$\{\Delta_{(\beta)}^{(a)}\} - D_{(\beta)}^{(a)} = \frac{1}{2N} \underset{i,j}{S} (\alpha_i \neq \beta_j) D_{\beta_1 \ldots \beta_i \ldots \alpha_i \ldots \beta_\mu}^{\alpha_1 \ldots \beta_j \ldots \alpha_j \ldots \alpha_\mu}$$

$$+ \frac{D_{(\beta)}^{(a)}}{N} [\Sigma'(\rho^2_{\kappa\lambda}) - \underset{\lambda}{\Sigma'} \underset{i}{S} (\rho^2_{\lambda\alpha_i} + \rho^2_{\lambda\beta_i}) + \tfrac{1}{4} \underset{i,j}{S'} (\rho^2_{\alpha_i\alpha_j} + 2\rho^2_{\alpha_i\beta_j} + \rho^2_{\beta_i\beta_j})$$

$$- \tfrac{1}{2}(n-\mu)(n-\mu-1) - \text{number of } \alpha\text{'s not equal to } \beta\text{'s}] \quad \ldots(8).$$

For example, $$\{\Delta\} - D = \frac{D}{N} [\underset{\kappa,\lambda}{\Sigma'} (\rho^2_{\kappa\lambda}) - \tfrac{1}{2} n (n-1)] \quad \ldots\ldots\ldots\ldots(9),$$

coinciding with the value obtained by Isserlis when $n = 3$, viz.

$$\frac{D}{N}(2\rho^2_{12} + 2\rho^2_{23} + 2\rho^2_{31} - 3) \quad \ldots\ldots\ldots\ldots(10).$$

Similarly for Δ_{11}, Δ_{12}^{12} and so on. While for Δ_{12}, we have

$$\{\Delta_{12}\} - D_{12} = \frac{D_{12}}{N} [\Sigma'^{\kappa \neq 1}_{\lambda \neq 2} \text{ of } (\rho^2_{\kappa\lambda}) - \tfrac{1}{2}(1 + \rho^2_{12}) - \tfrac{1}{2}(n-1)(n-2)] \quad \ldots(11).$$

To find $$\Sigma D_{\beta_1 \ldots \beta_\mu \tau}^{\alpha_1 \ldots \alpha_\mu \sigma} D_{b_1 \ldots b_\nu t}^{a_1 \ldots a_\nu s} \{\delta r_{st} \delta r_{\sigma\tau}\} = \Sigma D_{(\beta)\tau}^{(\alpha)\sigma} D_{(b)t}^{(a)s} \{\delta r_{st} \delta r_{\sigma\tau}\}.$$

We shall suppose that the suffixes α and β are so arranged that $\alpha_i = \beta_j$ implies that $i = j$, and that $\alpha_k = \beta_k$ for all $k < i$; and similarly for the a and b. This may always be secured at the cost of a possible change of sign. Further $D_{\beta_1 \ldots (\beta_i) \ldots \beta_\mu}^{\alpha_1 \ldots (\alpha_i) \ldots \alpha_\mu}$ will denote the minor (of order $n - \mu + 1$) obtained from $D_{(\beta)}^{(\alpha)} = D_{\beta_1 \ldots \beta_\mu}^{\alpha_1 \ldots \alpha_\mu}$ by omitting the suffixes α_i and β_i. With these conventions, we have

$$\underset{s}{\Sigma} D_{(b)t}^{(a)s} (\rho_{s\sigma} - \rho_{st}\rho_{t\sigma}) = (\epsilon_{\sigma t} - \rho_{\sigma t}) D_{(b)}^{(a)} - \underset{i}{S} (\epsilon_{\sigma b_i} D_{b_1 \ldots t \ldots b_\nu}^{a_1 \ldots a_i \ldots a_\nu})$$

and $$\underset{\tau}{\Sigma} D_{(\beta)\tau}^{(\alpha)\sigma} (\rho_{t\tau} - \rho_{t\sigma}\rho_{\sigma\tau}) = (\epsilon_{\sigma t} - \rho_{\sigma t}) D_{(\beta)}^{(\alpha)} - \underset{j}{S} (\epsilon_{t\alpha_j} D_{\beta_1 \ldots \beta_j \ldots \beta_\mu}^{\alpha_1 \ldots \sigma \ldots \alpha_\mu}).$$

Multiplying and summing over $\sigma \neq (\alpha)$, and $t \neq (b)$, will give

$$\Sigma D_{(b)t}^{(a)s} D_{(\beta)\tau}^{(\alpha)\sigma} P_{st,\sigma\tau}.$$

There will be four terms:

(i) $D_{(b)}^{(a)} D_{(\beta)}^{(\alpha)} \Sigma' \binom{\sigma \neq \alpha}{t \neq b}$ of $(\rho^2_{\sigma t})$,

(ii) $-D_{(b)}^{(a)} \underset{j}{S} \Sigma (\epsilon_{\sigma t} - \rho_{\sigma t})(\epsilon_{t\alpha_j} D_{\beta_1 \ldots \beta_j \ldots \beta_\mu}^{\alpha_1 \ldots \sigma \ldots \alpha_\mu})$

$= D_{(b)}^{(a)} \underset{j}{S} \Sigma \rho_{\sigma\alpha_j} D_{\beta_1 \ldots \beta_j \ldots \beta_\mu}^{\alpha_1 \ldots \sigma \ldots \alpha_\mu}$ for α_j not a b

$= - D_{(b)}^{(a)} D_{(\beta)}^{(\alpha)} \times$ number of α's not equal to b's

$+ D_{(b)}^{(a)} \times$ sum of those $D_{\beta_1 \ldots (\beta_j) \ldots \beta_\mu}^{\alpha_1 \ldots (\alpha_j) \ldots \alpha_\mu}$ for which $\alpha_j = \beta_j \neq$ a b.

(iii) From (ii) by interchanging the rôles of (a), (b) with those of (α), (β).

(iv) $\Sigma \underset{i,j}{S} \epsilon_{\sigma b_i} \epsilon_{t\alpha_j} D_{b_1 \ldots t \ldots b_\nu}^{a_1 \ldots a_i \ldots a_\nu} D_{\beta_1 \ldots \beta_j \ldots \beta_\mu}^{\alpha_1 \ldots \sigma \ldots \alpha_\mu}$

= sum of all the products obtained from $D_{(b)}^{(a)} D_{(\beta)}^{(\alpha)}$ by interchanging in every possible way a b which is not an α with an α which is not a b.

To obtain the remaining three terms of $\Sigma D^{(a)s}_{(b)t} D^{(\alpha)\sigma}_{(\beta)\tau} \{\delta r_{st}\, \delta r_{\sigma\tau}\}$ we must make the substitutions $(a \mid b)$, $(\alpha \mid \beta)$ and $(a \mid b)(\alpha \mid \beta)$.

Adding gives

$$\Sigma D^{(a)s}_{(b)t} D^{(\alpha)\sigma}_{(\beta)\tau} \{\delta r_{st}\, \delta r_{\sigma\tau}\} = \frac{2}{N} [D^{(a)}_{(b)} D^{(\alpha)}_{(\beta)} \times A + D^{(a)}_{(b)} \times B + D^{(\alpha)}_{(\beta)} \times B' + C] \quad \ldots(12),$$

where

$$A = \sum_{\kappa, \lambda}' (\rho^2_{\kappa\lambda}) - \tfrac{1}{2}\Sigma \{S(\rho^2_{\lambda a} + \rho^2_{\lambda b} + \rho^2_{\lambda\alpha} + \rho^2_{\lambda\beta})\} + \tfrac{1}{4}SS(\rho^2_{a\alpha} + \rho^2_{a\beta} + \rho^2_{b\alpha} + \rho^2_{b\beta}) \quad \ldots\ldots\ldots\ldots(13).$$

$B =$ the sum of all the $D^{\alpha_1 \ldots (\alpha_i) \ldots \alpha_\mu}_{\beta_1 \ldots (\beta_i) \ldots \beta_\mu}$ for which $\alpha_i = \beta_i$

minus $\tfrac{1}{2}$ of those for which $\alpha_i = \beta_i =$ a b

minus $\tfrac{1}{2}$ of those for which $\alpha_i = \beta_i =$ an a ....(14).

B' is analogous to B.

$C = \tfrac{1}{4}$ the sum of the products obtained from $D^{(a)}_{(b)} D^{(\alpha)}_{(\beta)}$ by interchanging an a or a b with an α or a β in every possible way ....(15).

Now it follows from equations (6), (7), that to our present approximation, if $\delta\Delta^{(\alpha)}_{(\beta)} = \Delta^{(\alpha)}_{(\beta)} - \bar{D}^{(\alpha)}_{(\beta)}$, then

$$\{\delta\Delta^{(\alpha)}_{(\beta)} \delta\Delta^{(a)}_{(b)}\} = \Sigma D^{(\alpha)s}_{(\beta)t} D^{(a)\sigma}_{(b)\tau} \{\delta r_{st}\, \delta r_{\sigma\tau}\}.$$

Hence we have e.g.

$$\{(\delta\Delta)^2\} = \frac{2}{N} D^2 \sum_{\kappa, \lambda}' (\rho^2_{\kappa\lambda}) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(16),$$

and similarly for $\{(\delta\Delta_{11})^2\}$ &c.,

while

$$\{\delta\Delta_{11}\, \delta\Delta_{22}\} = \frac{2D_{11} D_{22}}{N} [\Sigma'^{\kappa \neq 1}_{\lambda \neq 2} \text{ of } (\rho^2_{\kappa\lambda}) - 2] + \frac{2}{N} [D(D_{11} + D_{22}) + D^2_{12}].$$

Or, writing $\qquad \dfrac{D}{D_{11}} = 1 - \bar{R}_1^2$, and $\dfrac{-D_{12}}{\sqrt{D_{11}}\sqrt{D_{22}}} = \bar{R}_{12}$,

$$\{\delta\Delta_{11}\, \delta\Delta_{22}\} = \frac{2D_{11} D_{22}}{N} [\Sigma'^{\kappa \neq 1}_{\lambda \neq 2} \text{ of } (\rho^2_{\kappa\lambda}) + \bar{R}^2_{12} - \bar{R}_1^2 - \bar{R}_2^2] \quad \ldots\ldots\ldots(17).$$

Next

$$\{(\delta\Delta_{12})^2\} = \frac{2D^2_{12}}{N} [\Sigma'^{\kappa \neq 1}_{\lambda \neq 2} \text{ of } (\rho^2_{\kappa\lambda}) - \tfrac{1}{2}\rho^2_{12} - 1] + \frac{D_{11} D_{22}}{N}$$

$$= \frac{2D^2_{12}}{N} \left[\Sigma'^{\kappa \neq 1}_{\lambda \neq 2} \text{ of } (\rho^2_{\kappa\lambda}) + \tfrac{1}{2}\left(\frac{1}{\bar{R}^2_{12}} - \rho^2_{12}\right) - 1\right] \quad \ldots\ldots\ldots(18);$$

while

$$\{\delta\Delta_{11}\, \delta\Delta_{12}\} = \frac{2D_{11} D_{12}}{N} [\sum_{\kappa, \lambda}' (\rho^2_{\kappa\lambda}) - \tfrac{3}{2}\Sigma' \rho^2_{1\lambda} - \tfrac{1}{2}\Sigma' \rho^2_{2\lambda} + \tfrac{1}{2}(\rho^2_{12} - 1)] + \frac{DD_{12}}{N},$$

so that

$$\left\{\delta\Delta_{12}\left(\frac{\delta\Delta_{11}}{D_{11}} + \frac{\delta\Delta_{22}}{D_{22}}\right)\right\} = \frac{4D_{12}}{N} [\Sigma'^{\kappa \neq 1}_{\lambda \neq 2} \text{ of } (\rho^2_{\kappa\lambda}) - \tfrac{1}{2}\rho^2_{12} - \tfrac{1}{2}(\bar{R}_1^2 + \bar{R}_2^2)] \quad \ldots\ldots(19).$$

And finally

$$\{\delta\Delta\, \delta\Delta_{12}\} = \frac{2DD_{12}}{N} [\sum_{\kappa, \lambda}' (\rho^2_{\kappa\lambda}) - \tfrac{1}{2}\sum_{\lambda} (\rho^2_{1\lambda} + \rho^2_{2\lambda})] \quad \ldots\ldots\ldots\ldots\ldots(20)$$

and

$$\{\delta\Delta\, \delta\Delta_{11}\} = \frac{2DD_{11}}{N} [\sum_{\lambda \neq 1}' (\rho^2_{\kappa\lambda}) - \bar{R}_1^2] \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(21).$$

With these results, we can now proceed to evaluate the means of R_1^2 and R_{12}. We have

$$R^2_{1.23\,\ldots\,n} = R_1^2 = 1 - \frac{\Delta}{\Delta_{11}}$$

$$= 1 - \frac{D}{D_{11}}\left(1 + \frac{\bar{\delta}\Delta}{D}\right)\left(1 - \frac{\bar{\delta}\Delta_{11}}{D_{11}} + \frac{(\bar{\delta}\Delta_{11})^2}{D^2_{11}}\right) + \ldots,$$

where $\bar{\delta}$ denotes variations from the ρ-values *. Hence

$$\bar{\delta}R_1^2 = R_1^2 - \bar{R}_1^2 = (1 - \bar{R}_1^2)\left(\frac{\bar{\delta}\Delta_{11}}{D_{11}} - \frac{\bar{\delta}\Delta}{D} + \frac{\bar{\delta}\Delta\,\bar{\delta}\Delta_{11}}{D\,D_{11}} - \frac{(\bar{\delta}\Delta_{11})^2}{D^2_{11}}\right) \quad \ldots\ldots(9),$$

And

$$\sigma^2_{R_1^2} = \{(\bar{\delta}R_1^2)^2\} - \{\bar{\delta}R_1^2\}^2$$

$$= (1 - \bar{R}_1^2)^2\left\{\left(\frac{\bar{\delta}\Delta_{11}}{D_{11}} - \frac{\bar{\delta}\Delta}{D}\right)^2\right\} + O\left(\frac{1}{N^2}\right) \quad \ldots\ldots\ldots\ldots(10).$$

Substituting in these two equations after taking the mean, we get

$$\{\bar{\delta}R_1^2\} = \frac{1 - \bar{R}_1^2}{N}(n - 1 - 2\bar{R}_1^2) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

and

$$\sigma^2_{R_1^2} = \frac{4\bar{R}_1^2(1 - \bar{R}_1^2)^2}{N} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(25).$$

Next,

$$R_{12.34\ldots n} = -\frac{\Delta_{12}}{\sqrt{\Delta_{11}}\sqrt{\Delta_2}}$$

$$\frac{1}{\sqrt{\Delta_{11}}} = \frac{1}{\sqrt{D_{11}}}\left(1 - \tfrac{1}{2}\frac{\bar{\delta}\Delta_{11}}{D_{11}} + \tfrac{3}{8}\left(\frac{\bar{\delta}\Delta_{11}}{D_{11}}\right)^2 + \ldots\right),$$

therefore

$$\frac{\bar{\delta}R_{12}}{R_{12}} = \frac{\bar{\delta}\Delta_{12}}{D_{12}} - \tfrac{1}{2}\frac{\bar{\delta}\Delta_{11}}{D_{11}} - \tfrac{1}{2}\frac{\bar{\delta}\Delta_{22}}{D_{22}}$$

$$+ \tfrac{3}{8}\frac{(\bar{\delta}\Delta_{11})^2}{D^2_{11}} + \tfrac{3}{8}\frac{(\bar{\delta}\Delta_{22})^2}{D^2_{22}} - \tfrac{1}{2}\frac{\bar{\delta}\Delta_{12}}{D_{12}}\left(\frac{\bar{\delta}\Delta_{11}}{D_{11}} + \frac{\bar{\delta}\Delta_{22}}{D_{22}}\right) + \tfrac{1}{4}\frac{\bar{\delta}\Delta_{11}\,\bar{\delta}\Delta_{22}}{D_{11}D_{22}} + \ldots \quad \ldots\ldots(26),$$

giving in the mean

$$\{\bar{\delta}R_{12}\} = -\frac{\bar{R}_{12}}{2N}(1 - \bar{R}^2_{12}) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(27),$$

or

$$\{R_{12}\} = \bar{R}_{12}\left(1 - \frac{1 - \bar{R}^2_{12}}{2N}\right),$$

a generalised form of Soper's formula (1), and

$$\{(\bar{\delta}R_{12})^2\} = \{(\delta R_{12})^2\} = \left(\frac{\delta\Delta_{12}}{D_{12}} - \tfrac{1}{2}\frac{\delta\Delta_{11}}{D_{11}} - \tfrac{1}{2}\frac{\delta\Delta_{22}}{D_{22}}\right)^2_{\text{mean}} \times \bar{R}^2_{12} = \frac{(1 - \bar{R}^2_{12})^2}{N}.$$

And since

$$\{\delta R_{12}\}^2 = O\left(\frac{1}{N^2}\right),$$

$\sigma^2_{R_{12}} = \dfrac{(1 - \bar{R}^2_{12})^2}{N}$, which is the generalised form of the Pearson-Filon formula

$$\sigma^2_{r_{12}} = \frac{(1 - r^2_{12})^2}{N} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(28).$$

* It is to be observed that equation (12) and all its corollaries hold to our present order of approximation when we replace $\bar{\delta}$ by δ. In view of equation (8), it will still be true if we measure $\Delta^{(a)}_{(\beta)}$ and $D^{(a)}_{(b)}$ from their means.

This formula is only reached by neglecting terms in $\frac{1}{N^2}$ and therefore it cannot be certainly improved by writing $N-(n-2)$ for N. Accordingly the use of a form such as $\frac{(1-\bar{R}^2_{12})^2}{N-(n-2)}$ is to be deprecated.

That the forms (27) and (28) should be the same as the corresponding formulae for r_{12} is shown by G. U. Yule in the paper already cited. It may also be deduced from the expression for the frequency distribution, viz.

$$\frac{1}{2\pi\Sigma_1 \ldots \Sigma_n \sqrt{D}} e^{-\frac{1}{2D}\left[S\frac{D_{ss}}{\sigma_s^2}x_s^2 - 2S'\frac{D_{st}x_s x_t}{\sigma_s\sigma_t}\right]} \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}.$$

If in this expression we consider $x_3, \ldots x_n$ as constants, it becomes

$$\text{constant} \times e^{-\frac{1}{2D}\left[\frac{D_{11}X_1^2}{\sigma_1^2} + \frac{2D_{12}X_1X_2}{\sigma_1\sigma_2} + \frac{D_{22}X_2^2}{\sigma_2^2}\right]},$$

where $X_1 = x_1 + a_1$, $X_2 = x_2 + a_2$ and a_1 and a_2 are suitable constants; i.e.

$$\text{constant} \times e^{-\frac{1}{2(1-\bar{R}^2_{12})}\left[\frac{X_1^2}{\Sigma_1^2} - 2\frac{\bar{R}_{12}X_1X_2}{\Sigma_1\Sigma_2} + \frac{X_2^2}{\Sigma_2^2}\right]} \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)},$$

where
$$\Sigma_1^2 = \frac{\sigma_1^2(1-\bar{R}_1^2)}{1-\bar{R}^2_{12}} = \sigma_1^2(1-\bar{R}^2_{1.3\ldots n}),$$

and $\Sigma_2^2 = \sigma_2^2(1-\bar{R}^2_{2.3\ldots n})$. But this is exactly of the same form as (i) for the case $n=2$, viz.

$$\text{constant} \times e^{-\frac{1}{2(1-\rho^2_{12})}\left[\frac{x_1^2}{\sigma_1^2} - \frac{2\rho_{12}x_1x_2}{\sigma_1\sigma_2} + \frac{x_2^2}{\sigma_2^2}\right]},$$

when we write $\bar{R}_{12}$ for ρ_{12} and Σ_1, Σ_2 for σ_1 and σ_2. In view of the meaning of Σ_1 and Σ_2, it follows that the distribution of R_{12} has exactly the same form as that for r_{12}, viz. the curve given by R. A. Fisher*. This seems to be in conflict with Mr Fisher's recent paper in *Metron*†, where he suggests that in the partial coefficient of correlation N must be replaced by $N-(n-2)$.

Correlations of Partial and Multiple Correlation Coefficients.

Let
$$R^2_{1(\alpha)} = R^2_{1(\alpha_1\alpha_2\ldots\alpha_\mu)} = 1 - \frac{\Delta^{(\alpha)}_{(\alpha)}}{\Delta^{1(\alpha)}_{1(\alpha)}} = 1 - \frac{\Delta(\alpha)}{\Delta(1\alpha)}.$$

Similarly let

$$R_{12(\beta)} = R_{12(\beta_1\ldots\beta_\nu)} = \frac{-\Delta^{1(\beta)}_{2(\beta)}}{\sqrt{\Delta^{1(\beta)}_{1(\beta)}}\sqrt{\Delta^{2(\beta)}_{2(\beta)}}} = \frac{-\Delta^{\,1}_{\,2}(\beta)}{\sqrt{\Delta(1\beta)}\sqrt{\Delta(2\beta)}}.$$

In calculating e.g. $\{\delta R^2_{1(\alpha)}\,\delta R^2_{1(\beta)}\}$

we may write $(1-\bar{R}^2_{1(\alpha)})\left(\frac{\delta\Delta(1\alpha)}{D(1\alpha)} - \frac{\delta\Delta(\alpha)}{D(\alpha)}\right)$ for $\delta R^2_{1(\alpha)}$ to our approximation.

* *Biometrika*, Vol. x. p. 507.

† *Metron*, Vol. III. p. 329; but we have been unable to follow Mr Fisher's argument.

Hence, in view of the footnote, p. 104, and equation (25) we have

$$r_{R^2_{1(\alpha)}R^2_{1(\beta)}} = \frac{N}{4\bar{R}_{1(\alpha)}\bar{R}_{1(\beta)}}\left[\frac{\{\delta\Delta(1\alpha)\,\delta\Delta(1\beta)\}}{D(1\alpha)\,D(1\beta)} + \frac{\{\delta\Delta(\alpha)\,\delta\Delta(\beta)\}}{D(\alpha)\,D(\beta)} - \frac{\{\delta\Delta(1\alpha)\,\delta\Delta(\beta)\}}{D(1\alpha)\,D(\beta)} - \frac{\{\delta\Delta(\alpha)\,\delta\Delta(1\beta)\}}{D(\alpha)\,D(1\beta)}\right].$$

For the expression in the bracket, we must recur to equations (12), (13), (14) and (15).

The A-terms give $\frac{2}{N}$.

The B-terms give $-\frac{2}{N}\frac{D(\beta)}{D(1\beta)}$, and the B' give $-\frac{2}{N}\frac{D(\alpha)}{D(1\alpha)}$,

i.e. $\qquad -\frac{2}{N}(1-\bar{R}^2_{1(\beta)})$ and $-\frac{2}{N}(1-\bar{R}^2_{1(\alpha)})$ respectively.

Hence

$$r_{R^2_{1(\alpha)}R^2_{1(\beta)}} = \frac{1}{2\bar{R}_{1(\alpha)}\bar{R}_{1(\beta)}}[\bar{R}^2_{1(\alpha)} + \bar{R}^2_{1(\beta)} - 1 + \text{contribution of the } C\text{-terms}] \qquad \text{............(29).}$$

Similarly, $\qquad r_{R^2_{1(\alpha)}R^2_{2(\beta)}} = \frac{1}{2\bar{R}_{1(\alpha)}\bar{R}_{1(\beta)}}[\rho^2_{12} + B\text{-terms} + C\text{-terms}].$

There will only be B-terms if one of the β's (say β_1) $= 1$, and we then get $\bar{R}^2_{1(2\beta_2\ldots\beta_\nu)} - \bar{R}^2_{1(\beta_2\ldots\beta_\nu)}$. Similarly, B'-terms occur only if an α (say α_1) $= 2$, and then give $\bar{R}^2_{2(1\alpha_2\ldots\alpha_\nu)} - \bar{R}^2_{2(\alpha_2\ldots\alpha_\nu)}$. The C-terms can of course be expressed in terms of the $\bar{R}$, but the general form is complicated. However, any particular case may be written down at once from equation (15). For example, if there are no α's or β's

$$r_{R_1^2R_2^2} = \frac{1}{2\bar{R}_1\bar{R}_2}[\rho^2_{12} + \bar{R}^2_{12}] \qquad \text{..........................(30),}$$

corresponding to the value given by Isserlis for the case $n = 3$*. This quantity naturally cannot exceed 1 since both $\bar{R}_1$ and $\bar{R}_2$ are never less than $\bar{R}_{12}$ and ρ_{12}.

Other particular cases are:

$$r_{R_1^2R^2_{1(2)}} = \frac{\bar{R}_1^2 + \bar{R}^2_{1(2)} - \bar{R}^2_{12}}{2\bar{R}_1\bar{R}_{1(2)}} \qquad \text{........................(31),}$$

as may be obtained directly from the equation $1 - R_1^2 = (1 - R^2_{12})(1 - R^2_{1(2)})$. More generally,

$$r_{R_1^2R^2_{1(\beta)}} = \frac{1}{2\bar{R}_1\bar{R}_{1(\beta)}}\left[\bar{R}_1^2 + \bar{R}^2_{1(\beta)} - \underset{i}{S}\frac{D_{1\beta_i}}{D_{11}}\frac{D^{\beta_1\ldots\beta_i\ldots\beta_\mu}_{\beta_1\ldots1\ldots\beta_\mu}}{D(\beta)}\right] \qquad \text{.........(32).}$$

To find $\{\delta R^2_{1(\alpha)}\,\delta R_{12(\beta)}\}$ we may substitute for $\delta R_{12(\beta)}$ its linear part, viz.

$$\bar{R}_{12(\beta)}\left(\frac{\delta\Delta^{2(\beta)}_{1(\beta)}}{D^{2(\beta)}_{1(\beta)}} - \frac{1}{2}\frac{\delta\Delta(1\beta)}{D(1\beta)} - \frac{1}{2}\frac{\delta\Delta(2\beta)}{D(2\beta)}\right),$$

* It is easy to verify that $r_{R_1^2X} = r_{R_1X}$, to our order of approximation, X being any linear function of the δr_{st}.

and similarly for $\delta R^2{}_{1(\alpha)}$. Proceeding as before, we get, since the A-terms vanish, and the B-terms give $\frac{1}{2}(1-\bar{R}^2{}_{1(\beta)})$,

$$\{\delta R^2{}_{1(\alpha)}\,\delta R_{12(\beta)}\} = \frac{2}{N}\bar{R}_{12(\beta)}(1-\bar{R}^2{}_{1(\alpha)})\left[\tfrac{1}{2}(1-\bar{R}^2{}_{1(\beta)}) + C\text{-terms}\right].$$

When there are no α's or β's, the C-terms become $\frac{1}{2}(1-\bar{R}^2{}_{12})$. Hence

$$\{\delta R_1{}^2\,\delta R_{12}\} = \frac{\bar{R}_{12}(1-\bar{R}_1{}^2)}{N}(2-\bar{R}_1{}^2-\bar{R}^2{}_{12}) \quad\text{.................(33)},$$

and

$$r_{R_1{}^2 R_{12}} = \frac{\bar{R}_{12}}{2\bar{R}_1}\left(1+\frac{1-\bar{R}_1{}^2}{1-\bar{R}^2{}_{12}}\right) \qquad (\leqslant \tfrac{1}{2}) \quad\text{..............(34)}.$$

Similarly

$$\{\delta R_3{}^2\,\delta R_{12}\} = -\frac{\bar{R}_{13}(1-\bar{R}_3{}^2)}{N}\left(\bar{R}^2{}_{13}+\bar{R}^2{}_{23}+\frac{2\bar{R}_{13}\bar{R}_{23}}{\bar{R}_{12}}\right) \quad\text{.........(35)},$$

and

$$r_{R_3{}^2 R_{12}} = -\tfrac{1}{2}\frac{\bar{R}_{12}}{\bar{R}_3}\left(\bar{R}^2{}_{13}+\bar{R}^2{}_{23}+\frac{2\bar{R}_{13}\bar{R}_{23}}{\bar{R}_{12}}\right)\Big/(1-\bar{R}^2{}_{12}) \quad\text{...........(36)}.$$

Equation (34) may be also written in the form

$$r_{R_1{}^2 R_{12}} = \frac{\bar{R}^2{}_{12}+\bar{R}_1{}^2-\bar{R}^2{}_{1(2)}}{2\bar{R}_1\bar{R}_{12}} \quad\text{..........................(37)},$$

giving for $n=3$ the form found by Isserlis, *loc. cit.* p. 213.

Finally, to find $\{\delta R_{12(\alpha)}\,\delta R_{13(\beta)}\}$ and $\{\delta R_{12(\alpha)}\,\delta R_{34(\beta)}\}$.

In both cases, *the A, B and B' terms vanish*, and we are left with the C-terms, which can be written down at once from equation (15), and the formula

$$\frac{\{\delta R_{12(\alpha)}\,\delta R_{13(\beta)}\}}{\bar{R}_{12(\alpha)}\bar{R}_{13(\beta)}} = \left[\left(\frac{\delta\Delta_2{}^1(\alpha)}{D_2{}^1(\alpha)} - \tfrac{1}{2}\frac{\delta\Delta(1\alpha)}{D(1\alpha)} - \tfrac{1}{2}\frac{\delta\Delta(2\alpha)}{D(2\alpha)}\right)\right.$$
$$\left.\times\left(\frac{\delta\Delta_3{}^1(\beta)}{D_3{}^1(\beta)} - \tfrac{1}{2}\frac{\delta\Delta(1\beta)}{D(1\beta)} - \tfrac{1}{2}\frac{\delta\Delta(3\beta)}{D(3\beta)}\right)\right]_{\text{mean}}.$$

When there are no α's or β's, we get

$$\{\delta R_{12}\delta R_{13}\} = \frac{1}{2N}[\bar{R}_{12}\bar{R}_{13}(\bar{R}^2{}_{12}+\bar{R}^2{}_{13}+\bar{R}^2{}_{23}-1) - 2\bar{R}_{23}(1-\bar{R}^2{}_{13}-\bar{R}^2{}_{12})] \quad\text{...(38)},$$

and therefore

$$r_{R_{12}R_{13}} = -\bar{R}_{23} - \tfrac{1}{2}\frac{\bar{R}_{12}\bar{R}_{13}}{(1-\bar{R}^2{}_{12})(1-\bar{R}^2{}_{13})}\left|-\bar{R}_{ij}\right|_{i,j=1,2,3} \quad\text{...........(39)},$$

which is identical with the Pearson-Filon formula for $r_{r_{12}r_{13}}$ when we change the sign of all the R. Similarly

$$\{\delta R_{12}\delta R_{34}\} = \frac{1}{2N}[\bar{R}'_{(12,34)} + \bar{R}'_{(21,34)} + \bar{R}'_{(12,43)} + \bar{R}'_{(21,43)}] \quad\text{........(40)},$$

where

$$\bar{R}'_{(12,34)} = (\bar{R}_{13}+\bar{R}_{12}\bar{R}_{23})(\bar{R}_{24}+\bar{R}_{23}\bar{R}_{34}).$$

However

$$\{\delta R_{12(3)}\,\delta R_{13(2)}\} = \frac{1}{2N}[\bar{R}_{12(3)}\bar{R}_{13(2)}(\bar{R}^2{}_{12(3)}+\bar{R}^2{}_{13(2)}+\bar{R}^2{}_{23(1)}-1)$$
$$+ 2\bar{R}_{23(1)}(1-\bar{R}^2{}_{13(2)}-\bar{R}^2{}_{12(3)})] \quad\text{........(41)}$$

and $$\{\delta R_{12(34)}\,\delta R_{34(12)}\} = \frac{1}{2N}\left[\bar{R}_{(12,34)} + \bar{R}_{(21,34)} + \bar{R}_{(12,43)} + \bar{R}_{(21,43)}\right] \quad\ldots\ldots\ldots(42),$$

where $$\bar{R}_{(12,34)} = (\bar{R}_{13(24)} - \bar{R}_{12(34)}\,\bar{R}_{23(14)})(\bar{R}_{24(13)} - \bar{R}_{23(14)}\,\bar{R}_{34(12)}).$$

(41) and (42) are therefore identical in form with the Pearson-Filon formulae *without* a change of sign, just as we should expect from the particular case of $n = 4$.

It will be noticed that we have chosen to deal with $R^2_{1.23\ldots n}$ rather than with $R_{1.23\ldots n}$ itself. The former being a measure of reduced variability, the latter may be treated as always positive. Then if $\bar{R}_1$ is not too small, we have

$$\frac{\delta R_1}{\bar{R}_1} = \tfrac{1}{2}\frac{\delta R_1^2}{\bar{R}_1^2} - \tfrac{1}{8}\frac{(\delta R_1^2)^2}{\bar{R}_1^4},$$

therefore $$\{\delta R_1\} = \frac{1-\bar{R}_1^2}{2N\bar{R}_1}(n - 2 - \bar{R}_1^2) \quad\ldots\ldots\ldots(43),$$

where we have neglected terms in $\dfrac{1}{(N\bar{R}_1)^2}$. This gives the value obtained by Isserlis when we put $n = 3$. However the formula clearly cannot be true when $\bar{R}_1 \to 0$.

Similarly, we have from equations (24) and (25)

$$4\bar{R}_1^2\{(\delta R_1)^2\} + 4\bar{R}_1\{(\delta R_1)^3\} + \{(\delta R_1)^4\} = \frac{4\bar{R}_1^2(1-\bar{R}_1^2)^2}{N},$$

or again neglecting $\dfrac{1}{N^2\bar{R}_1^2}$

$$\sigma^2_{R_1} = \frac{(1-\bar{R}_1^2)^2}{N} \quad\ldots\ldots\ldots(44).$$

If, in order to obtain a formula for $\{\delta R_1\}$ when $\bar{R}_1$ is small, we were to assume that

$$\sigma^2_{R_1} = O\left(\frac{1}{N}\right),$$

then we should have

$$\{\delta R_1^2\} = 2\bar{R}_1\{\delta R_1\} + \{(\delta R_1)^2\},$$

and $$\sigma^2_{R_1} = \{(\delta R_1)^2\} - \{\delta R_1\}^2,$$

or, writing X for $\{\delta R_1\}$,

$$X^2 + 2\bar{R}_1 X = O\left(\frac{1}{N}\right) \quad\ldots\ldots\ldots(45),$$

or, if $\bar{R}_1 = 0$,

$$\{R_1\} = \{\delta R_1\} = O\left(\frac{1}{\sqrt{N}}\right) \quad\ldots\ldots\ldots(46).$$

Such an assumption would appear to remove the difficulty pointed out by Isserlis* of having $\{\delta R_1\}$ tending to ∞ as $\bar{R}_1$ tends to 0, as in equation (43). Equation (46) shows that the order of $\{\delta R_1\}$ is raised from -1 to $-\frac{1}{2}$ as $\bar{R}_1 \to 0$.

* *Loc. cit.* p. 217.

It is to be observed that when $\bar{R}_1 = 0$, then also $\rho_{12}, \rho_{13}, \dots \rho_{1n}$ all vanish and so do $D_{12}, D_{13}, \dots D_{1n}$. This follows from the equation

$$1 - \bar{R}_1^2 = (1 - \bar{R}^2_{12})(1 - \bar{R}^2_{1(2)}),$$

which shows that $\bar{R}_1 = 0$ implies $\bar{R}_{12} = \bar{R}_{1(2)} = 0$. Hence $D_{12} = 0$. Applying this to $\bar{R}_{1(2)}$, we get $D^{21}_{23} = 0$ and $\bar{R}_{1(23)} = 0$ and so on, until we arrive eventually at $\bar{R}_{1(23\dots n-1)} = 0$, or $\rho_{1n} = 0$. Since $\bar{r}_{12} = \rho_{12}\left(1 - \frac{1-\rho^2_{12}}{2N}\right)$ to our approximation, it follows that

$$\tilde{R}_1^2 = 1 - \frac{\tilde{D}}{\tilde{D}_{11}} = \frac{1}{\tilde{D}_{11}}\left(-\sum_{s=2}^{n} \bar{r}_{1s}\tilde{D}_{1s}\right) = 0,$$

when $\bar{R}_1 = 0$.

Finally we wish to express our very great indebtedness to Professor Pearson, without whose help this paper could not have been written.

Note: Since this paper has been in type, our attention has been drawn to a recent paper by Mr Fisher, *Phil. Trans.* B. Vol. 213, 1925, p. 91, in which he actually obtains the distribution of R_1 for the case $\bar{R}_1 = 0$, using his geometrical methods. In our notation, his results give

$$\{R\} = \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} \frac{\Gamma\left(\frac{N-1}{2}\right)}{\Gamma\left(\frac{N}{2}\right)} \sim \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} \sqrt{\frac{2}{N}} \text{ for large } N.$$

Further: $\{R^2\} = \frac{n-1}{N-1}$, agreeing with (24) for large N.

A STUDY OF THE BADARIAN CRANIA RECENTLY EXCAVATED BY THE BRITISH SCHOOL OF ARCHAEOLOGY IN EGYPT.

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CONTENTS.

1. *Introductory.*

In 1924—25 excavations by the British School of Archaeology in Egypt were made in the district of Badari, 30 miles south of Asyut. A settlement actually in Badari was worked by Miss Caton-Thompson, and cemeteries, a few miles to the north, exhibiting the same culture and of the same period, were worked by Mr and Mrs G. Brunton. Some sixty crania brought home were most kindly placed at the disposal of the Biometric Laboratory by Sir W. M. Flinders Petrie.

All authorities seem to agree that this Egyptian civilization is the most ancient yet discovered in Egypt, but further than this the actual dating has caused much controversy*. A detailed account of the excavations has not yet been published; some account was given in the *Proceedings of the British Association*, Sessions 1925 and 1926.

In general appearance, the skulls are very like the crania from Naqada; they are markedly dolichocephalic, smooth, fragile and very feminine in type. They are rather more prognathous than the Naqada skulls and the series may be slightly less homogeneous.

* See *Man*, Vol. xxv. No. 9, p. 129, *Nature*, Vol. cxviii. pp. 468 and 514.

2. *Measurements and Methods of Measurement.*

The direct measurements taken corresponded to those of other workers in the Biometric Laboratory, of which detailed accounts are given in *Biometrika*, Vol. I. pp. 412—419, and Vol. XIV. pp. 196—200; the definitions of cranial points given in the latter paper were followed.

F = Flower's Ophryo-occipital length. L' = Glabellar projective length. L = Glabellar occipital length. B = Maximum parietal breadth. B' = Least forehead breadth. H' = Basio-bregmatic height. H = Basion to point vertically above it (the "apex") with skull adjusted on craniophor to Frankfurt horizontal. OH = Craniophor auricular height. LB = Basion to nasion. Q = Craniophor transverse arc through "apex" and terminating at ear rods. Q' = Transverse arc terminating at auricular points. S = Arc from nasion to opisthion. S_1 = Arc nasion to bregma. S_2 = Arc bregma to lambda. S_3 = Arc lambda to opisthion. S_1' = Chord nasion to bregma. S_2' = Chord bregma to lambda. S_3' = Chord lambda to opisthion. U = Horizontal circumference. PH = Alveolar point to tip of anterior nasal spine. $G'H$ = Nasion to alveolar point. GB = Distance between points where zygomatic-maxillary sutures cross lower front ridges of zygomatic arches. J = Bizygomatic breadth. NH' = Nasion to base of anterior nasal spine. NH, R and L = Nasion to lowest edge, right and left, of pyriform aperture. NB = Greatest breadth of pyriform aperture. DS = Shortest subtense from bridge of nose to dacryal chord with Mérejkowsky's simometer. DC = Chord, dacryon to dacryon. DA = Arc, dacryon to dacryon. SS = Shortest subtense from nasal bridge to simotic chord with simometer. SC = Minimum chord between naso-maxillary sutures. O_1, R and L = Breadth of orbit, right and left, using curvature method. O_1' = Breadth of right orbit from dacryon. O_2, R and L = Height of orbit, right and left. EOW = Distance between points where the borders of the orbital ridges, right and left, meet the fronto-malar sutures. It is the same as Martin's "innere orbitale Gesichtsbreite." G_1 = Distance from point of spina nasalis posterior to imaginary line tangential to inner rims of alveoli of middle incisors. G_1' is similar to G_1, but taken from base of spine. G_2 = Distance between inner alveolar walls at 2nd molars. EH = Palate height; EB = Palate breadth, both taken with Pearson's uraniscometer. GL = Basion to alveolar point. fml = Basion to opisthion. fmb = Greatest breadth of foramen magnum. $P\angle$ = Profile angle, found by means of Ranke's goniometer when the skull is in the Frankfurt horizontal position on the craniophor. The angles ($N\angle$, $A\angle$, $B\angle$) of the fundamental triangle were calculated with the aid of Pearson's trigonometer. θ_1, the basio-nasal horizontal angle, is obtained by subtracting $N\angle$ from the supplement of $P\angle$; θ_2, the basio-alveolar horizontal angle, by subtracting $A\angle$ from $P\angle$. The Occipital Index $100\frac{S_3}{S_3'}\sqrt{\frac{S_3}{24(S_3-S_3')}}$ was obtained from Tildesley's table of this function*.

Capacity. Most of the skulls were much too fragile for the capacities to be

* *Biometrika*, Vol. XIII. p. 261.

found by any of the direct methods usually applied. The probable capacities were calculated from the following formulae:

$$\begin{array}{lll}
♂: & C = 317{\cdot}2 + {\cdot}018430\,(U \times Q' \times S) & \ldots\ldots \text{ I*.} \\
♂: & C = 524{\cdot}6 + {\cdot}000266\,(L \times B \times H') & \ldots\ldots \text{ II†.} \\
♀: & C = 812{\cdot}0 + {\cdot}000156\,(L \times B \times H') & \ldots\ldots \text{ III†.} \\
♂: & C = 372{\cdot}4 + {\cdot}000352\,(L \times B \times OH) & \ldots\ldots \text{ IV†.} \\
♀: & C = 206{\cdot}6 + {\cdot}000400\,(L \times B \times OH) & \ldots\ldots \text{ V†.} \\
♀: & C = 211{\cdot}64 + {\cdot}020{,}320\,(U \times Q' \times S) & \ldots\ldots \text{ VI‡.}
\end{array}$$

Here the arcs must be measured in *cm.* and the diameters in *mm.*, while the capacity will be given in *cm.*^{3}

The most satisfactory is perhaps No. I, based on the arcual products of the Naqada race. The mean capacity for the Badarian males was calculated from the mean of the probable capacities of individual skulls (see Appendix I) as well as from the product of the means of the direct measurements. The two values were in very close agreement, though theoretically the first should be used. Unfortunately this formula had not been calculated for the Naqada females.

Nos. II and III, the Pearson-Lee reconstruction formulae, based on eleven different races, were used to calculate the male capacities from the means of direct measurements. These values were used in calculating the Coefficients of Racial Likeness. It was not altogether a happy choice, but the values for males from this and the arcual product formula did not differ widely and probably the coefficients were not affected significantly. For the females an arcual formula for the long Egyptian "*E* Series" was used, namely No. VI above.

The formulae Nos. IV and V are based on the direct measurements of the Naqada race. The values based on the mean direct measurements were lower partly owing to a much smaller auricular height. As there may be error due to personal equation in this measurement, the capacities based on it were not used.

The following are the results obtained:

Formula	Capacity on mean direct measurements
I (♂)	1355·1 c.c.
II (♂)	1370·7 c.c.
III (♀)	1274·1 c.c.
[IV (♂)	1304·1 c.c.]
[V (♀)	1226·4 c.c.]
VI (♀)	1280·8 c.c.

The mean of individual male capacities from formula No. I = 1354·8 c.c. and for the females from formula No. VI = 1281·4 c.c.

* *Biometrika*, Vol. III. p. 370.
† *Phil. Trans.* Vol. 196, pp. 243—247.
‡ Unpublished.

Weight. The skulls were mostly complete but too fragile to be thoroughly cleaned for weighing. They had also been dipped in a preparation of wax to preserve them.

Sex. The skulls were sexed by Professor Pearson and Dr Morant; the latter also repaired skulls that had been damaged in transit.

3. *Remarks on Individual Crania.*

The skulls were examined with the assistance of Dr Morant for the following characters and anomalies which are recorded in Appendix I; where no mention is made of an anomaly it may be assumed not to be present in that skull.

Age. Where no mention is made of age, the skull may be assumed adult, the sutures closed but not obliterated, and the third molars normally erupted. The other classifications are: *child*, all sutures open, 3rd molars below alveolar border, 2nd molars not fully erupted; *young adult*, sutures not completely synostosed, 3rd molars not fully erupted; *beginning to age*, sutures beginning to be obliterated; *aged*, sutures obliterated, falling in or thinning of calvaria, loss of teeth and absorption of alveolus. About 25 °/₀ of the skulls were aged or beginning to age but no cases of thinning of calvaria and very few of absorption of alveolus were noted.

Teeth. Undeveloped or imperfectly developed 3rd molars, loss of teeth (during life), carious teeth and general condition of teeth were noted.

As with other early Egyptian races, the teeth were as a rule large, strong and regular, though remarkably worn even down to the pulp-cavity in several cases; an exceptionally large proportion of the skulls had complete dentition even when ageing. One skull had three carious molars and three had a single one each.

Two rarer anomalies in teeth were noted; in one upper jaw (No. 5123 *b*) there were only two incisors, one right and one left, while there were the normal four in the lower jaw; in the upper and lower jaws of another skull (No. 5441) three premolars were missing and some of the milk teeth had persisted in the adult, while their successors are visible embedded between alveolar border and palate. (See Plate VII.)

Palate. It has been usual in the Biometric Laboratory to examine the skulls for a bridge over the inner palatine grooves leading from the pterygo-palatine canals, but it was considered that the bridge would not have been preserved well enough in the present fragile series of skulls to give reliable percentages. One case of a remarkably strong bridge was however noted (No. 5808).

Precondyles. A pair of precondyles (No. 5400 *b*) and a single precondyle (No. 5400 *a*) were recorded and also a pair of osseous nodules on the foraminal border to the right and left of the opisthion, possibly but not certainly associated with a shallow opisthial notch* (No. 5370).

* See *Biometrika*, Vol. VIII. p. 259.

On the left margin of one foramen magnum there was an osseous bridge (No. 5808).

Tympanic Perforations. Including the smallest visible perforations, there were four cases of perforation of right and left tympanic plates and five of single perforations, right or left. As the variation in percentages of tympanic perforations among different races seemed to be so wide on the limited data available the subject was investigated a little further. We may arrange our material as follows:

One child: Central perforation on *R.*

Persistent perforations: 3 *R.* + *L.* small single central perforations, 1 *L.* ditto (possibly an ageing skull), 1 *R.* large central perforation.

Perforations in Ageing Skulls (all towards edge of plate): 1 *L.*, 1 *R.*, 1 *L* + ? *R.*

The Eastern races agree fairly closely in having about 20 °/₀ of perforated plates, while the Egyptian had only about half this number and the English Farringdon Street about 4 °/₀. The other two English series measured had not been examined for this anomaly. Miss Tildesley in her Burmese Study* was the first in the Laboratory to record it. To corroborate this low percentage in English skulls the Whitechapel series was examined and found to have 12 °/₀ of perforated plates, but this included the perforations of age as well as the persistent perforations.

This difference was so marked that it seemed probable that it was due to some difference in interpretation of tympanic perforation, for in the Farringdon Street records we have mainly tympanic "foramens" or "cribriform" tympanic

TABLE I.

The Occurrence of various Anomalies in different Races.

Race and total number of crania		Percentage Perforated Tympanic Plates	Percentage Metopic	Percentage Epipteric bones	Percentage Fronto-Temporal Articulation		
	♂ and ♀					♀	♂
Farringdon Street	(299)	14	10	6	1+†	4 cases	2 cases
Whitechapel	(254)	12	8	7	1+†	4 cases	3 cases
Motley's 1st Dynasty Egyptians	(47)	8·3	0·0	5·2	3·2	10·0 (10)	1·4 (37)
Badarians	(59)	11·0	5·1	7·6	4·2	10·9 (23)	0·0 (36)
Tibetan A	(17)	20·6	18·8	35·9	0·0	—	0·0
Tibetan B	(15)	26·7					
Burmese A	(91)	18·7					
Burmese B	(24)	22·0	1·4	13·6	4·0	6·0 (75)	1·6 (61)
Burmese C	(26)	17·3					
Nepalese	(56)	23·0	0	25·9	2·7	0·0 (7)	3·1 (49)

* *Biometrika*, Vol. XIII. p. 235.

† The exact number of crania on which it was possible to make observations is not known for Farringdon Street and Whitechapel. The number of cases only is given for separate sexes. For other races numbers of crania examined are put in brackets after percentages.

plates, the latter all due to ageing. On this account the series was re-examined and the smallest perforations recorded so that the results might be comparable with the present series and the Eastern series of Morant. By doing this the Farringdon Street percentage was increased to 14. The lower percentage would probably be the one recognised by anatomists; the higher one is inserted for comparison with the Eastern series.

The tympanic plates though frail do not appear to have been broken in many cases except at the outer rim, so that these observations are fairly reliable and do suggest (see Table I) that there are a larger number of (? persistent) perforated plates in the Eastern than in the Western races.

Base of Pyriform Aperture. It has been usual to note the slope of the base of the nasal aperture with respect to the Frankfurt Horizontal as "upwards" if it is upwards and forwards towards the pyriform border, "flat" if it is parallel to that Horizontal, and "downwards" if downwards and forwards, and the anterior border as "sharp," "blunt" or "rounded."

Base of Aperture.

Border of Aperture.		Upwards	Flat	Downwards	?	Totals
	Sharp ...	6	15	4	1	26
	Blunt ...	1	8	13	4	26
	Rounded	0	1	3	0	4
	Totals	7	24	20	5	56

Asymmetry. *SL*, *SE*, *SR* and *JL*, *JE*, *JR* have been used to denote whether the sylvian depression and the jugular foramen respectively are greater on the "left," "equal" or greater on the "right" side. The difference in the size of the sylvian depression was very slight and difficult to appreciate by touch; here again the agreement of two independent observers was not good. My records were as follows:

	Sylvian Depression	Jugular Foramen
Greater on right side	5	23
Equal	40	6
Greater on left side...	5	12

There were a few cases of slight calvarial and facial asymmetry, mainly perhaps post-mortem, but no marked asymmetry.

Malar bones. There was one case of a faint horizontal suture across the malar bones (No. 5806).

Metopism. Three skulls (Nos. 5367, 5439, 5805) had a persistent frontal suture. Apart from the remarkably high percentage in the Tibetan series, very little can be deduced from such limited material.

Ossicles in Sutures. Several cases of ossicles of the lambda and in the lambdoid suture and at the asteria were noted but no interparietal bones. In the child's skull there were ossicles in the sagittal suture.

Conformation of the Pterion. The pterion was examined for epipteric bones, and for fronto-temporal articulation. In the percentage of epipteric bones there seems to be a marked racial difference, the English Farringdon Street being rather low and the Eastern races high. The Farringdon Street and Whitechapel series were re-examined and by including any small ossicles in the neighbourhood of the pterion a rather higher percentage was found. Here also the higher percentage is only given for comparison with the Egyptian and Eastern series which had been examined in that way.

The percentage of fronto-temporal articulation seems to differ sexually as well as racially. Le Double* gives observations on large numbers of many races compiled by J. Ranke and Anutchine. Both series of figures show that the percentage varies from 1·5 in European races to 4 in Mongoloid and 12 in the most primitive races. Le Double does not mention a sexual difference. The data of this laboratory are very limited but they show a decidedly higher percentage for females in all instances except that of the Nepalese, which gives rather more cases than we should expect for males (on 49 skulls) and none for females (on 7 skulls only).

Other Features. There was one case of a triple division of the jugular foramen (No. 5436), two of a simple division (Nos. 25/5816, 5400 *a*). There were single cases of a flattened obelion (No. 5439), a sagittal groove (No. 24/5100 *B*) and of a rounded transverse occipital protuberance (No. 5718), and also a very large ivory exostosis (No. 5100): see Plate VII. Several skulls showed marked subnasal prognathism but the anterior nasal spine was quite prominent and from the pyriform border and general appearance no skull could be singled out as distinctly negroid.

In general there were rather few anomalies in this sample of the Badarian race.

4. *On Comparative Material.*

A detailed study of the craniometric material of Egypt from Predynastic to Roman times has been made by Dr G. M. Morant†. He concludes (pp. 4—10) that in Early Predynastic times, there were two distinct races of man living in Egypt, which he calls the Upper and Lower Egyptian types. The craniometric material from the district of Abydos and Thebes is more abundant than that from any other district of similar area; it shows that there has been a gradual transition from the Upper Egyptian type to the Lower, beginning from the earliest Predynastic times of which we have record up to the Late Dynastic, when the

* *Traité des Variations des Os du Crâne de l'Homme*, pp. 302—305.

† *Biometrika*, Vol. XVII, pp. 1—52.

TABLE II. *Comparative Table of Means of Male and Female Egyptians.*

	Badarian	Early Predynastic	Naqada *A* and *Q*	Late Predynastic	1st Dynasty Private Tombs	6th—12th Dynasties	El Kubania, North	El Kubania, South	1st Dynasty Royal Tombs	"*E* Series" 26th—30th Dynasties	Badarian	Early Predynastic	Naqada All graves	Late Predynastic	1st Dynasty Private Tombs	El Kubania, North	El Kubania, South	"*E* Series" 26th—30th Dynasties
	♂	♂	♂	♂	♂	♂	♂	♂	♂	♂	♀	♀	♀	♀	♀	♀	♀	♀
F	182·0 (36)	182·6 (53)	184·1 (71)	183·5 (127)	182·4 (43)	181·4 (175)	—	—	182·3 (36)	182·8 (895)	177·5 (22)	—	177·4 (168)	—	—	—	—	175·9 (590)
L'	182·1 (34)	—	185·4 (87)	—	—	—	181·8 (37)	182·5 (70)	185·5 (31)	184·6 (872)	176·1 (21)	—	177·9 (159)	—	—	—	—	176·4 (579)
L	182·3 (36)	183·5 (54)	184·7 (101)	185·1 (127)	183·8 (43)	182·5 (175)	182·2 (37)	182·7 (66)	184·9 (36)	185·3 (895)	176·7 (22)	177·1 (59)	177·5 (185)	177·5 (136)	177·9 (60)	175·9 (22)	176·7 (50)	177·1 (589)
B	130·8 (36)	131·4 (53)	132·7 (88)	135·4 (125)	133·7 (41)	134·3 (173)	134·2 (37)	135·9 (70)	136·2 (37)	138·9 (896)	130·3 (21)	128·9 (61)	131·5 (185)	130·7 (134)	130·9 (59)	131·3 (22)	133·6 (50)	135·6 (587)
B'	91·1 (36)	—	92·6 (101)	—	—	—	90·4 (37)	92·7 (70)	93·0 (35)	94·8 (935)	89·4 (22)	—	88·2 (181)	—	—	89·5 (22)	90·8 (49)	92·3 (628)
H	133·4 (32)	—	136·4 (71)	—	—	—	—	—	133·8 (30)	134·1 (884)	129·1 (21)	—	—	—	—	—	—	—
H'	132·9 (34)	134·0 (40)	133·8 (70)	132·7 (107)	135·1 (38)	134·8 (171)	135·9 (35)	132·9 (65)	132·1 (35)	132·4 (884)	129·1 (22)	128·9 (49)	129·5 (169)	128·4 (127)	128·6 (57)	130·0 (19)	128·6 (43)	129·2 (582)
OH	111·0 (34)	—	115·6 (98)	—	—	—	116·1 (36)	115·0 (66)	110·3 (31)	113·4 (887)	108·6 (21)	—	113·1 (174)	—	—	—	—	110·0 (588)
LB	99·3 (35)	102·0 (36)	101·4 (69)	101·6 (99)	102·3 (30)	100·8 (167)	101·6 (34)	100·1 (63)	102·0 (33)	101·6 (896)	96·1 (22)	97·5 (46)	94·9 (141)	96·8 (116)	96·5 (55)	96·1 (19)	95·3 (42)	96·5 (595)
Q	299·8 (34)	—	—	—	—	—	—	—	304·9 (31)	307·5 (876)	296·1 (20)	—	—	—	—	—	—	298·3 (576)
Q'	302·0 (34)	—	302·9 (87)	—	—	—	—	—	305·5 (30)	307·2 (878)	298·8 (20)	—	296·5 (151)	—	—	—	—	296·4 (579)
S	372·0 (35)	—	372·6 (90)	—	—	—	372·7 (33)	371·8 (69)	369·0 (29)	371·9 (884)	363·4 (22)	—	363·6 (151)	—	—	362·5 (20)	364·4 (45)	359·8 (583)
S_1	127·0 (35)	—	—	—	—	—	126·0 (36)	128·2 (69)	123·7 (32)	127·4 (928)	123·2 (22)	—	—	—	—	—	—	123·9 (617)
S_2	129·1 (35)	—	—	—	—	—	130·9 (33)	127·8 (67)	129·4 (34)	128·5 (908)	128·5 (22)	—	—	—	—	—	—	123·9 (606)
S_3	115·7 (34)	—	116·9 (85)	—	—	—	115·1 (30)	116·0 (66)	116·9 (32)	115·2 (889)	111·7 (22)	—	—	—	—	117·5 (20)	111·2 (42)	111·6 (593)
S_3'	96·8 (34)	—	96·5 (84)	—	—	—	97·3 (30)	97·8 (66)	96·0 (32)	97·1 (885)	93·8 (22)	—	—	—	—	98·8 (20)	93·8 (42)	94·7 (594)
U	501·3 (36)	—	510·4 (90)	—	—	—	505·6 (35)	506·2 (69)	513·5 (30)	518·7 (876)	489·0 (22)	—	493·7 (146)	—	—	490·2 (20)	495·8 (50)	499·7 (583)
PH	19·5 (34)	—	20·1 (56)	—	—	—	—	—	21·3 (34)	—	19·1 (19)	—	—	—	—	—	—	—
G'H	67·1 (34)	69·2 (39)	69·7 (52)	70·0 (97)	70·6 (29)	69·9 (169)	70·1 (29)	70·2 (62)	74·7 (31)	70·4 (845)	64·8 (20)	67·0 (48)	65·8 (119)	66·7 (111)	67·5 (52)	67·3 (15)	66·4 (40)	66·7 (567)
GB	94·6 (34)	—	95·9 (60)	—	—	—	94·9 (29)	95·2 (57)	97·3 (33)	95·2 (877)	89·6 (18)	—	92·5 (113)	—	—	—	—	91·6 (581)
J	122·5 (32)	124·3 (34)	125·9 (37)	127·0 (87)	130·9 (25)	124·5 (157)	125·5 (26)	125·7 (54)	129·7 (31)	128·7 (785)	117·7 (13)	118·9 (45)	117·0 (63)	120·7 (97)	121·7 (50)	117·6 (15)	119·7 (32)	20·9 (550)
NH'	47·2 (34)	50·1 (37)	50·0 (54)	50·1 (102)	51·4 (30)	51·2 (168)	49·8 (37)	49·4 (65)	53·7 (32)	—	44·9 (20)	47·8 (46)	—	47·9 (112)	48·2 (54)	47·7 (17)	47·7 (42)	—
NH (*R*)	48·4 (33)	—	50·2 (54)	—	—	—	—	—	53·9 (32)	51·5 (898)	46·0 (21)	—	—	—	—	—	—	} 49·0 (593)
NH (*L*)	48·4 (34)	—	—	—	—	—	—	—	54·0 (32)	—	45·6 (20)	—	46·7 (123)	—	—	—	—	
NB	24·9 (34)	25·2 (37)	25·0 (64)	25·5 (102)	25·8 (30)	25·8 (167)	24·8 (36)	25·0 (63)	25·8 (30)	24·4 (893)	23·6 (20)	24·1 (45)	24·3 (119)	24·7 (112)	25·0 (54)	23·4 (20)	24·5 (41)	23·6 (591)
DS	11·7 (32)	—	—	—	—	—	—	—	12·2 (29)	—	11·1 (18)	—	—	—	—	—	—	—
DC	22·4 (32)	—	—	—	—	—	23·4 (36)	23·3 (59)	22·8 (34)	—	20·6 (21)	—	—	—	—	—	—	—
DA	34·5 (32)	—	—	—		—	—	—	35·8 (29)	—	32·5 (18)	—	—	—	—	—	—	—
SS	4·2 (34)	—	—	—		—	—	—	4·6 (30)	—	4·0 (19)	—	—	—	—	—	—	
SC	11·1 (34)	—	—	—		—	—	—	10·9 (34)		10·6 (20)	—	—	—	—	—	—	—
O_1 (*R*)	40·6 (33)	—	41·6 (57)	—	—	—	10·8 (36)	41·3 (66)	42·8 (33)	41·1 (886)	40·0 (20)	—	41·3 (113)	—	—	—	—	39·9 (589)
O_1 (*L*)	40·0 (34)	—	—	—	—	—	—	—	42·3 (31)	40·8 (880)	39·4 (18)	—	41·7 (118)	—	—	—	—	39·5 (587)
O_2 (*R*)	32·0 (34)	—	32·2 (57)	—			32·3 (35)	32·8 (66)	33·5 (33)	33·7 (888)	31·3 (21)	—	31·9 (116)	—	—	33·1 (20)	33·4 (44)	33·7 (590)
O_2 (*L*)	32·1 (33)	—	—	—		—	—	—	33·9 (31)	33·8 (888)	31·4 (18)	—	32·2 (117)	—	—	—	—	33·5 (595)
O_1'	38·4 (33)	—	39·1 (57)	—	—	—	38·1 (35)	38·4 (64)	40·0 (32)	38·4 (886)	37·6 (20)	—	—	—	—	37·4 (20)	37·9 (43)	—
EOW	94·2 (34)	—	96·1 (56)	—	—	—	—	—	98·4 (37)	—	90·5 (21)	—	—	—	—	—	—	—
G_1	51·2 (34)	—	52·5 (49)	—	—	—	—	—	54·6 (32)	49·7 (817)	48·8 (18)	—	53·9 (103)	—	—	—	—	47·4 (552)
G_1'	46·3 (34)	—	48·0 (49)	—		—	46·4 (25)	47·0 (55)	49·5 (32)	46·3 (821)	44·7 (17)	—	—	—	—	44·0 (14)	44·5 (33)	44·5 (560)
G_1''	55·7 (34)	—	—	—	—	—	—	—	—	—	53·3 (18)	—	—	—	—	—	—	—
GL	95·0 (33)	98·8 (36)	96·7 (46)	98·3 (97)	98·6 (29)	96·5 (166)	96·5 (25)	98·1 (55)	96·2 (30)	95·0 (832)	92·6 (20)	—	91·0 (106)	—	93·7 (53)	93·0 (13)	93·7 (40)	90·8 (563)
EH	12·1 (29)	—	—	—		—	—	—	14·7 (29)	—	11·1 (16)	—	—	—	—	—	—	—
G_2	37·8 (30)	—	40·7 (48)	—	—	—	—	—	42·7 (29)	38·8 (703)	36·5 (17)	—	38·9 (103)	—	—	—	—	37·0 (462)
fml	35·5 (35)	—	35·6 (70)	—	—	—	34·9 (32)	34·7 (64)	37·0 (33)	35·6 (894)	35·2 (22)	—	—	—	—	33·3 (18)	34·2 (42)	34·4 (602)
fmb	28·7 (33)	—	29·5 (70)	—	—	—	29·2 (32)	28·9 (64)	30·0 (32)	30·0 (905)	27·5 (22)	—	—	—	—	28·4 (19)	28·6 (42)	28·8 (607)
100 *B*/*L'*	71·7 (34)	—	71·6 (87)	—	—	—	73·7 (37)	74·5 (66)	73·4 (31)	75·3 (866)	73·8 (20)	—	73·9 (146)	—	—	—	—	76·9 (574)
100 *H*/*L'*	73·3 (32)	—	73·6 (71)	—	—	—	—	—	72·1 (30)	72·7 (863)	73·4 (21)	—	73·1 (141)	—	—	—	—	73·3 (570)
100 *B*/*L*	71·8 (36)	71·7 (53)	71·8 (88)	72·1 (125)	72·7 (41)	73·6 (173)	73·7 (37)	74·4 (70)	73·8 (36)	75·1 (884)	73·8 (21)	72·8 (59)	74·2 (169)	73·6 (135)	73·6 (59)	74·7 (22)	75·5 (50)	76·6 (584)
100 *H*/*L*	73·2 (32)	—	73·8 (71)	—	—	—	—	—	72·3 (30)	72·5 (871)	73·1 (21)	—	73·2 (166)	—	—	—	—	73·1 (570)
100 *B*/*H*	—	—	—	—	—	—	—	—	102·1 (30)	103·7 (873)	—	—	101·6 (163)	—	—	—	—	105·0 (570)
100 *H'*/*L*	73·1 (34)	73·3 (40)	72·4 (70)	71·8 (105)	73·6 (38)	73·9 (171)	74·7 (35)	72·9 (65)	71·3 (34)	71·5 (884)	73·1 (22)	72·8 (49)	—	72·3 (130)	72·3 (59)	73·9 (19)	73·0 (43)	73·1 (570)
100 *B*/*H'*	98·3 (34)	98·1 (40)	99·2 (70)	100·5 (107)	99·0 (38)	99·6 (171)	98·7 (35)	102·3 (65)	103·6 (35)	—	101·0 (21)	100·0 (49)	—	101·8 (130)	101·8 (58)	101·2 (19)	103·9 (43)	—
100 (*B* − *H'*)/*L*	−1·3 (34)	−1·6 (40)	−·6 (70)	·3 (105)	−·9 (38)	−·3 (171)	−·9 (35)	1·6 (65)	2·6 (34)	3·2 (884)	0·7 (21)	0·0	1·1	1·3	1·3	0·7	2·8	3·6 (586)
100 *G'H*/*GB*	70·9 (34)	—	72·7 (52)	—	—	—	73·9 (29)	73·7 (57)	76·8 (30)	74·1 (823)	71·7 (17)	—	70·4 (106)	—	—	73·3 (15)	71·5 (38)	73·0 (547)
100 *NB*/*NH* (*R*)	51·5 (33)	—	49·8 (54)	—	—	—	—	—	47·5 (31)	47·3 (881)	51·6 (20)	—	} 52·3 (113)	—	—	—	—	} 47·7 (580)
100 *NB*/*NH* (*L*)	51·6 (34)	—	—	—	—	—	—	—	—	—	51·8 (20)	—		—	—	—	—	
100 *NB*/*NH'*	53·0 (34)	50·3 (37)	50·0 (54)	51·0 (101)	50·4 (30)	50·4 (167)	49·9 (36)	50·8 (63)	47·7 (32)	—	52·7 (20)	50·4 (45)	—	51·6 (112)	51·9 (54)	49·2 (17)	51·3 (41)	—
100 O_2/O_1 (*R*)	78·7 (33)	—	77·4 (46)	—	—	—	79·2 (35)	79·4 (66)	78·2 (33)	82·2 (878)	78·1 (20)	—	76·9 (111)	—	—	—	—	84·0 (585)
100 O_2/O_1 (*L*)	80·4 (33)	—	—	—	—	—	—	—	80·4 (31)	83·1 (876)	79·3 (17)	—	77·3 (113)	—	—	—	—	85·4 (584)
100 O_2/O_1'	83·3 (33)	—	82·4 (57)	—	—	—	84·9 (35)	85·7 (64)	86·1 (32)	—	83·4 (20)	—	—	—	—	88·6 (20)	88·2 (43)	—
100 *fml*/*fmb*	82·1 (33)	—	82·9 (70)	—	—	—	84·0 (31)	83·4 (63)	81·5 (32)	84·5 (889)	78·4 (22)	—	—	—	—	85·1 (18)	84·1 (40)	83·9 (597)
100 *DS*/*DC*	52·6 (32)	—	—	—	—	—	—	—	53·5 (29)	—	53·7 (18)	—	—	—	—	—	—	—
100 *SS*/*SC*	38·7 (34)	—	—	—	—	—	—	—	42·4 (30)	—	37·5 (18)	—	—	—	—	—	—	—
100 G_2/G_1	75·0 (30)	—	77·5 (48)	—	—	—	—	—	78·6 (29)	83·9 (703)	74·5 (15)	—	72·3 (113)	—	—	—	—	78·4 (432)
100 *EH*/*EB*	31·8 (29)	—	—	—	—	—	—	—	34·4 (29)	—	30·0 (16)	—	—	—	—	—	—	—
P∠	84·0 (33)	—	84·6 (50)	—	—	—	85·6 (28)	84·9 (58)	86·1 (32)	85°·8 (795)	83·3 (20)	—	84·5 (89)	—	—	85·9 (15)	84·3 (37)	84°·9 (529)
N∠	66·4 (33)	67·0 (36)	65·5 (46)	66·8 (97)	66·2 (29)	65·7 (166)	65·2 (25)	67·5 (55)	63·4 (30)	63°·9 (818)	67·1 (20)	67·2 (46)	72·6 (93)	69·0 (113)	66·8 (53)	66·4 (16)	68·2 (40)	64°·5 (550)
A∠	73·6 (33)	72·6 (36)	73·3 (46)	72·4 (97)	72·8 (29)	72·8 (166)	73·4 (25)	71·0 (55)	72·4 (30)	74°·1 (818)	73·0 (20)	71·8 (46)	66·4 (93)	70·1 (113)	71·8 (53)	72·1 (16)	70·9 (40)	73°·8 (550)
B∠	40·1 (33)	40·0 (36)	41·2 (46)	40·8 (97)	41·0 (29)	41·5 (166)	4 ·4 (25)	41·5 (55)	44·2 (30)	41°·9 (818)	39·7 (20)	—	41·0 (93)	—	—	41·5 (16)	40·9 (40)	41°·6 (550)
θ_1	29·6 (33)	—	29·9 (46)	—	—	—	29·2 (25)	27·6 (55)	30·3 (30)	30°·0 (795)	29·5 (20)	—	29·1 (89)	—	—	27·7 (15)	27·5 (38)	30°·3 (550)
θ_2	10·4 (33)	—	11·3 (46)	—	—	—	12·2 (25)	13·9 (55)	13·9 (30)	11°·9 (795)	10·4 (20)	—	11·8 (89)	—	—	13·8 (15)	13·4 (38)	11°·3 (550)
Occ. I.	60·6 (34)	—	60·2 (84)	—	—	—	62·3 (30)	62·0 (66)	59·9 (32)	61·5 (884)	61·1 (22)	—	—	—	—	60·9 (20)	61·1 (42)	62·0 (592)
Capacity	1370·7 (35)	1382·8 (61)	1388·6 (61)	1396·2 (107)	—	—	1355·0 (27)	1366·1 (67)	1364·3 (24)	1438·9 (753)	1274·1 (22)	1271·0 (56)	1287·9 (123)	1265·2 (125)	1274·8 (56)	1280·4 (21)	1285·6 (48)	1301·1 (472)

To face p. 117.

type is almost pure Lower Egyptian. The Royal Retainers of the 1st and 2nd Dynasties buried in Abydos are of the Lower Egyptian type but are distinct from the native population and therefore are probably intrusive in Upper Egypt. No population of the Upper type has been found in Lower Egypt, but from this district the material is much less abundant. The type however is well defined by the long "*E* Series" of skulls of the 26th—30th Dynasties from Gizeh. There are slight divergences in a few of the series, but none that are at all marked except in a primitive dolichocephalic race from Upper Egypt, called the Aeneolithic, which disappeared before the opening of the historic era.

The present series from Badari in the extreme north of Upper Egypt, a district which has not previously provided a series of skulls, is probably on topographical and cultural evidence earlier than any other Predynastic series so far discovered.

In the comparison which follows the prehistoric series will be represented by:

(1) The divergent Aeneolithic type. A short series of skulls from Abydos district measured by Dr Fouquet*. It is reasonable to suppose them Predynastic but it is not possible to assign to them any more definite period.

(2) Early Predynastic type.

(*a*) Skulls from near Abydos measured by Thomson and McIver†.

(*b*) Skulls from Naqada district excavated by Petrie and measured by Fawcett‡. These came from four distinct cemeteries and were at first considered to be of the same period, from investigation of culture and skull measurements. Morant, on comparing the male crania with later material and using the coefficient of racial likeness, divided the males into two slightly divergent series§. In this paper only the *A* and *Q*, or Early Predynastic series, according to Morant, with his revised means are considered. The remainder, the *B*, *T* and *R* series, are too short to give satisfactory comparison with the short Badari series. For females the means based on all the skulls have been used.

(3) The Late Predynastic series from the Thebaid measured by Thomson and McIver†.

The following series have been selected to represent the later period, the long "*E* Series" as Lower Egyptian type and the others as transitional or slightly divergent types:

(4) 1st Dynasty Private Tombs from Abydos measured by Thomson and McIver‖.

(5) 6th—12th Dynasties from the Thebaid‖.

* Quoted from De Morgan, *Biometrika*, Vol. XVII. pp. 13—17. See also p. 52, References (10) and (11).

† *The Ancient Races of the Thebaid.* Means given, *Biometrika*, Vol. XVII. p. 14.

‡ *Biometrika*, Vol. I. pp. 408—467.

§ *Biometrika*, Vol. XVII. pp. 14, 15.

‖ *The Ancient Races of the Thebaid.* Means given, *Biometrika*, Vol. XVII. pp. 24, 38.

(6) Skulls from two graveyards in the extreme south of Egypt (El Kubania) belonging to early and middle dynasties measured by Toldt*.

(7) Skulls of Royal Retainers of 1st Dynasty from Abydos measured by Motley*.

(8) Long "*E* Series" from Gizeh (26th—30th Dynasties) reduced by Davin and Pearson*.

The mean values for these series are given in Table II.

5. *The Coefficient of Racial Likeness.*

The coefficient of racial likeness

$$\frac{1}{M}\left\{\frac{(M_s - M_s')^2}{\dfrac{\sigma_s^2}{n_s} + \dfrac{\sigma_s'^2}{n_s'}} - 1,\right.\dagger$$

provides a measure of the resemblance between two races based on the means of a number of characters which should have little—or theoretically no—inter-racial correlation. For discussion of theory and precautions in use see *Biometrika*, Vol. XVIII. pp. 105—117, Vol. XVI. pp. 11—14. All of the thirty-one characters selected by Morant that were available have been used in calculating the coefficients, but unfortunately for the series measured by Thomson and M^cIver only fifteen were given. The number of characters on which the coefficient is based has been placed in brackets after it in the Tables. Frequently it was necessary to replace NH' by $NH(R)$, H by H', $O_1'(R)$ by $O_1(R)$ and correspondingly for the indices.

Fouquet's Aeneolithic series gave $17{\cdot}27 \pm {\cdot}21$ (20) for "all characters" and $2{\cdot}13 \pm {\cdot}39$ (6) for "angles and indices." The difference between the two coefficients shows that the races diverge mainly in size as was found for the Aeneolithic with other Egyptian races. The divergence for "all characters" is fairly marked.

Table III*a* gives the coefficients of racial likeness between the various Predynastic series.

These results show that the Badarian is undoubtedly of the Upper Egyptian type, and conforms to the general sequence of the Predynastic series when placed as the most primitive. But whereas the Naqada, intermediate between the Early and Late Predynastic, is scarcely racially distinct from either, the Badarian is somewhat different from the Early Predynastic and more so from the Late. The characters causing this divergence will be considered in the next section.

Taking only the first row in Table III*a* and Table III*b* we have the Coefficients of Racial Likeness between the Badarian and other Egyptian races placed in the sequence of Upper to Lower Egyptian type as deduced by Morant. Except for the two series from El Kubania North and South these coefficients corroborate the sequence, although of course too much weight must not be placed on the

* *Biometrika*, Vol. XVII. pp. 24, 36. See also p. 52, Reference (20).

† Throughout this paper the σ's of the "*E* Series" have been used in calculating the coefficients. See *Biometrika*, Vol. XVI. p. 338.

TABLE III *a*.

Coefficients of Racial Likeness between some Predynastic Egyptian Series (Males).

Race		Badarian	Early Predynastic	Naqada *A* and *Q*	Late Predynastic
	Number of skulls	33	41	66	106
Badarian	All Characters Angles and Indices	— —	2·19 ± ·25 (15) 1·00 ± ·39 (6)	3·02 ± ·17 (31) 1·01 ± ·39 (12)	6·46 ± ·25 (15) 3·17 ± ·39 (6)
Early Predynastic	All Characters Angles and Indices	2·19 ± ·25 (15) 1·00 ± ·39 (6)	— —	·31 ± ·26 (14) ·53 ± ·38 (6)	2·17 ± ·26 (14) 2·09 ± ·38 (6)
Naqada *A* and *Q*	All Characters Angles and Indices	3·02 ± ·17 (31) 1·01 ± ·39 (12)	·31 ± ·26 (14) ·53 ± ·38 (6)	— —	·73 ± ·26 (14) 1·62 ± ·38 (6)
Late Predynastic	All Characters Angles and Indices	6·46 ± ·25 (15) 3·17 ± ·39 (6)	2·17 ± ·26 (14) 2·09 ± ·38 (6)	·73 ± ·26 (14) 1·62 ± ·38 (6)	— —

TABLE III *b*.

Coefficients of Racial Likeness between the Badarian and Dynastic Egyptians (Males).

Race	1st Dynasty Private Tombs	6th—12th Dynasties	El Kubania North	El Kubania South	1st Dynasty Royal Tombs	"*E* Series" 26th—30th Dynasties
Number of skulls	34	169	33	63	33	885
All Characters ... Angles and Indices	7·68 ± ·25 (15) ·90 ± ·39 (6)	8·14 ± ·25 (15) 4·66 ± ·39 (6)	2·59 ± ·18 (28) 3·09 ± ·29 (11)	4·65 ± ·18 (28) 6·90 ± ·29 (11)	11·69 ± ·17 (31) 9·52 ± ·28 (12)	20·97 ± ·17 (31) 21·62 ± ·28 (12)

exact values of the coefficients as they are based on series of very different length and on different numbers of characters. The El Kubania North series is of the same length as the 1st Dynasty Private Tombs but has a coefficient for all characters of only half the size even when both are based on the same characters. For six indices the coefficient for the 1st Dynasty is curiously low. The El Kubania South, but not the North, diverged slightly from other series of the same period. The resemblance with the Badarian cannot arise from geographical proximity since they come one from the extreme South and the other from the extreme North of Upper Egypt. However the numbers of skulls are too small to stress very much this resemblance.

The Badarian was not found similar to any other aberrant Egyptian series nor to the allied Abyssinian and Sardinian races.

The number of female Badarian skulls is too small for reliable conclusions to be drawn from them. Table IV is given to show that they more or less support those drawn from the male coefficients.

TABLE IV.

Coefficients of Racial Likeness between the Badarian Females (20) *and other Egyptian Series of Females.*

Race	Early Predynastic	Naqada All graves	Late Predynastic	1st Dynasty Private
Number of skulls	50	128	120	55
All Characters ...	1·62 ± ·25 (15)	2·63 ± ·18 (27)	3·33 ± ·25 (15)	2·85 ± ·25 (15)
Angles and Indices	·72 ± ·39 (6)	– ·14 ± ·30 (10)	2·64 ± ·39 (6)	– ·24 ± ·39 (6)

Race	El Kubania North	El Kubania South	"*E* Series" 26th–30th Dynasties
Number of skulls	18	43	566
All Characters ...	1·91 ± ·18 (28)	3·30 ± ·18 (28)	8·18 ± 17 (31)
Angles and Indices	3·16 ± ·29 (11)	3·16 ± ·29 (11)	10·8 ± 28 (12)

6. *Comparison of Mean Direct Measurements.*

The Coefficients of Racial Likeness suggest that the Badarian series is definitely Upper Egyptian in type and most like the earliest Predynastic from which they scarcely differ significantly when the personal equations of the recorders and errors of random sampling are taken into account. The means of the direct measurements and (following the usual notation) the values of "α" show where the main differences lie:

$$= \frac{n_s n_s'}{n_s + n_s'}\left(\frac{M_s - M_s'}{\sigma_s}\right)^2 \quad (\text{C.R.L.} = \Sigma(\alpha) - 1).$$

$\sqrt{\alpha}$ is the ratio of a difference to its standard deviation in sampling on the supposition that the means of the two populations are the same. Referring $\sqrt{\alpha}$ to the normal scale, we find that deviations of 2, 3 and 4 times the probable error correspond to values of α of 1·8, 4·1 and 7·3 respectively. Following earlier usage values of α of over 6·0 may be picked out as indicating clearly significant differences in the mean characters of two races.

For divergent characters some account must be taken of the number of skulls

when considering the value of α. Thus for the Lower Egyptian type represented by the "*E* Series" the values will be magnified*.

Table V gives the values of α between the Badarian and other Egyptian types for characters used in calculating the coefficient of racial likeness. For characters which have significant α's, the values increase fairly regularly from the Predynastic to the 1st Dynasty (Private Tombs); the Middle Dynasties series present slight differences and are on the whole rather similar to each other, but the relative values of α in the long "*E* Series" are in general typical of the relative degree of variation from character to character through all the types. The means of some characters in Motley's 1st Dynasty Royal Retainers show extraordinarily great divergence.

There are four significant α's between the Badarian and Early Predynastic series. One is the skull measurement LB, the nasio-basal length, and the others are facial. Comparing the α's with the mean direct measurements repeated for convenience in Table VI, we see that LB is significantly smaller in the Badarian than in any other race except the El Kubania South. The others do not differ widely among themselves and show no regular variation. The highest value of α between the Badarian and Early Predynastic is that for the nasal height NH' ($\alpha = 17{\cdot}5$). The values increase to the 6th—12th Dynasties, are smaller for the El Kubania series and very large for Motley's 1st Dynastic. The corresponding nasal index NB/NH' varies similarly but the α's are rather less. There is also this type of variation in the facial height $G'H$ and facial index $G'H/GB$. (GB unfortunately is not given by Thomson and M^cIver.) The mean direct measurements show that the Badarians have shorter facial and nasal heights than any other series and that Motley's 1st Dynastic have considerably longer ones. The variations in the other types are not very marked but there is just a tendency for the heights to increase towards the Lower Egyptian type. The breadth measurements increase also very slightly so that the divergence in direct length measurements is greater than in indices. For these divergent facial characters we have on the whole the Badarian and Motley's 1st Dynastic series forming extremes and the other types collected almost midway between them.

The palatine width G_2 is only given for the Naqada, Motley's 1st Dynasty and the "*E* Series"; the first two diverge markedly from the Badarian but the "*E* Series" does not. The index does not diverge significantly in any series so that the difference may be merely one of size.

The characters which have no marked or regular variation are:

$$Occ.\ I.,\ \frac{fml}{fmb},\ \frac{G_2}{G_1},\ H',\ S,\ NB,\ fml,\ O_1',\ G_1,\ C,\ A\ \angle;$$

* For example in Table V in comparing the 33 Badarian crania with the 33 1st Dynasty Royal Tombs, the 169 6th to 12th Dynasties and the 885 "*E* Series" 26th to 30th Dynasties crania we must bear in mind that $n_s n_{s'}/(n_s + n_{s'})$ takes the values 16·5, 27·6 and 31·8 respectively. A comparison of the values recorded in Table V shows at once, however, that other factors are far more influential than these relative numbers.

TABLE V. *Values of* $\alpha = \frac{n_s n_s'}{n_s + n_s'} \left(\frac{M_s - M_s'}{\sigma_s}\right)^2$ *between Badarian and Egyptian Series.*

	Early Pre-dynastic	Naqada *A* and *Q*	Late Pre-dynastic	1st Dynasty Private Tombs	6th–12th Dynasties	El Kubania North	El Kubania South	1st Dynasty Royal Tombs	"*E* Series" 26th–30th Dynasties	Early Pre-dynastic	Naqada All graves	Late Pre-dynastic	1st Dynasty Private Tombs	El Kubania North	El Kubania South	"*E* Series" 26th–30th Dynasties
Sex	♂	♂	♂	♂	♂	♂	♂	♂	♂	♀	♀	♀	♀	♀	♀	♀
No. of skulls	41	66	106	34	169	35	64	33	885	49	128	120	55	18	43	566
B/L	·03	·00	1·91	2·16	13·45	9·18	22·38	10·03	58·97	2·37	·45	·11	·09	1·33	6·53	24·26
H'/L	·09	1·30	5·02	·52	2·10	5·11	·10	6·37	9·70	·16	·02	1·48	1·26	·80	·02	0·00
B/H'	·04	1·00	6·76	·48	2·59	·15	19·32	26·20	77·14	·89	·40	·70	·60	·02	7·16	19·57
Occ. I.	—	·36	—	—	—	4·23	4·04	·74	2·44	—	—	—	—	·04	·00	1·68
$G'H/GB$	—	2·71	—	—	—	5·73	6·79	22·55	13·59	—	1·23	—	—	1·02	·23	1·39
O_2/O_1'	—	·67	—	—	—	1·71	·42	5·00	14·42	—	·52	—	—	12·83	14·93	31·95
NB/NH'	8·85	12·87	6·98	7·38	13·09	11·52	7·33	29·39	39·58	5·16	·59	1·45	·66	7·92	1·85	17·63
fml/fmb	—	·43	—	—	—	1·72	1·09	·17	5·47	—	—	—	—	14·07	14·60	21·10
G_2/G_1	—	2·50	—	—	—	—	—	4·15	3·59	—	1·45	—	—	—	—	5·09
$P\angle$	—	·68	—	—	—	3·69	1·62	6·82	9·78	—	2·70	—	—	6·68	1·49	5·67
$N\angle$	·57	1·40	·36	·06	1·23	1·87	2·38	12·90	18·09	·01	·76	5·77	·12	·41	1·52	12·27
$A\angle$	1·41	·14	2·96	·83	1·47	·05	11·65	1·89	6·63	1·73	·23	12·28	1·80	·62	5·06	1·06
L	·95	4·67	6·43	1·35	·04	·01	·12	3·72	9·52	·72	1·73	1·67	2·34	·02	·25	·24
B	·31	4·07	8·34	7·11	16·11	9·31	27·29	23·48	100·21	1·50	1·33	·14	·27	·53	7·88	27·88
B'	—	3·64	—	—	—	·55	3·71	3·91	28·81	—	1·97	—	—	·01	2·07	12·45
H'	·88	·14	·04	3·43	2·09	6·13	·21	·44	·32	·03	·16	·48	·21	·43	·19	·01
LB	8·21	6·50	8·68	9·22	4·13	5·79	·91	7·86	11·31	2·35	2·21	·73	·20	·00	·75	·27
Q'	—	·20	—	—	—	—	—	2·00	9·05	—	1·20	—	—	—	—	1·43
S	—	·13	—	—	—	·05	·01	·91	·00	—	·01	—	—	·08	·13	2·51
U	—	11·23	—	—	—	1·73	3·00	12·85	55·22	—	3·05	—	—	·11	5·11	17·61
$G'H$	4·65	8·07	12·30	11·13	12·89	8·18	12·26	54·39	20·67	4·83	1·21	4·33	7·45	3·93	2·41	4·93
J	2·56	9·50	22·69	47·43	5·09	6·18	9·85	39·09	56·61	·76	·28	5·38	8·61	·00	1·93	6·77
NH'	17·48	19·18	25·14	32·96	53·04	14·04	12·66	81·65	39·33	17·34	·74	22·59	23·51	10·66	15·71	21·87
NB	·51	·07	2·93	4·12	7·13	·56	·07	1·93	2·62	1·29	3·12	7·63	10·63	·14	4·90	·00
O_1'	—	3·67	—	—	—	·55	3·86	14·91	2·85	—	11·82	—	—	·17	·23	·08
O_2	—	·23	—	—	—	·43	3·93	10·32	25·93	—	1·88	—	—	9·79	18·49	28·96
G_1	—	3·06	—	—	—	·01	·93	17·18	·00	—	45·49 ?	—	—	·44	·05	3·90
G_2	—	22·44	—	—	—	—	—	54·30	4·16	—	13·13	—	—	—	—	·64
fml	—	·04	—	—	—	·99	2·37	6·27	·05	—	—	—	—	7·45	3·01	2·83
fmb	—	3·11	—	—	—	·88	·19	5·94	11·65	—	—	—	—	2·06	4·37	8·97
C	·26	·56	1·36	2·04	2·47	·30	·04	·05	12·40	·00	·43	·18	·00	·05	·24	1·85

In computing this table of the values of α and of the corresponding coefficients small allowances have been made when the characters measured were not quite identical, i.e. Q for Q', NH for NH', H for H', etc.

TABLE VI.

Some Mean Direct Measurements of Characters which diverge in the Badarian Series.

	Badarian	Early Pre-dynastic	Naqada *A* and *Q*	Late Pre-dynastic	1st Dynasty*	6th–12th Dynasties	El Kubania North	El Kubania South	"*E* Series" 26th–30th Dynasties	Motley's 1st Dynasty
LB	99·3	102·0	101·4	101·6	102·3	100·8	101·6	100·1	101·6	102·0
NH'	47·2	50·1	50·0	50·1	51·4	51·2	49·8	49·4	51·5	53·7
NB/NH'	53·0	50·3	50·0	51·0	50·4	50·4	49·9	50·8	47·4	47·7
$G'H$	67·1	69·2	69·7	70·0	70·6	69·9	70·1	70·2	70·4	74·7
$G'H/GB$	70·9	—	72·7	—	—	—	73·9	73·7	74·1	76·8

* From private tombs.

characters increasing towards Lower Egyptian type:

$$B/L,\ B/H',\ B,\ U,\ G'H,\ J,\ NH',\ O_2R,\ \frac{G'H}{GB},\ Q'(?),\ G_2(?);$$

and increasing less significantly:

$$L(?),\ \frac{O_2}{O_1}(R),\ P\angle,\ B',\ fmb,\ LB;$$

characters decreasing: $\frac{NB}{NH'}$ (markedly), H'/L, $N\angle$, just significantly.

Thus of the 31 characters, 11 were practically constant, 17 increasing, and 3 decreasing.

Of the characters not used in the Coefficient of Racial Likeness, GB is constant, $\frac{B-H'}{L}$ increases from negative in Predynastic to positive in the "*E* Series."

Thus the direct measurements of the Badarian race support Morant's conclusions that the breadth of the skull and of the face show steady and considerable increase as we pass from Predynastic to late Dynastic times; the length and height measurements of the skull show very little change but the height measurements of the face increase.

The nasal and orbital breadths have remained practically constant but the heights of each have increased; the measurements of the foramen magnum have varied very little.

The comparison between the series from El Kubania and the Badarian showed rather lower coefficients than for the other Dynastic series. This is due mainly to facial measurements; for while the parietal breadths and corresponding indices would place them in the Middle Dynasties, the bizygomatic breadth is much nearer the Naqada. The length of skull and measurements involving it are almost identical with the Badarian.

There is a curious difference in the Coefficients of Racial Likeness based on "angles and indices" and those based on "all characters" for the Predynastic and Early Dynastic series. The skulls are extremely difficult to sex, being all

of a frail and feminine type and this difference may possibly be due to sexing, or it may be that the Badarian type is proportionally smaller. There is a similar tendency towards smallness in the primitive Eastern races, which also appears in the difference between the two coefficients.

Modern Abyssinians* of the Tigre District are less dolichocephalic than the Badarians owing to a much greater parietal breadth; characters involving this measurement largely account for the fairly high value of the coefficient, 7·07 for "all characters" and 7·74 for "angles and indices." Modern Sardinians† give coefficients of the same order but diverge mainly in facial characters and horizontal circumference. Both of these races are closely allied to the Egyptians, the latter to the Naqadas in particular.

A Negro admixture in Egyptian races has often been discussed. The Badarian skulls appeared to be slightly prognathous, to have a short $G'H$ and NH', and in general to be somewhat nearer to a negroid type. Since cephalic index and other calvarial characters vary over a wide range for Negro races, therefore Egyptian series may easily fall within their limits. Dr Morant has selected some characters which are fairly constant for the Negro races and decidedly different for Egyptian. He has then given a sequence of Egyptian and Negro races for these characters‡. In Table VII part of his table is reproduced, replacing the Early Predynastic by the Badarians.

TABLE VII.

A Comparison of Means of Certain Characters in some Egyptian and Negroid Races.

Race	No.	NH, R	NB	$NB/NH, R$	$P\angle$	$N\angle$	$B\angle$	GL	$G'H$	$G'H/GB$	H'/L	S_3'	S_3	$Occ. I.$
"*E* Series" ...	850	51·7	24·4	47·3	85·8	63·9	41·9	95·0	70·4	74·1	71·5	94·7	115·2	61·5
Naqada *A* and *Q*	70	50·2	25·0	49·8	84·6	65·5	41·2	96·7	69·7	72·7	72·4	96·5	116·9	60·2
Badarian ...	33	48·4	24·9	51·5	84·0	66·4	40·1	95·0	67·1	70·9	73·1	96·8	115·7	60·6
Gaboon 1864 ...	50	48·2	26·6	55·3	—	70·5	38·6	100·5	66·4	69·7	75·5	95·9	109·1	68·3
Northern Negro	30	49·5	27·3	55·4	82·0	72·4	38·0	105·3	68·0	70·4	74·2	—	—	—

By introducing the Badarians the sequence becomes decidedly more marked, for several of the characters are ones in which the Badarians diverge from the other Early Egyptians. The Gaboon§ and Northern Negro§ races were selected because they are the races represented by an adequate number of skulls which are nearest in type to the Badarian. The second series does not diverge significantly in B/L, L, B, or H'. To show, however, how distantly related the types really are, the coefficients of racial likeness were calculated:

* Means in *Biometrika*, Vol. XVII. p. 36.

† *Zeitschrift für Morphologie*, Vol. XIII. p. 440.

‡ *Biometrika*, Vol. XVII. pp. 8—10.

§ For means see *Biometrika*, Vol. VIII. pp. 298, 299.

	Badari		Naqada	
	All Characters	Angles and Indices	All Characters	Angles and Indices
Gaboon	15·07 ± ·18	20·97 ± ·29	21·01 ± ·18	40·81 ± ·29
Northern Negro...	10·48 ± ·19	13·22 ± ·30	13·37 ± ·19	21·87 ± ·30

Thus, although the Badarian is nearer to these Negro types than the Dynastic series in all the characters given above except the occipital index, the basio-alveolar length, *GL*, and the nasal breadth, *NB*, it is still widely divergent and this result would suggest that if there be any real relation of the Badarian to a Negro type it is a long way back in evolutionary history.

7. *Comparison of Badarian and Primitive Indian Races.*

The Badarian is distinguished from most other races, whose mean skull measurements have been found, by the small mean parietal and bizygomatic breadths and the marked dolichocephaly. To a less marked degree, it is distinguished from races of the European type by the mean facial and nasal heights and indices also.

Of the races outside Egypt whose mean skull measurements were available, the primitive Indian were the only ones at all similar to the Badarian as measured by the Coefficient of Racial Likeness; this was only 2·40 ± ·19 for Veddah and Badarian males, that is of the same order as between the Badarian and Early Predynastic Egyptians or between the Farringdon Street and Whitechapel series both of the same period and district. Unfortunately the means of the two races were based on too few skulls (about 30 each) to lay great stress on the similarity, but the apparent agreement suggested that it was worth while collecting further data.

Anthropologists have frequently drawn attention to the similarity in the appearance of the skulls of the Hindu races of India and the Early Egyptians. In a paper "Sur l'Origine de l'Ancienne Race Égyptienne," published in the first volume of the *Mémoires de la Société d'Anthropologie de Paris*, pp. 410—422, Pruner-Bey notes this similarity only to reject it; each race, he says, has a skull, small and oval in shape; the body and extremities are for both races beautifully proportioned; but there is a marked difference in the fleshy parts, the Ancient Egyptian resembling the modern Berber, while the Hindu is bronze to bistre in colour. Finally he dismisses the idea of direct relationship on the grounds of linguistic differences, an argument which would scarcely now-a-days be advanced.

Comparative Material. To the mean skull measurements of various Indian races compiled by Tildesley and Morant have been added some short series by Turner, Mantegazza and Danielli giving the following material for comparison.

(1) *Veddahs.* The means compiled by Morant* are based on the Sarasin Collection of 22 ♂ and 11 ♀ (Lüthy and Sarasin), the Oxford Collection (Thomson), Royal College of Surgeons (Flower), Barnard Davis Collection and three measured by Virchow.

Turner† gives measurements of 7 ♂ and 2 ♀ Veddahs, three in the Henderson Trust Collection in the Edinburgh University Museum, one in the India Museum, three in Museum of Trinity College, Dublin, and two from East Ceylon in Dublin.

(2) *Dravidians.* Tildesley‡ gives the means for forty skulls belonging to the Maravar tribe, a Dravidian stock from the Madras Presidency. One half of these were sent to Paris and measured by Callamand and the other half to the Royal College of Surgeons and measured by Flower. Thirty-seven of these have been taken as male.

A second series of 15 ♂ Dravidians is given by Turner† and includes various tribes from the Central Provinces and Orissa and one Tamil from Madras. Three are from the Edinburgh University Museum, and the rest from the India Museum.

(3) *Kolarians.* Turner† gives measurements of 18 ♂ skulls in the India Museum classed as Kolarian on linguistic grounds, belonging chiefly to the Munda and Kol tribes. One of these was pathological (scaphocephalic) and has been excluded from the means. Turner excluded another on account of its high cephalic index; more recent research has shown that a distinction on such a ground only is fallacious and accordingly measurements of this skull have been included in the means.

(4) *Hindu.* (*a*) The material compiled by Tildesley§ from Flower, Barnard Davis and the German Catalogues together with a short series in the Biometric Laboratory all relate to the N.E. of India, chiefly Bengal. Care was taken to eliminate any of Dravidian origin or admixture.

(*b*) 42 ♂ skulls of the Bengalee Hindus measured by Danielli ||.

(*c*) 24 ♂ skulls of Hindus from Southern Indian measured by Mantegazza ¶. He does not state precisely the region from which they came; he separates them into tribes of high and low caste, but does not consider, himself, that this affects the type to any considerable extent. The numbers are too small to consider this division in the present paper.

(*d*) 25 ♂ skulls from the India Museum marked Uriyá**. Uriyá is the mother tongue of 90 °/ₒ of the Hindu population of Orissa who provide most of the domestic servants of N.E. India. The skulls came to the museum from the Calcutta Medical School.

* *Biometrika*, Vol. XVI. pp. 49—50.

† *Transactions of Royal Society of Edinburgh*, Vol. XV. pp. 64—110.

‡ *Biometrika*, Vol. XIII. p. 238.

§ *Biometrika*, Vol. XIII. pp. 237—238.

|| *Archivio per l' Antrop. e Etnol.*, Firenze, Vol. XXII. pp. 371—448.

¶ *Ibid.* Vol. XIII. pp. 177—241.

** *Transactions of Royal Society of Edinburgh*, Vol. XV. pp. 92—99.

On account of the wide variation in cephalic index, Turner assumed that the skulls had probably been carelessly labelled and divided them into three groups based on cephalic index—dolichocephalic, metasicephalic and brachycephalic—and he concluded that the Uriyás were a mixed race of Hinduised Dravidians and Aryans with possibly some Chinese admixture also. Since the variation does not seem more marked than we know it to be to-day in a homogeneous race, the three groups were mixed.

(5) *Nepalese**. These consist of 48 ♂ adult skulls from the Hodgson Collection and include most representatives of the native tribes of the country.

(6) *Tamils*. 35 ♂ skulls of coolies who had migrated from Southern India to Singapore were measured by Harrower†.

The corresponding female series are all very short and so have not been considered in this paper.

Comparison of Series of similar Type. For most of these series an adequate number of characters have been measured to make a comparison of means by coefficients of racial likeness fairly satisfactory. It is necessary to group the data where possible.

For Morant's series of Veddahs (24), and Turner's series (7), the coefficient for "all characters" is $1{\cdot}11 \pm {\cdot}20$ (23) and for "angles and indices" alone $1{\cdot}40 \pm {\cdot}36$. Although these coefficients are rather greater than we should expect from random sampling—about $5\frac{1}{2}$ times the probable error for "all characters"—the numbers of skulls are too few to keep the series separate at present; the divergence is probably due to the personal equation of observers. For Q', NH', O_2, O_2/O_1', and $A\angle$ the value of α was > 3 but O_2 ($\alpha = 8{\cdot}60$) and O_2/O_1' ($\alpha = 9{\cdot}24$) gave the only values of $\alpha > 5$. The means of Turner's measurements were on the whole smaller than Morant's. The combined means of the two series are given in Table VIII and have been used in this paper for comparison with other races.

Turner and others considered that the distinction between Dravidians and Kolarians of the Central Provinces and Orissa was linguistic and not physical. Comparing the mean measurements by the coefficient of racial likeness, we have for "all characters" C.R.L. $= {\cdot}03 \pm {\cdot}20$ (23) and for "angles and indices" C.R.L. $= {\cdot}54 \pm {\cdot}36$ (7) showing that there is no significant difference. The only means that differed were B ($\alpha = 3{\cdot}17$), B' ($\alpha = 3{\cdot}18$) and B/H' ($\alpha = 5{\cdot}99$), none of which show racial distinction. The two series were accordingly pooled for further comparisons.

On the other hand the skulls of the Maravars of the Madras Presidency were, from the mean skull measurements, obviously more closely allied to the various Hindu series than to the Dravidians and Kolarians of Central and N.E. India. The means of these series are given in Table VIII.

* *Biometrika*, Vol. XVI. pp. 14—26.

† *Transactions of Royal Society of Edinburgh*, Vol. LIV. Part III. p. 574.

TABLE VIII. *Some Mean Direct Measurements of the Badarian and various Indian Races.*

Character	Badarian	Dravidian	Kolarian	Dravidian & Kolarian	Veddah	Tamil	Maravar	Uriyá	Hindu (*a*)	Hindu (*b*)	Hindu (*c*)	Nepalese
L	182·3 (36)	180·1 (15)	180·8 (17)	180·5 (32)	178·9 (39)	179·6 (35)	175·6 (21)	176·8 (25)	175·4 (33)	176·3 (24)	176·0 (42)	176·9 (47)
B	130·8 (36)	128·3 (15)	131·3 (17)	129·9 (32)	127·9 (28)	131·5 (35)	131·4 (38)	132·4 (25)	132·3 (69)	132·7 (24)	132·1 (42)	132·6 (47)
B'	91·1 (36)	90·4 (15)	93·2 (17)	91·9 (32)	90·9 (37)	95·2 (35)	93·2 (21)	93·3 (25)	92·4 (10)	91·5 (24)	91·7 (42)	90·9 (48)
H'	132·9 (34)	133·5 (15)	131·5 (17)	132·4 (32)	132·7 (44)	136·3 (35)	132·5 (38)	134·8 (25)	131·5 (10)	133·7 (24)	132·1 (42)	132·8 (47)
LB	99·3 (35)	100·0 (15)	98·5 (17)	99·2 (32)	97·8 (47)	101·5 (35)	98·8 (38)	99·9 (25)	99·2 (44)	—	97·4 (42)	98·0 (47)
Q'	302·0 (34)	295·3 (15)	296·8 (16)	302·0 (34)	297·0 (38)	313·2 (35)	—	306·2 (25)	302·9 (10)	—	—	300·7 (46)
S	372·0 (35)	368·3 (15)	368·4 (17)	368·3 (32)	363·2 (36)	367·6 (35)	—	368·0 (25)	363·5 (33)	361·4 (24)	356·9 (41)	364·9 (46)
U	501·3 (36)	498·2 (15)	502·5 (17)	500·5 (32)	496·2 (35)	498·0 (35)	488·7 (38)	495·7 (25)	493·5 (69)	490·3 (24)	488·3 (42)	497·4 (47)
$G'H$	67·1 (34)	62·4 (13)	63·0 (15)	62·7 (28)	62·0 (28)	64·8 (35)	—	65·0 (23)	63·8 (9)	—	67·3 (38)	67·9 (43)
GB	94·6 (34)	—	—	—	91·6 (20)	—	—	—	95·2 (9)	—	—	97·6 (47)
J	122·5 (32)	126·5 (15)	128·2 (15)	127·3 (30)	124·4 (36)	127·8 (35)	124·4 (21)	124·8 (25)	126·8 (32)	123·6 (21)	124·4 (33)	127·2 (44)
NH, R & L	—	—	—	—	—	—	46·5 (38)	—	49·0 (45)	—	—	49·7 (48)
NH'	47·2 (34)	47·2 (15)	47·4 (16)	47·3 (31)	46·5 (27)	49·4 (35)	—	48·0 (25)	—	47·8 (22)	49·1 (41)	—
NB	24·9 (34)	25·2 (15)	24·8 (16)	25·0 (31)	24·3 (47)	24·6 (35)	24·0 (38)	24·4 (25)	24·2 (43)	24·0 (22)	24·8 (41)	25·7 (48)
$O_1'R$	38·4 (33)	38·0 (15)	38·3 (16)	38·2 (31)	37·9 (28)	39·7 (35)	36·6 (17)	37·4 (25)	37·6 (46)	38·1 (22)	37·4 (38)	39·7 (48)
O_2R	32·0 (34)	30·9 (15)	31·4 (16)	31·2 (31)	32·7 (46)	32·2 (35)	31·7 (38)	32·2 (25)	32·4 (45)	32·3 (22)	33·1 (38)	33·1 (48)
G_1	51·2 (34)	—	—	—	49·5 (10)	50·2 (35)	50·5 (21)	—	46·4 (5)	—	—	48·6 (42)
G_2	37·8 (30)	—	—	—	39·0 (1)	38·8 (35)	—	—	35·9 (9)	—	—	41·3 (45)
GL	95·0 (33)	96·6 (13)	95·9 (16)	96·2 (29)	93·0 (33)	96·4 (35)	96·0 (36)	96·8 (23)	95·1 (39)	—	—	94·5 (42)
fml	35·5 (35)	34·1 (15)	33·8 (16)	33·9 (31)	35·0 (15)	37·6 (35)	33·9 (21)	33·1 (25)	35·5 (8)	—	34·4 (41)	36·0 (47)
fmb	28·7 (33)	—	—	—	30·0 (2)	29·5 (35)	28·2 (21)	—	27·4 (8)	—	29·3 (42)	29·6 (46)
$100\ B/L$	71·8 (36)	71·3 (15)	72·7 (17)	72·0 (32)	71·8 (39)	73·5 (35)	74·6 (21)	74·9 (25)	75·8 (33)	75·3 (24)	75·1 (42)	75·1 (47)
$100\ H'/L$	73·1 (34)	74·2 (15)	72·8 (17)	73·4 (32)	74·3 (39)	76·2 (35)	75·2 (21)	76·2 (25)	75·8 (10)	75·9 (24)	75·2 (42)	75·3 (46)
$100\ B/H'$	98·3 (34)	96·1 (15)	99·8 (16)	98·1 (32)	96·4 (36)	96·1 (35)	99·1 (21)	98·2 (25)	99·8 (10)	99·3 (24)	100·0 (42)	99·6 (46)
$100(B-H')/L$	−1·3 (34)	−2·9 (15)	−·1 (17)	−1·4 (32)	−2·7 (37)	−2·7 (35)	—	−1·4 (25)	·5 (10)	−·6 (24)	0 (42)	−·2 (46)
$100\ G'H/GB$	70·9 (34)	—	—	—	67·6 (20)	66·7 (35)	—	—	68·0 (8)	—	—	69·5 (42)
$100\ NB/NH'$	53·0 (34)	53·4 (15)	52·5 (16)	53·0 (31)	52·6 (27)	51·6 (35)	*51·7 (38)	50·9 (25)	† 49·0 (45)	50·4 (22)	50·6 (41)	†51·7 (48)
$100\ O_2/O_1$	83·3 (33)	81·5 (15)	82·1 (16)	81·8 (31)	87·3 (28)	81·0 (35)	84·4 (17)	85·9 (25)	86·3 (45)	84·9 (22)	88·5 (38)	85·3 (47)
$100\ fml/fmb$	82·1 (33)	—	—	—	85·7 (8)	80·2 (35)	83·3 (21)	—	77·3 (8)	—	85·2 (41)	82·4 (46)
$100\ G_2/G_1$	75·0 (30)	—	—	—	77·1 (5)	79·7 (35)	—	—	77·8 (5)	—	—	85·2 (40)
$P\angle$	84·0 (33)	—	—	—	88·0 (1)	85·5 (35)	—	—	86·1 (9)	—	—	85·2 (43)
$N\angle$	66·4 (33)	68·8 (13)	68·7 (16)	68·8 (30)	66·8 (21)	65·9 (35)	—	68·0 (24)	67·2 (9)	—	—	66·5 (42)
$A\angle$	73·6 (33)	74·3 (13)	73·6 (16)	74·1 (30)	76·4 (21)	75·5 (35)	—	73·5 (24)	74·8 (9)	—	—	72·8 (42)
$B\angle$	40·1 (33)	36·9 (13)	37·7 (16)	37·1 (30)	36·8 (21)	38·1 (35)	—	38·5 (24)	38·0 (9)	—	—	40·7 (42)
Capacity	1370·7 (35)	1296·4 (14)	1307·5 (17)	1302·5 (31)	1271·1 (36)	1350·3 (35)	1289·7 (17)	1344·1 (24)	1319·9 (34)	1340·8 (22)	1368·0 (37)	1436·2 (47)

* NB/NH, R or L.

TABLE IX.

Coefficients of Racial Likeness between the Various Hindu Series and the Maravars.

Race		Maravar	Uriyá	Hindu (*a*)	Hindu (*b*)	Hindu (*c*)
	No. of Skulls	27	25	26	23	40
Maravar	All Characters ...	—	·19 ± ·24	1·80 ± ·21	·01 ± ·25	1·46 ± ·22
	Angles and Indices	—	− ·49 ± ·48	2·03 ± ·39	− ·60 ± ·48	1·48 ± ·43
Uriyá	All Characters ...	·19 ± ·24	—	·13 ± ·21	− ·63 ± ·23	1·74 ± ·21
	Angles and Indices	− ·49 ± ·48	—	− ·31 ± ·39	− ·20 ± ·43	·75 ± ·43
Hindu (*a*)	All Characters ...	1·80 ± ·21	·13 ± ·21	—	− ·09 ± ·25	1·95 ± ·21
	Angles and Indices	2·03 ± ·39	− ·31 ± ·39	—	− ·50 ± ·48	2·74 ± ·43
Hindu (*b*)	All Characters ...	·01 ± ·25	− ·63 ± ·23	− ·09 ± .25	—	·45 ± ·23
	Angles and Indices	− ·60 ± ·48	− ·20 ± ·43	− ·50 ± ·48	—	·72 ± ·43
Hindu (*c*)	All Characters ...	1·46 ± ·22	1·74 ± ·21	1·95 ± ·21	·45 ± ·23	—
	Angles and Indices	1·48 ± ·43	·75 ± ·43	2·74 ± ·43	·72 ± ·43	—

The general low values of these coefficients are very striking for tribes coming from all parts of the densely populated east coast of India, said to be some of Aryan and others of Non-Aryan descent and in several instances without the bond of a common culture. Unfortunately Danielli's series of Bengalees (Hindu (*c*)) is the only one of at all adequate length and this differs slightly but just significantly from all but Mantegazza's series from Southern India (Hindu (*b*)). Mantegazza and Danielli are both Italians and are using Italian methods of measurement.

TABLE X.

Giving all Values of $\alpha = \dfrac{n_s n_s'}{n_s + n_s'}\left(\dfrac{M_s - M_s'}{\sigma_s}\right)^2 > {}^{\cdot}4$ *between Various Groups of Indian Crania.*

	O_2/O_1'	fml/fmb	H'	LB	S	U	$G'H$	J	NB	O_1'	O_2	fml	fmb	C
Hindu (*c*) and Maravar	7·74	3·48	·13	2·48	—	·02	—	·00	4·03	2·69	10·20	·57	3·67	5·69
" " Uriyá	4·00	—	4·52	6·22	12·23	4·53	4·40	·15	·79	·00	3·35	4·30	—	·65
" " Hindu (*a*)	3·97	12·46	·12	4·42	4·99	3·93	5·18	5·95	2·41	·30	2·77	2·15	5·25	3·18
" " Hindu (*b*)	7·08	1·55	1·55	—	1·96	·32	—	·39	2·93	2·45	2·44	—	—	·79
Maravar and Uriyá	·89	—	3·15	1·16	—	3·90	—	·09	·77	2·32	1·03	1·20	—	2·29
" " Hindu (*a*)	1·81	6·24	·13	·20	—	2·99	—	3·50	·26	4·44	2·76	2·44	·80	·80
" " Hindu (*b*)	·09	—	·84	—	—	·20	—	·32	·00	7·73	1·37	—	—	1·94
Uriyá and Hindu (*a*)	·10	—	3·07	·50	1·84	·47	·15	1·04	·40	·25	·53	·12	—	·07
" " Hindu (*b*)	·46	—	·59	—	3·41	1·88	—	·79	·60	2·06	·03	—	—	·01
Hindu (*a*) and Hindu (*b*)	1·14	—	1·35	—	·39	·96	—	6·22	·19	1·33	·04	—	—	·46

Of the measurements normally selected for finding the coefficient of racial likeness those which do not appear in this table were either not available for the series used or did not vary significantly for any pair of means.

The palatal measurements G_1, G_2 and the index G_1/G_2, the profile angle P, the occipital index and facial index $G'H/GB$, were not given or only given for a few crania. LB and the angles N and A were not given for the Italian data, Q was only given for the Uriyá and Hindu (*a*). For the Maravar and Hindu (*a*) series, the nasal height was only measured to the lower border of the pyriform aperture, right or left, and they could therefore only be compared with each other. It is the divergence in these measurements that accounts largely for the higher coefficient between the two series (for NH: $\alpha = 15{\cdot}08$ and for NB/NH: $\alpha = 5{\cdot}53$). No value of α for NH' or NB/NH' was significant and it is unfortunate that we cannot compare the Maravar and Hindu (*a*) for these characters with the other groups. The calvarial characters B, L, B/L, H'/L, B/H' do not differ significantly in any of the types.

For certain characters, it is only the Danielli series which differs significantly from any other and on the whole the values of α for this series are of rather higher order. Examination of the means shows somewhat how far this depends on the number of crania on which the means are based.

Most of the high values of α can be accounted for by the following marked divergences in the different types:

1. Maravar. A low orbital breadth O_1'.
2. Urijá. A rather high basio-bregmatic height.
3. Hindu (*a*). A low foraminal index (but fmb/fml is based on 8 crania only). A high bizygomatic breadth J.
4. Hindu (*b*). No marked divergence.
5. Hindu (*c*). A high orbital height O_2 and index O_2/O_1'. Rather low basi-nasal height and sagittal arc.

There are a few other significant α's, but these are due to slight differences in all the means which become marked between the greatest and least. There is not any obvious racial order in these differences. The Hindu (*c*) series perhaps diverges most from the rest; but it is curious that it should diverge most from Hindu (*a*) which came from the same district. The Hindu (*b*) series does not diverge significantly from all the others for any character. Whether the type is the same for all the series and the differences are due to random sampling and personal equation, it is not possible to assert on the basis of so few crania; but considering the wide area which has been sampled and the different laboratories in which investigations have been made, the agreement is extraordinarily close. Moreover even the Maravars said to be of Non-Aryan origin have been found similar to the so-called Aryan Hindus.

In comparing the "Hindu" data with Badarian and other Indians, the Uriyá and Hindu (*c*) series have been used separately to represent the Hindus since

there is a slight divergence in their means; the difference in number of crania magnifies this somewhat.

Comparison of Badarian and Primitive Indian Races. The Coefficients of Racial Likeness between the Badarians and various Indian races will be found in Table XI. They show that the Badarians, judged from skull measurements, are as closely allied to the Dravidians and Kolarians as to the Early Predynastic Egyptians and are almost as close to the Veddahs. They are more divergent from the Hindus, the Nepalese and the Tamils.

The coefficients do not show any very regular sequence from the Dravidians as the most primitive type. The Dravidians and Veddahs differ slightly from each other but seem to be as closely allied to the Hindus as to the Nepalese. The Tamils diverge from all the races considered, being nearest to the Uriyás for which the means however are based on small numbers. The means do not seem to fall in with the sequence at all (see Table VIII). The Nepalese likewise are not very close to the other races but they are more allied to the Hindu (*c*), since both races are based on considerable numbers. A comparison of mean direct measurements is more profitable in this case.

The Badarians are significantly different—i.e. have $\alpha > 6$—from Dravidians for six characters only, none of which are indices. See Table XII. The facial height $G'H$ and bizygomatic breadths differ most ($\alpha = 17$); Q', fml and C and $N\angle$ have differences which might possibly be racial. There are more significant differences between Veddahs and Badarians although the only characters with α's exceeding six are O_2/O_1', L, S, $G'H$ and C; the difference is mainly one of size, the Veddahs being smaller than the Badarians. The Hindus show divergence in other characters. They are a more brachycephalic race, with skull of very much shorter length and correspondingly shorter circumference and sagittal arc. There is a marked difference in orbital index and slight differences in other facial characters but not a marked difference in facial height and bizygomatic breadth.

From skull measurements alone it would be difficult to choose between the primitive Indian and Egyptian series as the group to which the Badarians are closer. Unfortunately it is not possible to carry the analogy further and find low coefficients between the later Egyptian series and the Indian series as the types diverge in different directions.

The Dravidians diverge from the Veddahs mainly in orbital height and index and bizygomatic breadth.

The Hindus are more brachycephalic due to greater breadth and shorter length; the transverse arc is greater and the sagittal and horizontal are less; the orbital index and facial height are slightly greater.

The Veddahs differ from the Hindus in having a much smaller parietal breadth and rather greater length, a smaller Q' and a much shorter facial and nasal height.

The Tamils have several marked divergences from all the Indian series and in characters which do not usually enable us to distinguish between races. Thus

TABLE XI.

Coefficients of Racial Likeness between the Badarians and Various Primitive Indian Races.

Race	Badarian	Dravidian	Veddah	Uriyá	Hindu (*c*)	Nepalese	Tamil
Number of Skulls	33	31	35	25	40	46	35
Badarian	— —	2·10 ± ·20 (23) ·47 ± ·36 (7)	3·32 ± ·19 (25) 2·26 ± ·34 (8)	3·59 ± ·20 (23) 5·86 ± ·36 (7)	7·02 ± ·20 (22) 11·21 ± ·39 (6)	6·05 ± ·17 (31) 6·32 ± ·28 (12)	5·69 ± ·17 (31) 6·17 ± ·28 (12)
Dravidian	2·10 ± ·20 (23) ·47 ± ·36 (7)	— —	1·85 ± ·20 (23) 2·84 ± ·36 (7)	3·16 ± ·20 (23) 5·24 ± ·36 (7)	7·96 ± ·21 (20) 13·16 ± ·43 (5)	7·60 ± ·21 (21) 8·26 ± ·39 (6)	6·67 ± ·20 (23) 4·95 ± ·36 (7)
Veddah	3·32 ± ·19 (25) 2·26 ± ·34 (8)	1·85 ± ·20 (23) 2·84 ± ·36 (7)	— —	3·48 ± ·20 (23) 4·24 ± ·36 (7)	5·67 ± ·21 (20) 10·94 ± ·43 (5)	8·96 ± ·21 (21) 8·29 ± ·39 (6)	7·69 ± ·19 (25) 4·31 ± ·34 (8)
Uriyá	3·59 ± ·20 (23) 5·86 ± ·36 (7)	3·16 ± ·20 (23) 5·24 ± ·36 (7)	3·48 ± ·20 (23) 4·24 ± ·36 (7)	— —	1·74 ± ·21 (20) ·75 ± ·43 (5)	4·44 ± ·21 (21) ·36 ± ·39 (6)	4·73 ± ·20 (22) 3·60 ± ·36 (7)
Hindu (*c*)	7·02 ± ·20 (22) 11·21 ± ·39 (6)	7·96 ± ·21 (20) 13·16 ± ·43 (5)	5·67 ± ·21 (20) 10·94 ± ·43 (5)	1·74 ± ·21 (20) ·75 ± ·43 (5)	— —	4·46 ± ·21 (20) 1·75 ± ·43 (5)	9·52 ± ·20 (22) 12·38 ± ·39 (6)
Nepalese	6·05 ± ·17 (31) 6·32 ± ·28 (12)	7·60 ± ·21 (21) 8·26 ± ·39 (6)	8·96 ± ·21 (21) 8·29 ± ·39 (6)	4·44 ± ·21 (21) ·36 ± ·39 (6)	4·46 ± ·21 (20) 1·75 ± ·43 (5)	— —	7·92 ± ·18 (29) 5·75 ± ·29 (11)
Tamil	5·69 ± ·17 (31) 6·17 ± ·28 (12)	6·67 ± ·20 (23) 4·95 ± ·36 (7)	7·69 ± ·19 (25) 4·31 ± ·34 (8)	4·73 ± ·20 (22) 3·60 ± ·36 (7)	9·52 ± ·20 (22) 12·38 ± ·39 (6)	7·92 ± ·18 (29) 5·75 ± ·29 (11)	— —

TA L XI

*Values of a for Badarian and Indian Races**.

	Badarian with				Dravidian with			Veddah with		Tamil with				Nepalese with			
	Dravidian	Veddah	Uriyá	Hindu (c)	Veddah	Uriyá	Hindu (c)	Uriyá	Hindu (c)*	Dravidian	Veddah	Uriyá	Hindu (c)	Dravidian	Veddah	Uriyá	Hindu (c)
B/L	·10	·00	19·75	29·40	·10	16·44	24·31	20·39	30·67	5·24	7·43	3·98	6·81	25·48	32·33	·91	·00
H'/L	·17	3·03	16·03	9·59	1·65	12·74	6·81	6·37	1·90	5·17	7·71	·00	2·21	7·89	2·44	1·52	·03
B/H'	·04	3·41	·01	2·94	2·65	·01	3·55	2·59	13·59	3·62	·09	3·48	15·70	2·30	11·18	1·72	·19
O_2/O_1'	1·41	9·51	3·77	18·73	17·45	9·12	29·15	1·02	9·10	·41	24·21	13·73	40·19	8·97	2·74	·23	8·43
NB/NH'	·00	·17	4·36	7·34	·16	4·18	6·97	2·57	4·46	2·21	1·04	·49	1·29	—	—	—	—
fml/fmb	—	—	—	5·24	—	—	—	—	—	—	—	—	14·08	—	—	—	5·07
$P\angle$	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
$N\angle$	8·26	·58	3·25	—	3·65	·78	—	1·02	—	12·40	1·45	5·73	—	8·44	·32	3·14	—
$A\angle$	·33	3·78	6·01	—	1·25	·40	—	2·70	—	2·65	·10	4·76	—	2·47	6·74	·63	—
Occ. I.	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
L	1·67	7·83	13·60	23·51	1·94	5·87	11·24	1·51	4·18	4·15	5·64	3·49	7·56	7·54	1·88	·01	·55
B	·61	5·85	1·67	1·45	2·64	3·87	3·88	11·80	13·08	1·89	8·90	·52	·30	6·13	17·11	·03	·24
B'	·66	·04	4·35	·43	1·05	1·68	·04	5·24	·77	11·10	20·28	3·21	14·26	·12	·00	5·77	·87
H'	·16	·03	2·06	·48	·07	3·19	·07	2·78	·31	10·05	10·00	1·30	13·31	·12	·01	2·58	·43
LB	·01	2·86	·33	4·37	2·37	·44	3·73	4·57	·23	5·61	17·43	2·37	20·36	1·74	·06	3·73	·51
Q'	5·97	4·59	2·60	—	·17	14·72	—	13·05	—	49·72	48·89	—	—	4·82	2·91	5·01	—
S	1·46	8·78	1·40	27·51	2·82	·01	14·93	2·17	4·86	·05	2·20	·02	13·81	1·40	·37	·46	8·86
U	·06	2·44	2·44	17·28	1·63	1·71	14·26	·02	6·28	·55	·30	·41	9·47	·97	·15	·25	9·69
$G'H$	17·26	23·19	3·51	·04	·40	3·88	19·81	6·60	26·30	3·98	7·08	·03	6·61	26·63	34·28	7·32	·42
J	17·09	2·93	3·56	2·81	6·59	4·08	6·33	·11	·00	·19	9·82	6·29	9·40	·01	7·44	4·40	7·08
NH'	·02	·86	1·08	7·87	1·08	·79	6·71	4·72	12·90	8·50	15·03	3·35	·20	—	—	—	—
NB	·05	2·27	1·15	·06	2·92	1·59	·23	·05	1·75	·84	·58	·19	·24	2·95	14·87	8·88	5·72
O_1R	·23	1·36	5·10	6·33	·48	3·18	3·92	1·18	1·44	13·26	18·06	27·65	34·55	15·19	20·54	31·17	40·21
O_2R	2·84	2·62	·16	5·95	11·42	3·79	16·88	1·11	·91	4·50	1·36	·00	4·04	18·63	1·03	3·65	·00
G_1	—	—	—	—	—	—	—	—	—	—	·35	—	—	—	—	—	—
G_2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
fml	6·09	·43	13·77	3·75	2·01	1·45	·72	5·55	·65	21·13	4·98	45·14	16·17	13·51	1·86	22·41	9·19
fmb	—	—	—	1·44	—	—	—	—	—	—	—	—	·17	—	—	—	4·28
C	6·01	13·86	·79	·61	1·08	1·82	5·62	6·03	·11	2·92	9·20	·04	·44	25·92	50·84	10·46	7·47

* For Hindu (c) and Uriyá see Table X.

B', H', LB, Q', $O_1'R$, and fml are one to three millimetres outside the range of the other types and greater in each case. In the cranial measurements B', H', LB, and Q' the Uriyás diverge in the same direction but less markedly. The two types only show racial distinction in fml, $O_1'R$, and $O_2/O_1'R$ but the α's for these are very high. Except for $O_2/O_1'R$ the indices show much less divergence than the mean direct measurements.

The Nepalese are like the Hindus in all calvarial measurements except in the foraminal length and the arcs but they differ in some of the facial characters, largely in the orbital breadth and index. The differences between the Nepalese and the Veddahs and Dravidians are very much more marked; both calvarial and facial characters differ. The Nepalese are less dolichocephalic, this being due to both decrease in length and increase in breadth, and have a longer and broader face.

The resemblance between the primitive Indian and Egyptian races as judged by the coefficients of racial likeness is based on rather slender material. But the mean direct measurements show interesting phases also, as they branch outwards in two directions from the very similar primitive Indian and Egyptian types. The Indians, who are represented by samples of the primitive populations of to-day spread over wide areas of the country, do not show the same regular sequence as the Egyptians spread over a vast number of years.

The Badarian type is smaller than the other Egyptian types in practically every direct measurement and in most characters the Dravidians and Veddahs are slightly smaller still. Table XII *a* shows the type of differences in the means of a number of the direct measurements in the two series. In the cephalic index, the breadth-height index, the parietal breadth, and the nasal and facial height there

TABLE XII *a*.

Some Mean Direct Measurements of Indian and Egyptian Races.

	B/L	B/H'	B	L	H'	U	S	NB/NH'	O_2/O_1'	NB	NH'	$G'H$	J	$N\angle$	$A\angle$
"Series *E*" ...	75·1	104·9	138·9	185·3	132·4	518·7	371·9	47·4	87·8	24·4	51·5	70·4	128·7	63°·9	74°·1
1st Dynasty .	72·7	99·0	133·7	183·8	135·1	—	—	50·4	—	25·8	51·4	70·6	130·9	66°·2	72°·8
Late Pre-dynastic	72·1	100·5	135·4	185·1	132·7	—	—	51·0	—	25·5	50·1	70·0	127·0	66°·8	72°·4
Naqada ...	71·8	99·2	132·7	184·7	133·8	510·4	372·6	50·0	82·4	25·0	50·0	69·7	125·9	65°·5	73°·3
Early Pre-dynastic	71·7	98·1	131·4	183·5	134·0	—	—	50·3	—	25·2	50·1	69·2	124·3	67°·0	72°·6
Badarian ...	71·7	98·3	130·8	182·3	132·9	501·3	372·0	53·0	83·3	24·9	47·2	67·1	122·5	66°·4	73°·6
Dravidian ...	72·0	98·1	129·9	180·5	132·4	500·5	368·3	53·0	81·8	25·0	47·3	62·7	127·3	68°·8	74°·1
Veddah ...	71·8	96·4	127·9	178·6	132·7	496·2	363·2	52·6	87·3	24·3	46·5	62·0	124·4	66°·8	76°·4
Uriyá ...	74·9	98·2	132·4	176·8	134·8	495·7	368·0	50·9	85·9	24·4	48·0	65·0	124·8	68°·0	73°·5
Hindu (*c*) ...	75·1	100·0	132·1	176·0	132·1	488·3	356·9	50·6	88·5	24·8	49·1	67·3	124·4	—	—
Nepalese ...	75·1	99·6	132·6	176·9	132·8	497·4	364·9	51·7	85·3	25·7	—	67·9	127·2	66°·5	72°·8

is in both series a marked increase in the values for the less over the more primitive types. But some characters, notably the skull length and horizontal circumference, vary in opposite directions in the two series; and others, including the basio-bregmatic height, the least forehead and the nasal breadth, vary very little in either series and over a similar range. In some of the facial and arcual characters the tendency to converge is less marked but in no case is there marked divergence.

8. *Comparison of Type Contours.*

Transverse, horizontal and sagittal contours were drawn with the Klaatsch contour tracer; for nearly one-half of the contours the original instrument with ink tracer was used, but comparison of contour and direct measurements showed that the pointer had been worn back and the contour measurements were a little less than the direct. Also the position of the tracer was not vertically below the pointer for all positions of the instrument. This was allowed for by taking measurements from the outside of the circular spot (made by the tracer) on the diameter parallel to the position of the axis of the pointer when the spot was made. For the other half of the contours an instrument adopted by Morant with steel pointer was used. This leaves a fine tracing which can be inked over. Comparison of mean direct measurements and type contour measurements gave reasonably close agreement. The process described in Benington's paper* with additions made by later workers in the Biometric Laboratory was followed.

The Transverse Vertical Contour. The method of constructing the mean transverse vertical type contour is described in *Biometrika*, Vol. XIV. p. 227. The means from which the type contour was plotted are given in Table XIII in which right and left correspond to R and L in the contour diagrams shown in Figs. I and II. Benington gives the range of variation for the type contour from 100 English skulls as 1·4 mm. at the apex tapering to ·8 mm. near the auricular points.

TABLE XIII.

Mean Measurements of Transverse Vertical Contours.

Sex	MA	$1R=1L$	$\frac{1}{4}L$	$\frac{1}{4}R$	$2L$	$2R$	$3L$	$3R$	$4L$	$4R$	$5L$	$5R$	$6L$	$6R$
♂	112·9	52·3	56·7	56·7	57·5	57·1	60·1	60·1	62·4	62·2	63·4	63·1	63·6	62·6
♀	109·1	49·6	55·2	54·8	56·2	55·3	59·8	58·5	62·5	61·2	63·6	61·5	63·7	61·1

Sex	$7L$	$7R$	$8L$	$8R$	$9L$	$9R$	$10L$	$10R$	$A\frac{1}{4}L$	$A\frac{1}{4}R$	ZR, L		ZR, R	
											y	x	y	x
♂	62·4	60·8	58·4	56·3	50·5	47·9	36·8	33·7	19·7	14·2	57·3	3·5	57·3	3·6
♀	62·4	59·4	58·7	55·3	51·1	47·3	36·8	33·5	18·4	14·9	55·7	3·5	55·4	3·8

* *Biometrika*, Vol. VIII. pp. 123—188.

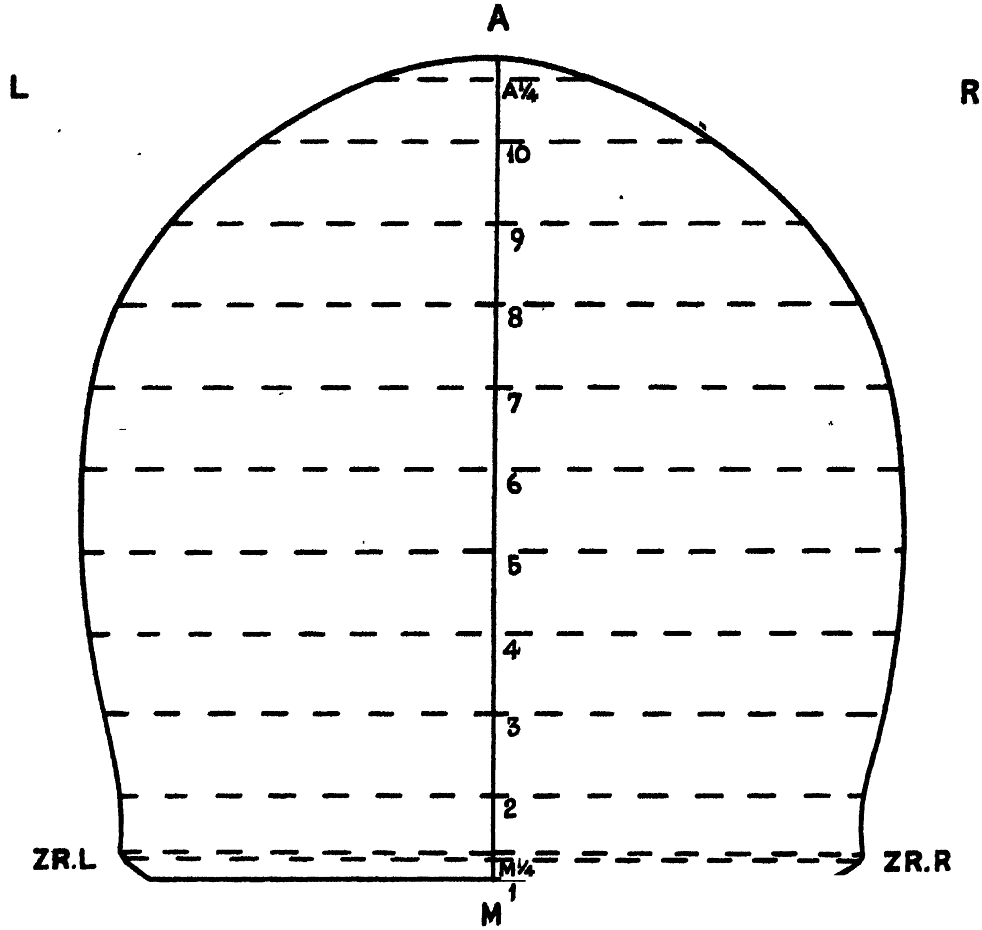


FIG. I Transverse Type Contour of 34 ♂ Badarian Skulls.

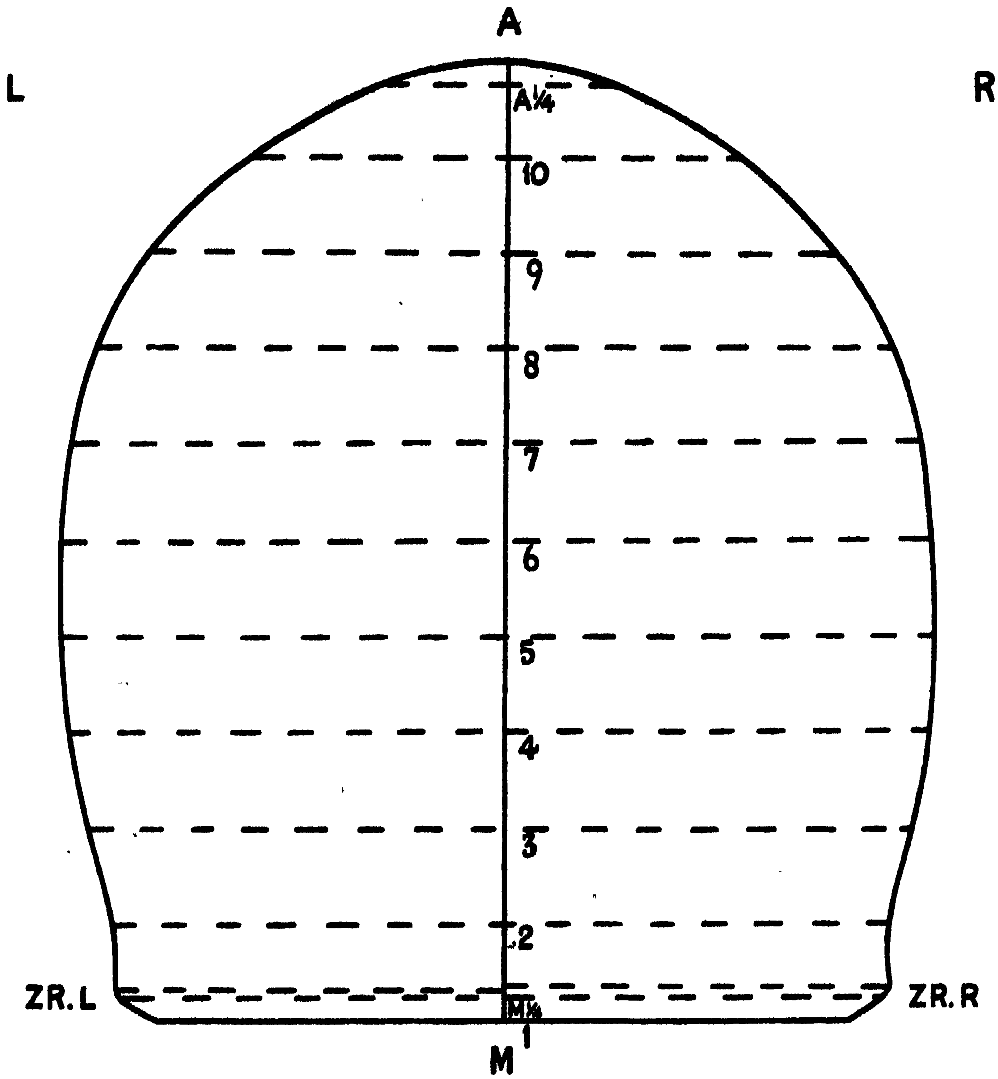


Fig. II Transverse Type Contour of 21 ♀ Badarian Skulls.

Taking the range to vary inversely as the square root of the number of crania this would give 2·4 mm. at the apex tapering to 1·4 mm. at the auricular points for the 34 male Badarians and 3·1 mm. tapering to 1·7 mm. for the 21 females. These type zones are not given in the diagrams, to avoid overcrowding them.

The only Egyptian type contours are those for the long "*E* Series," 100 males taken by Benington and also for Motley's series of the 1st Dynasty Royal Retainers.

Benington did not mark *ZR, L* and *ZR, R* on his contours and so the auricular points are joined directly to the parallels 2, 3, 4 and the contour gives no idea of the curvature above the auricular points. Also his diagrams show the frontal view so that my contour must be reversed before a comparison is made between the "*E* Series" and the Badarian.

The Motley and "*E* Series" are not significantly different from each other but they are both just significantly lower and broader than the Badarian which is 3 mm. higher at the apex and tapers to 3 mm. less at the zygomatic ridge and auricular points. The right side of the 1st Dynasty skulls and the left side of the Badarian are the more developed, which magnifies the difference on one side and reduces it on the other, but on each side it is significant; the contours cross in the region of the 8th horizontal on the left and 10th on the right. The difference is in the same direction but more marked in the females. This divergence is what we should expect from direct measurements though the difference in height of contour is rather greater than the difference in basio-bregmatic and auricular heights.

Comparing the Badarian with the type contours of the Eastern races, Hindu (*a*), Nepalese and Tibetan, we see that the first is lower vaulted but has no significant difference in breadth, the second is the same height but about 1·5 mm. broader at the auricular points, and the third is the same height and gets broader towards the auricular region where the difference is 4·5 mm. The female type contours are based on too few crania for comparison to be profitable.

TABLE XIV*.

Measurements of Transverse Type Contour (Males).

Race	Number	Inter-Auricular Length	Auricular Height	Index *Ht/L*
Tibetan A ...	35	118·4	113·1	95·5
Nepalese... ...	46	113·6	113·2	99·6
Hindu (*a*) ...	10	106·8	109·6	102·6
Badarian ...	34	104·6	112·9	107·9
1st Dynasty ...	32	114·2	110·6	96·8

* Measurements for the Indian races taken from *Biometrika*, Vol. XVI. p. 20.

TABLE XV.

Measurement of Curvature.

Race	Apex to Right Auricular Point	Subtense y	Index
Tibetan A ...	127·6	37·5	29·4
Nepalese	126·4	35·8	28·3
Hindu (a) ...	121·7	35·9	29·5
Badarian ...	124·3	37·1	29·8
1st Dynasty ...	124·1	39·3	31·8

The slope of the auricular orifices causes a slight variation in the index. On the whole the transverse contour supports the similarity of the Badarian type and the primitive Indian.

The Glabella Horizontal Section. The method of construction of the horizontal type contour is given in *Biometrika,* Vol. XIV. p. 234. The parallels $2\frac{1}{2}R$ and $2\frac{1}{2}L$ were added later to define more exactly the curvature in the temporal regions. The mean measurements from which the types were constructed are given in Table XVI, the R and L of the table corresponding to the R and L of the diagrams (Figs. III and IV), i.e. to the aspect from above.

The type zone of Benington for 100 English crania has an approximate width of 1·2 mm. all round. This would correspond to 2·1 mm. for Badarian males and 2·6 mm. for females.

The comparative material for the Horizontal section is the same as for the Vertical. Here again the asymmetry is on the opposite side for the Badarian, the right being the greater.

TABLE XVI.

Mean Measurements for Horizontal Type Contour.

Sex	FO*	$F\frac{1}{4}L$	$F\frac{1}{4}R$	$F\frac{1}{2}L$	$F\frac{1}{2}R$	$2L$	$2R$	$2\frac{1}{2}L$	$2\frac{1}{2}R$	$3L$	$3R$	$4L$	$4R$	$5L$	$5R$
♂	181·8	21·5	22·4	33·6	35·2	45·4	46·1	45·8	46·3	47·3	47·6	51·0	53·1	57·1	59·6
♀	175·7	21 7	22·6	32·0	32·7	43·3	44·0	45·1	45·4	47·2	47·3	51·1	52·3	57·1	58·6

Sex	$6L$	$6R$	$7L$	$7R$	$8L$	$8R$	$9L$	$9R$	$10L$	$10R$	$O\frac{1}{4}L$	$O\frac{1}{4}R$	T_L†		T_R†	
													y	x	y	x
♂	61·2	64·6	63·1	65·9	61·1	63·0	54·8	55·9	43·0	43·1	25·4	24·8	47·3	20·1	47·8	19·4
♀	61·3	63·4	62·6	64 7	60·2	62·0	53·8	54·7	42·1	42·7	24·7	24·6	45·3	19·8	45·8	19·8

* F and O = Glabellar and Occipital points in median plane. FO = Glabellar-occipital line.

† T_R and T_L: points right and left in which plane of contour cuts temporal ridges.

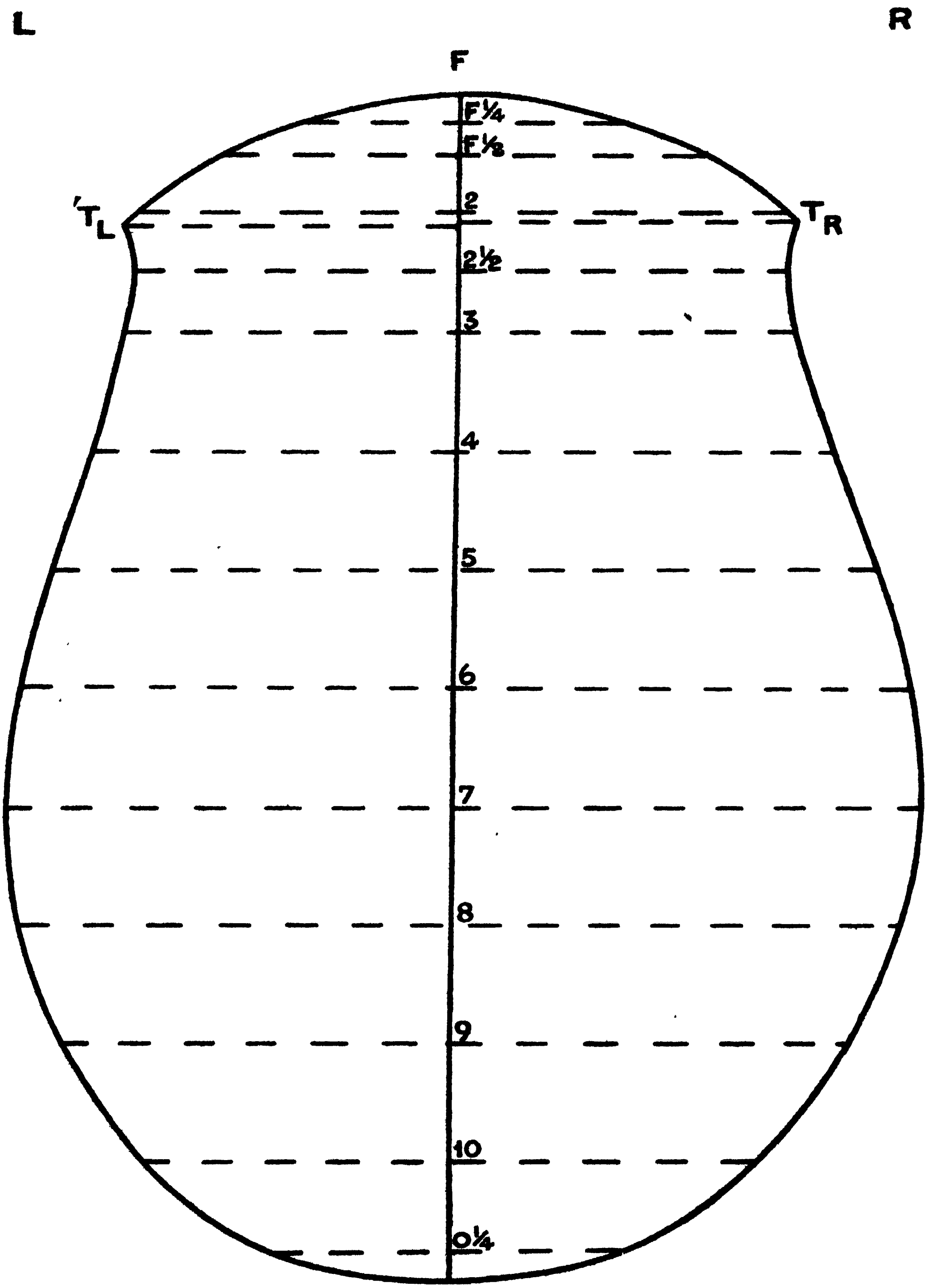


FIG. III Horizontal Type Contour of 34 ♂ Badarian Skulls.

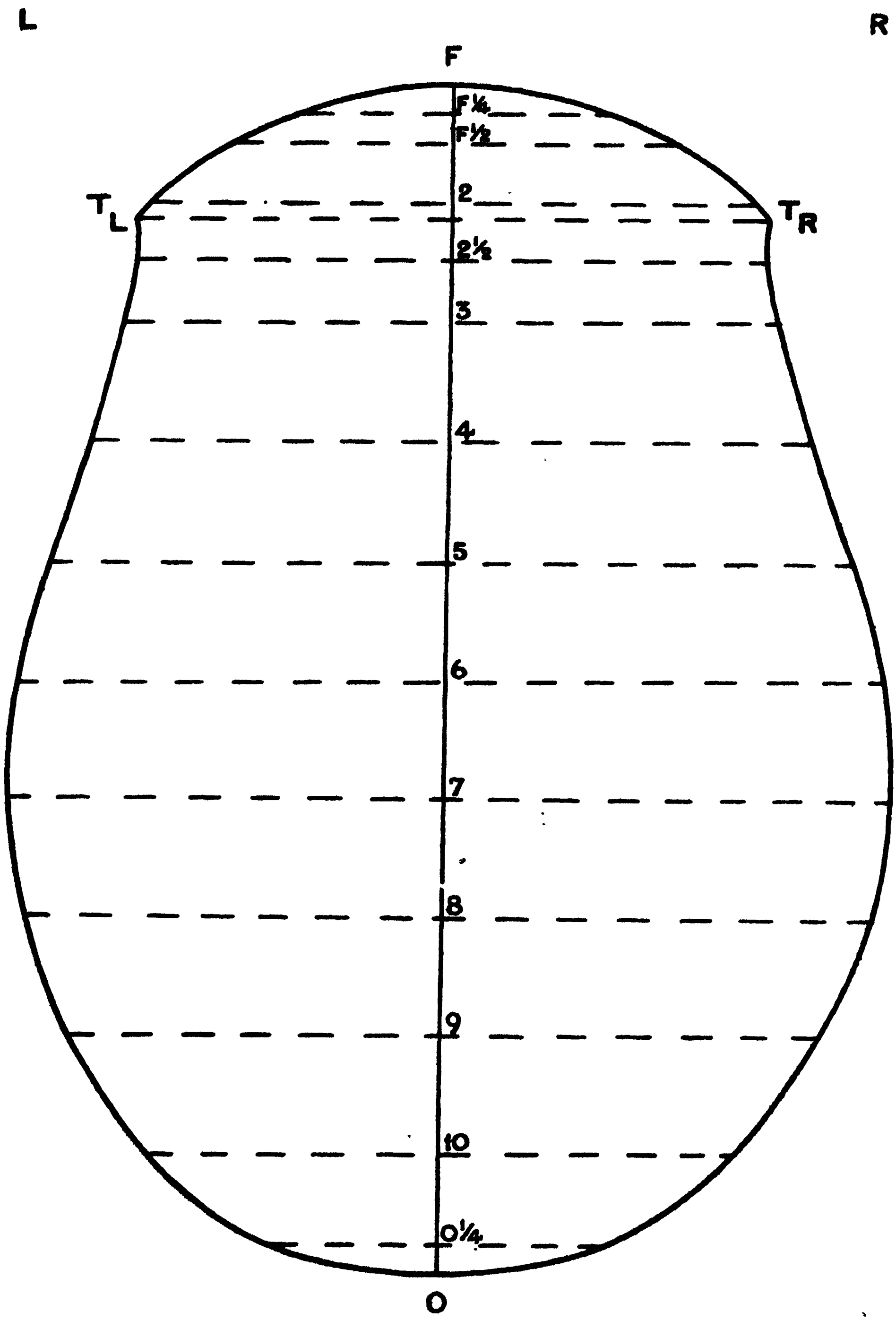


FIG. IV Horizontal Type Contour of 21 ♀ Badarian Skulls.

The frontal and temporal regions in the Predynastic and Motley's Dynastic types are not significantly different but the former lies inside the latter for the rest of the contour when F and FO are made to coincide, varying from 1 to 3 mm. on the right and from 2 to 4 mm. on the left. The contour for the "E Series" is of the same length as the Predynastic, wider in the temporal and parietal regions but hardly significantly different behind the 9th horizontal.

The difference in the Badarian and Indian type contours is much more marked because there is a marked difference in length, the Hindu being 8·3 mm. shorter, the Nepalese 5·5 and the Tibetan 7·4. The frontal region is similar in all, but the temporal and parietal regions become increasingly wider from Badarian to Tibetan.

TABLE XVII.

Temporal Index.

Race	Number of Skulls	Ordinate 3	Length of section FO	Temp. Index
Tibetan A ...	35	100·7	174·4	59·0
Nepalese ...	46	97·8	176·3	55·5
Hindu ...	10	93·3	173·5	53·8
Badarian ...	34	94·9	181·8	52·2
1st Dynasty	32	95·9	183·9	52·2

In this table there is the usual order both for index and two components.

TABLE XVIII.

Index of Frontal Flattening.

Race	$\frac{1}{2}\{T_R(x)+T_L(x)\}$	FO	Index
Tibetan A ...	18·15	174·4	10·4
Nepalese ...	18·85	176·3	10·7
Hindu ...	22·00	173·5	12·7
Badarian ...	19·75	181·8	10·9
1st Dynasty	20·20	183·9	11·0

Sagittal Type Contour or Median Section. The methods used in preparing this contour are described in *Biometrika*, Vol. XIV. pp. 239, 240. The means from which the type contour was constructed are given in Table XIX.

TABLE XIX. *Means of Measurements for Sagittal Type Contours.*

Sex	No. of Cases	Ordinates above $N\gamma$													
		$0=N$	$N\frac{1}{4}$	1	2	3	4	5	6	7	8	9	$\gamma\frac{1}{4}$	$\gamma\frac{1}{8}$	$\gamma 10$
♂	33	22·1	39·1	59·5	72·3	79·5	83·7	85·6	85·9	82·4	73·1	53·5	25·4	18·8	3·2
♀	21	23·8	42·5	59·7	71·4	78·1	82·1	83·8	84·4	81·4	72·9	53·9	25·7	18·8	5·8

Sex	No. of Cases	Ordinates below $N\gamma$									Vertex		β		Glabella		Occipital Point		λ		Sub-Orbital	
		$0=N$	$N\frac{1}{4}$	1	2	8	9	$\gamma\frac{1}{4}$	$\gamma\frac{1}{8}$	$N\gamma$	x from N	y	x from N	y	x from N	y	x from γ	y	x from γ	y	x from N	y
♂	33	61·7	58·2	51·2	50·5	46·1	35·7	20·2	14·8	180·3	100·4	86·5	71·0	84·0	1·7	10·6	·4	1·7	5·8	28·4	8·5	27·0
♀	21	58·8	55·3	48·7	47·5	45·7	34·3	18·4	12·7	175·2	101·3	84·7	69·0	81·9	1·5	12·5	·5	2·7	5·4	27·2	8·0	26·8

Sex	No. of Cases	Auricular Point		Opisthion		Inion		Basion		Alveolar Point		Nose					Palate			
															Max. Sub.		P'		P	
		x from N	y	x from γ	y	x from γ	y	x from N	y	from N	from Basion	NL	NL'	$LNY\angle$	x from N	y	x from N	y	x from Alv.	y
♂	33	87·5	27·4	58·0	52·9	13·9	32·9	99·5	105·0	66·6	95·6	19·6	15·7	121·2	2·0	7·2	46·1	48·4	37·1	15·7
♀	21	84·6	27·1	54·6	52·4	14·7	31·8	96·6	101·4	64·7	92·5	18·3	15·3	121·1	2·2	7·1	44·1	45·6	35·6	16·6

Sex	No. of Cases	Frontal Max. Sub. to $N\beta$		Occipital Max. Sub.		Max. Sub. to $N\lambda$		Max. Sub to GI		Sp.		Mid. Sub. from Bas. Sp. Chord		$N.S.$		Max. Sub. of Alv. $N.S.$ Arc	
		x from N	y	x from λ	y	x from N	y	x from G	y	x from N	y	N from Bas.	y	x from N	y	x from N	y
♂	33	49·7	26·5	45·3	27·2	86·1	71·9	101·6	101·7	67·3	34·3	12·1	·6	6·6	47·5	5·4	51·7
♀	21	47·2	26·1	43·3	25·6	81·8	70·3	99·2	99·6	65·7	33·2	11·5	·6	6·8	45·7	5·4	49·8

Abbreviations used in Table XIX and Figs. V and VI are:

N = Nasion. G = Glabella. *Bas.* or B = Basion. β = Bregma. V = Vertex. A or *Alv.* = Alveolar Point. *Sub-Orb.* = Left infra-orbital point. *Aur.* = Right auricular point. I = Inion. *Op.* = Opisthion. *Sp.* = Sphenoidal point, i.e. point of intersection of the median plane and the suture between the sphenoid and basi-occipital bones. P = Point of intersection of the palatine sutures. P' = Extremity of spina nasalis posterior. *N.S.* = Extremity of anterior nasal spine. L = Tip of nasal bone. L' = Point at which NL first meets the outline of the nasal bone.

The type zone for 100 ♂ English skulls given by Benington has a width of 1·6 mm. for greater part of length tapering to ·8 mm. at gamma and ·9 mm. at nasion. Unfortunately Benington did not give the contour below the $N\gamma$ line. Benington's widths would correspond in the Badarians to

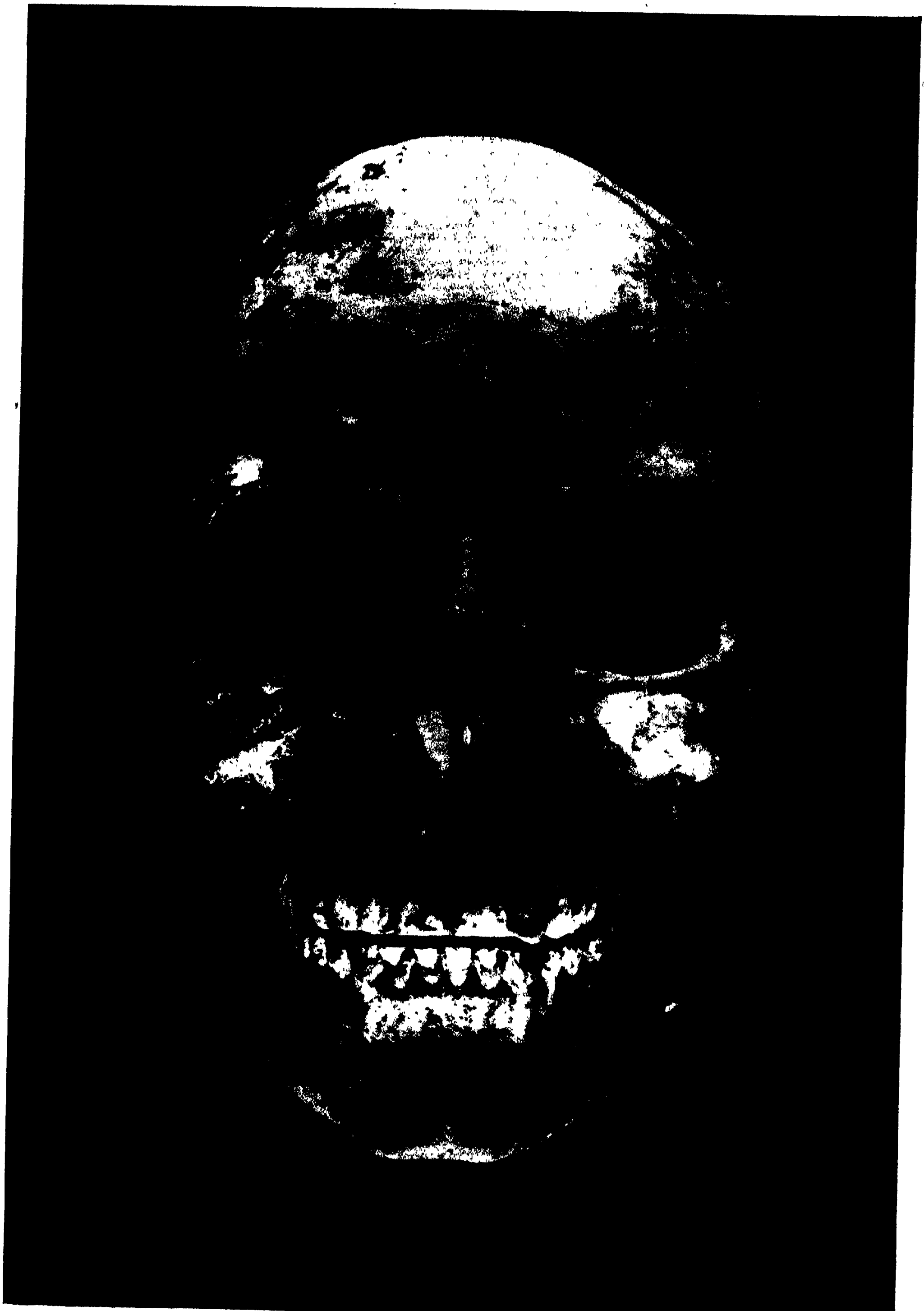
Sex	Vault mm.	Gamma mm.	Nasion mm.
♂	2·7	1·4	1·5
♀	3·5	1·7	2·0

Superposing the $N\gamma$ line of the Badarian and 1st Dynasty ♂ contours the latter is seen to be just significantly lower and longer. The ordinates $N\frac{1}{4}$, $N\frac{1}{2}$, ... 8 are all shorter but from the 9th ordinate the contours merge into one another and coincide in the occipital region. The difference in length of skulls seems to depend largely on the distance apart of the bregma and vertex and of the basion and opisthion. Allowing for this there is very little difference in the abscissae of the calvarial points. The alveolar point and the whole face is shorter and more prognathous in the Badarian; the palate is lower, the suborbital and auricular points are nearer the $N\gamma$ line. The nasal bone has more pronounced curvature below the nasion and makes a greater angle with $N\gamma$.

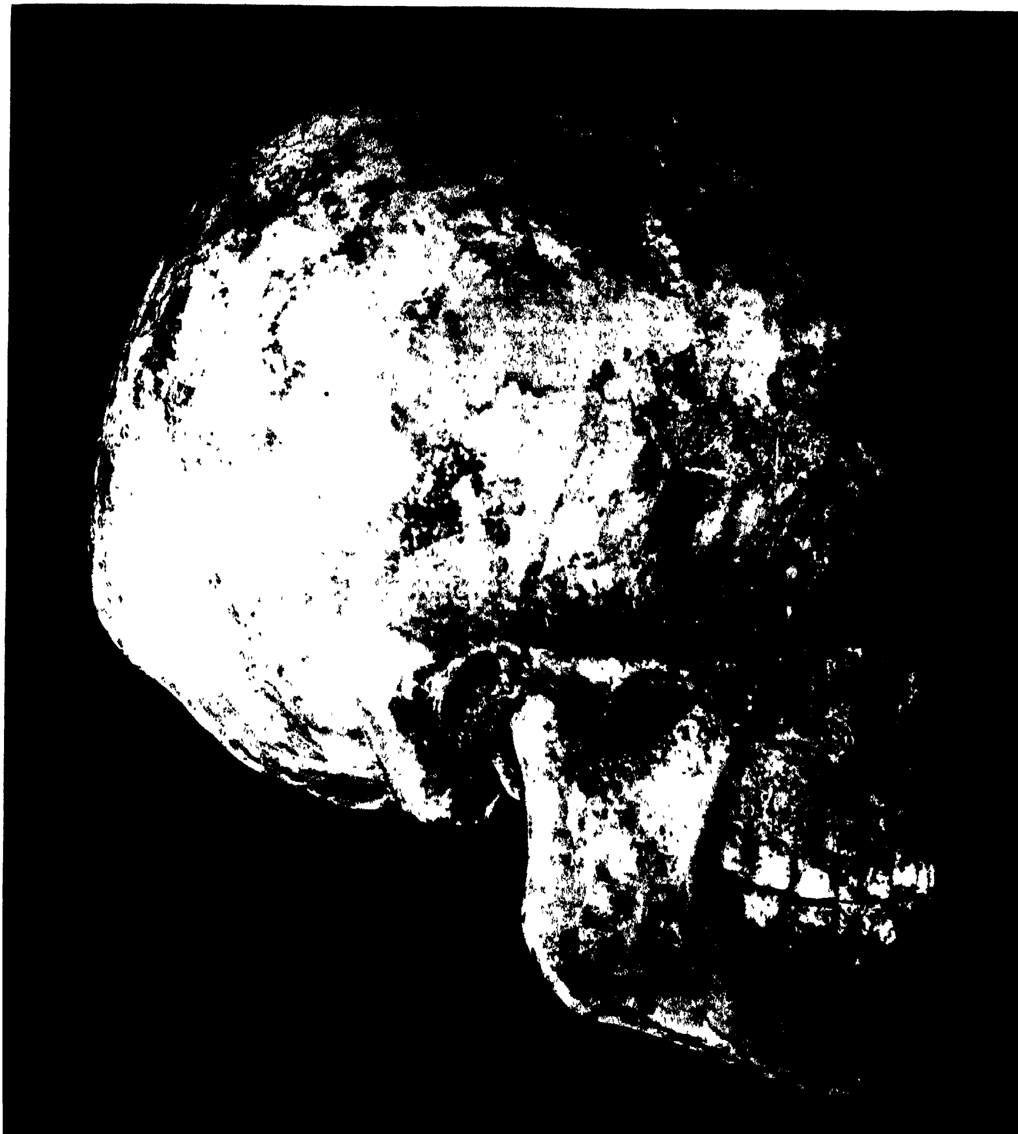
The difference in cranial length in the Badarian and Indian races is very marked. The Hindu is altogether very much smaller than any of the others. The Tibetan and Nepalese are both higher than the Badarian, but the Nepalese is almost as prognathous.

Conclusions. The type contours confirm the general conclusions reached from the study of direct measurements, that the Badarian race is somewhat more dolichocephalic and more prognathous than the Lower Egyptian type, and that it is narrower in the parietal region and has a shorter face.

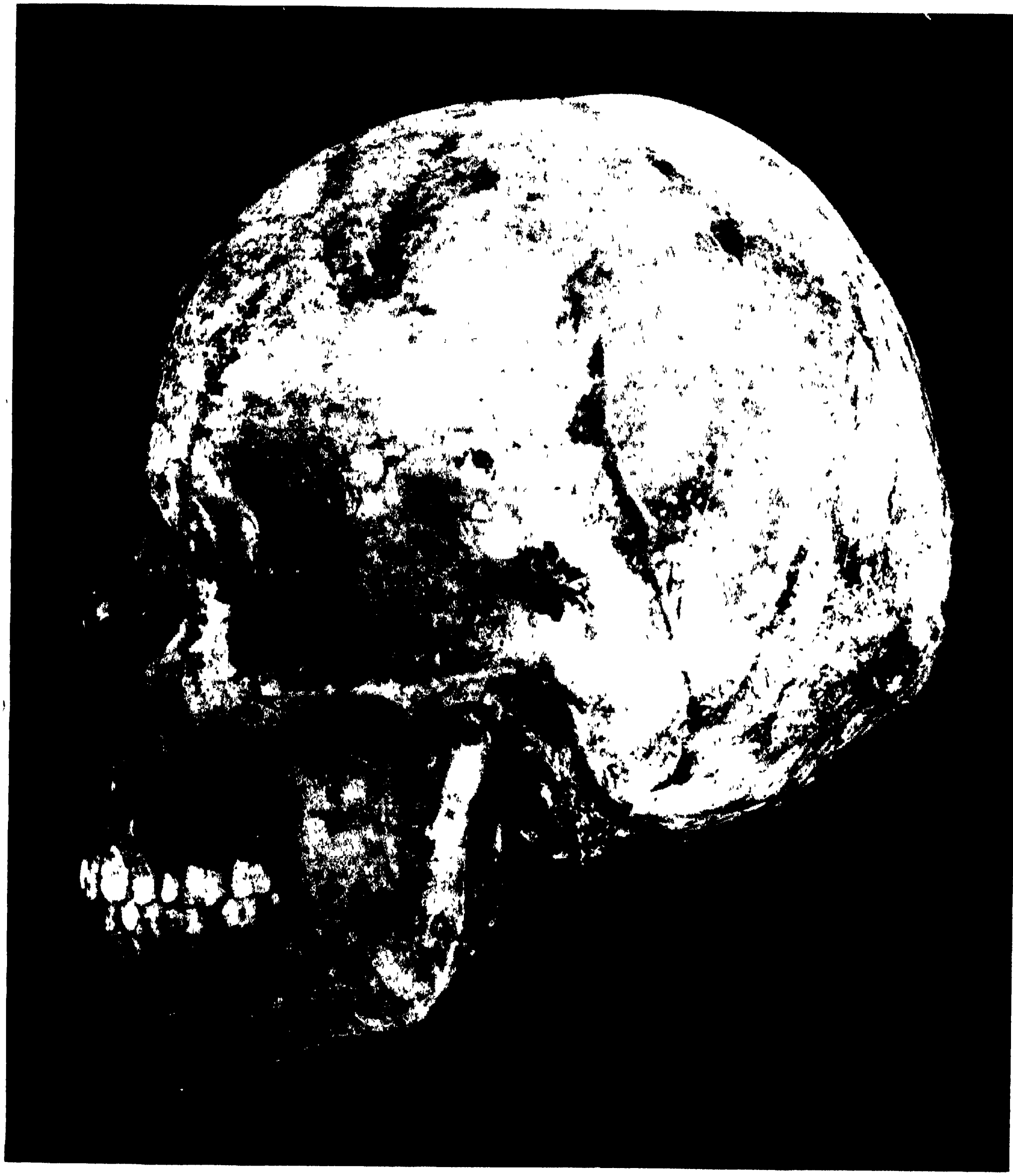
The contours of the oriental races do not diverge very widely except in the cranial length. They are more brachycephalic, but in all characters the Nepalese is nearer the Badarian than the Tibetan. The Hindu is a smaller type than any other.



Typical Badarian Male (5400 a). *Norma facialis* (circa natural size).



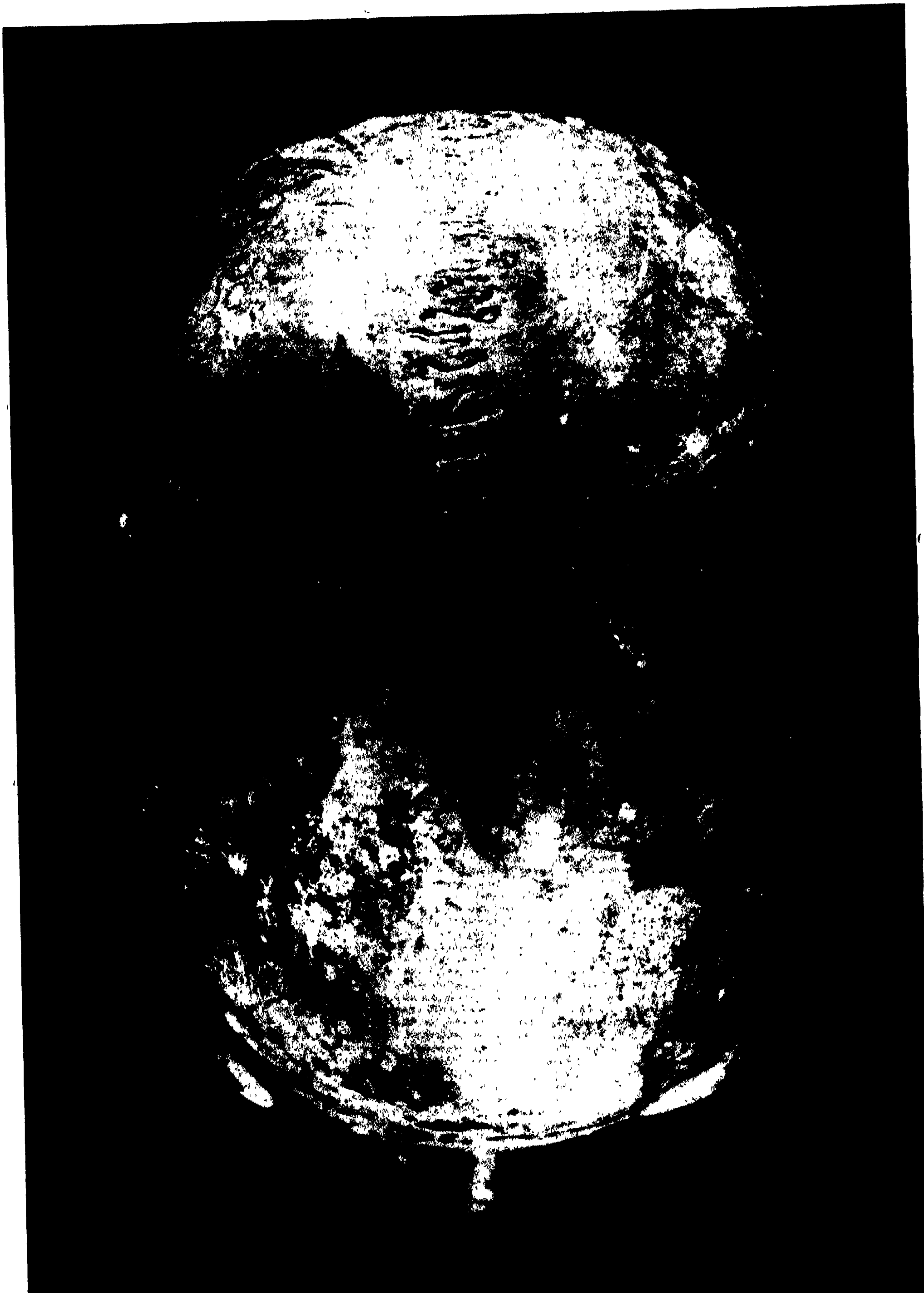
Typical Badarian Male (5400 a). *Norma lateralis dextralis* (circa two-thirds natural size).



Typical Badarian Male (5400 a). *Norma lateralis sinistralis* (circa two-thirds natural size).

Typical Badarian Male (5400 a). *Norma occipitalis* (circa seven-eighths natural size).

Typical Badarian Male (5400 a). *Norma basalis* (circa natural size).



Typical Badarian Male (5400 a). *Norma verticalis* (circa natural size).

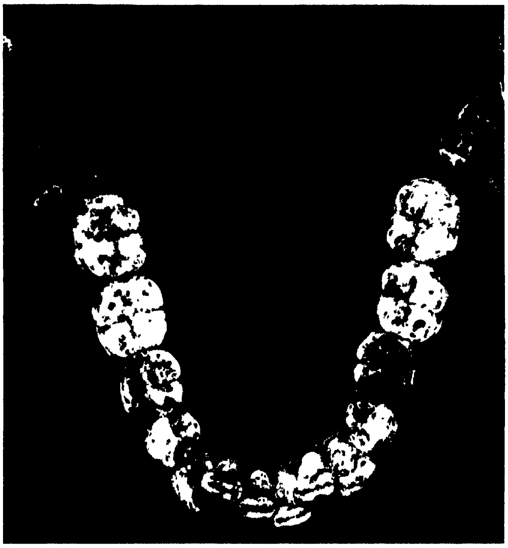
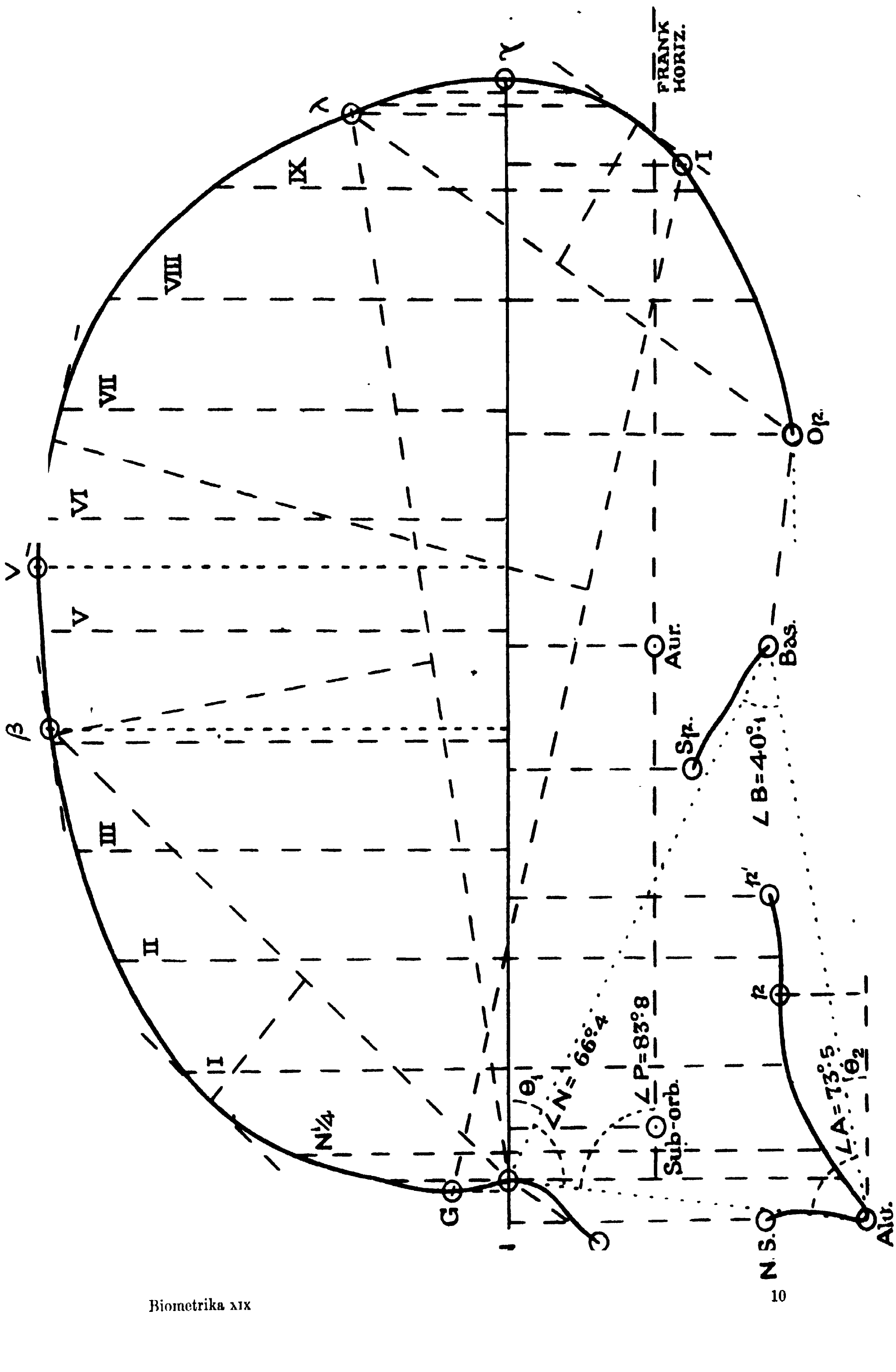


Fig. a.

Fig. b.

Fig. c.

Figs. a and b. Dental anomalies of Female (5441). For fuller description, see p. 113.
Fig. c. Ivory exostosis, R. temporal and parietal. 30 × 25 mm. at base. Male (5100): see p. 116.


FRANK HORIZ.
Op.
Aur.
Bas.
Sp.
∠B=40°·1
∠P=83°·8
∠N=66°·4
∠A=73°·5
Sub-orb.
Alv.
N.S.
G
N¹/4
θ1
θ2
β
λ
V
I
II
III
V
VI
VII
VIII
IX

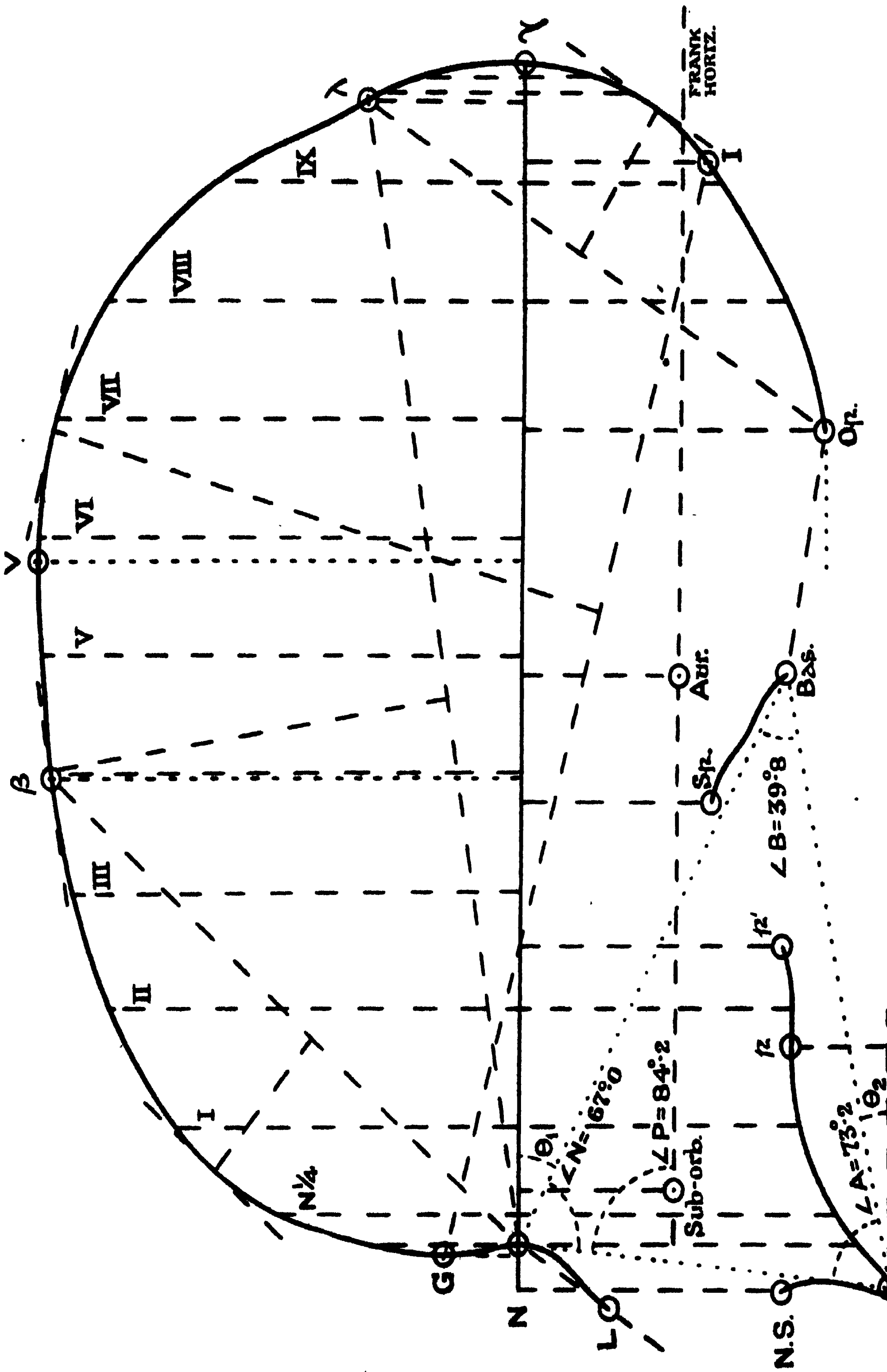


FIG. VI Sagittal Type Contour of 21 ♀ Badarian Skulls.

9. *General Conclusions.*

This study confirms the conclusions based on cultural and topographical evidence that the Badarian skulls are early Predynastic Egyptian but if anything more primitive in type than the other series of this period, though the mean direct measurements differ very little from them. The early and late Predynastic types however do show a significant difference which, if we may assume the races to be divided by a period of four or five thousand years or more, may be accounted for by slight evolutionary changes or a gradual infusion of races.

How far do these results confirm Sir Flinders Petrie's theory of a Caucasian origin? When we compare the Badarian race with others outside Egypt, it is not the Mediterranean or any Negro type which it resembles most closely but the primitive Indian, the Dravidian and the Veddah. Thus they do not oppose the suggestion of a common origin in the Caucasus from a race sending one branch westward to Egypt and Europe and another south-eastward to India. To confirm this, however, we should need series of ancient skulls from Palestine, Persia, and Western India.

10. *Measurements of Mandibles.*

Most of the Badarian skulls were complete with mandibles, but there were a few skulls and mandibles which it was impossible to pair. Several had been broken into two or more pieces and were mended by Dr Morant, but otherwise they were in good condition and on many a complete set of measurements could be taken. The separate mandibles were sexed by Dr Morant; the condition of teeth and any anomalies are noted with the remarks on cranial anomalies in the Appendix.

For a detailed description of measurements and definition of mandible points see *Biometrika*, Vol. XIV. pp. 253—260. The measurements taken with skull and mandible adjusted to Frankfurt plane were omitted as too risky for this fragile material.

Unfortunately the comparative material for mandibles is very limited. No series long enough to give reliable standard deviations, coefficients of variation or intra-racial correlations has yet been measured, so that it is impossible to compare races by means of a coefficient of racial likeness. The only series of means so far calculated for a considerable number of characters are for the Indian races, Nepalese and Tibetan A and B and for the Anglo-Saxon; although all are based on small numbers a comparison of the five types does suggest that mandibular measurements may become a reliable aid in the classification of races.

There is a wide variation in the individual measurements and in the indices, it is much greater than in the corresponding skull measurements. Thus the length-breadth index $g_o g_o / c_p l$ varies between 93 and 124 °/₀ in the males, i.e. a range of 31 °/₀, and there is a range of 47 °/₀ in the females. Similarly for $c_r h/ml$ the range is 23 °/₀. The condylar length-breadth index varies from 35 to 67. On the average the range for skull measurements is about half as great, e.g. for

the length-breadth index B/L it is 12 °/₀ in the present series. It is obvious that with such wide variations as these for a fairly homogeneous population, it will be necessary to have series of considerable length before any conclusions concerning race can be drawn from mandibular measurements.

Comparing the Badarian series with the Anglo-Saxons and Eastern races, there seems to be a consistent difference in the means depending largely on the breadth measurements (i.e. measurements perpendicular to the median plane). The Badarians have the smallest breadth and the Anglo-Saxons the greatest. In this they follow the trend of differences in parietal and bizygomatic breadths. See Table XX, $c_r c_r$ and $g_o g_o$.

TABLE XX.

Comparison of some Mandibular Measurements for Different Races (Males).

	Badarian	Nepalese	Tibetan A	Tibetan B	Anglo-Saxon
Number of Mandibles	33	19	25	12	44
w_1	109·5	115·0	117·0	122·0	123·7
w_2	88·8	96·2	96·2	99·4	103·2
zz	43·4	46·2	45·7	47·0	45·3
$c_r c_r$	86·8	93·4	93·8	93·9	100·3
$g_o g_o$	83·9	92·2	92·8	96·5	100·4
$c_p l$	76·2	74·3	74·6	78·9	77·7
ml	101·2	104·5	105·2	109·6	107·2
h_1	32·6	32·0	30·6	35·9	33·1
ih	44·3	42·2	41·8	43·6	47·9
rl	57·6	58·6	58·4	61·0	64·0
$g_o g_o/c_p l$	110·4	124·6	124·9	121·7	129·0
$c_r h/ml$	61·0	58·8	57·8	59·7	60·9
rb'/rl	58·6	55·6	55·3	56·3	51·5
$M\angle$	120°·0	124°·8	125°·3	123°·0	120°·3
$G\angle$	60°·8	66°·8	67°·1	65°·9	68°·6
$C'\angle$	73°·2	66°·4	62°·9	63°·1	68°·3

The measurements parallel to the median plane are much less variable (Table XX, $c_p l$ and ml). The Tibetan B series is on the whole larger than the other Eastern series; this appears also in the height measurements which do not vary very much. The Anglo-Saxon has a rather higher ramus (rl). The slope of the alveolar margin varies with the race quite significantly.

The differences in the indices are also fairly consistent. Thus the length-breadth index shows a marked increase from Badarian to Anglo-Saxon ($g_o g_o/c_p l$). The coronal height-length index is fairly constant ($c_r h/ml$), but the ramus length-breadth index decreases from Badarian to Anglo-Saxon (rb'/rl). There is a marked difference in the gnathio-gonial angle due to the difference in gonial breadths. The mandibular angle between the standard basal and rameal planes is rather higher for Indian races. The mental angles are difficult to measure and vary

TABLE XXI.

Badarian Mean Mandibular Measurements.

		Lengths with Callipers														
		w_1	w_2	h_1	zz	$c_r c_r$	rb	rb'	G_2'	$c_y c_r$	$g_o g_o$	$g_n g_o\ l$	$g_n g_o\ r$	$c_y l$	$c_y b$	$m_2 p_1$
Badarian	Mean	109·5	88·8	32·6	43·4	86·8	36·7	33·6	41·1	33·8	83·9	82·0	82·4	20·3	9·8	27·3
♂	No.	30	32	34	36	29	34	39	29	37	31	31	31	36	37	33
Badarian	Mean	105·2	86·2	31·5	42·0	86·3	35·6	32·0	41·0	33·8	80·4	79·2	79·8	18·8	9·7	27·1
♀	No.	16	21	18	22	17	21	23	21	21	20	20	20	18	21	22

		Lengths with Callipers				Tape	Lengths and Heights on Mandible Board									
		$p_a d_t$	$p_a g_n$	$p_a d_f$	$g_n d_f$	$g_o p_a g_o$	ih	ih'	$c_r h$	$c_y h$	$d_t h$	$m_2 h$	$p_1 h$	$c_p l$	rl	ml
Badarian	Mean	26·7	8·5	23·4	29·6	189·5	44·3	12·2	61·8	53·8	36·1	24·5	30·4	76·2	57·6	101·2
♂	No.	32	36	31	31	31	35	33	33	35	31	31	30	33	33	33
Badarian	Mean	26·8	7·5	23·5	28·8	184·4	40·2	11·6	57·6	48·8	34·4	24·0	29·4	74·1	53·2	99·8
♀	No.	18	20	18	18	20	21	16	21	20	19	20	19	19	18	19

		Indices						
		$100\ c_r h/ml$	$100 c_r c_r/ml$	$100\ g_o g_o/c_p l$	$100\ rb'/rl$	$100\ c_y b/c_y l$	$100\ g_o g_o/c_r c_r$	$100\ c_y h/c_r h$
Badarian	Mean	61·0	86·2	110·4	58·5	48·4	97·4	86·9
♂	No.	32	29	30	33	36	28	32
Badarian	Mean	57·2	86·2	108·8	60·6	51·1	92·9	85·1
♀	No.	19	16	19	18	18	16	20

		Indices		Angles				
		$100\ ih'/c_y c_r$	$100 d_t h/c_r h$	$M\angle$	$R\angle$	$G\angle$	$C\angle$	$C'\angle$
Badarian	Mean	36·9	58·6	120°·0	73°·7	60°·8	71°·7	73°·2
♂	No.	32	29	34	31	31	28	28
Badarian	Mean	34·9	60·3	123°·3	70°·9	61°·0	70°·0	71°·6
♀	No.	16	19	19	14	18	19	17

very widely, so that the means on small numbers may not be very reliable. The Badarian chin seems least salient and the Tibetan chin most ($M\angle$, $G\angle$ and $C'\angle$).

The similarity of the means for the Nepalese and Tibetan A is very striking. The divergences in the Tibetan B group are largely those of size, as was noted in comparing skull measurements. For nearly all measurements the Nepalese and Tibetan races lie between the Badarian and Anglo-Saxon.

The mean measurements for the Badarians are given in Table XXI.

My thanks are due to Miss McLearn for drawing the type contours, to Dr E. S. Pearson for photographing the crania and to Professor Pearson and Dr Morant for giving me references to other data and for much helpful criticism.

ERRORS OF ROUTINE ANALYSIS.

By "STUDENT."

Introduction. Dr E. S. Pearson, *Biometrika*, Vol. XVIII. p. 192, has given the moment coefficients of the distributions of range in small samples drawn from the normal population when the number in the sample lies between 2 and 6. Mr L. H. C. Tippett, *Biometrika*, Vol. XVII. pp. 364—387, had already provided similar data for samples of 10, 20, and 60, but Dr Pearson gives improved values in his Table VIII which I have used.

These constants provide a means of drawing curves which approximate closely to the actual frequency curves of the distribution of Ranges, apparently sufficiently closely for us to use their integrals as probability integrals for the occurrence of ranges of fairly large size.

Thus the real frequency curve for range in samples of two is known to be a half normal curve of S.D. $\sqrt{2}\sigma$, whereas the Pearson curve found from the moments is a Type I with equation

$$y = 543{\cdot}062\left(1+\frac{x}{\cdot574}\right)^{\cdot569}\left(1-\frac{x}{6{\cdot}623}\right)^{6{\cdot}569}.$$

If they be drawn on the same scale, Figure 1, we see that for the greater part of the way the two curves are practically identical.

Assuming then, as seems likely, that the approximation in the case of the larger samples is even closer than in the case of samples of two, we have here a means of determining the probability of occurrence of ranges of given size in the case of quite small samples, assuming as always a normal population. Now it is just in the case of these small samples that most of the tests which have been proposed for the rejection of observations fail; there is no possibility of finding the true mean of the population. Mr J. O. Irwin, *Biometrika*, Vol. XVII. pp. 238—250, has, it is true, proposed to use Galton's differences for this purpose, but on the other hand, there are cases in which the true standard deviation of the population is known with some approach to accuracy, and it seemed in such cases Dr Pearson's work should enable us to reject determinations so widely spread as to render the occurrence of the observed range unlikely to any specified degree.

Happening to mention to Dr Pearson that I proposed to apply his work to the rejection and repetition of analytical results, he suggested that the readers of this Journal might be interested both in the application, and, indeed, in a description of the errors of routine analysis from which the necessity of rejection arises.

In endeavouring to fall in with this suggestion, I propose to set out, firstly, what routine analyses are, and to what sort of errors they are liable; secondly, the advantages that accrue from a statistical examination of these errors; and,

lastly, the bearing of Dr Pearson's paper on the vexed question of the repetition and rejection of results.

At the outset I may state that, though no analyst, I have been in close touch for some years with a routine laboratory, the authorities of which have very kindly supplied me with some of their results for the purpose of the present paper.

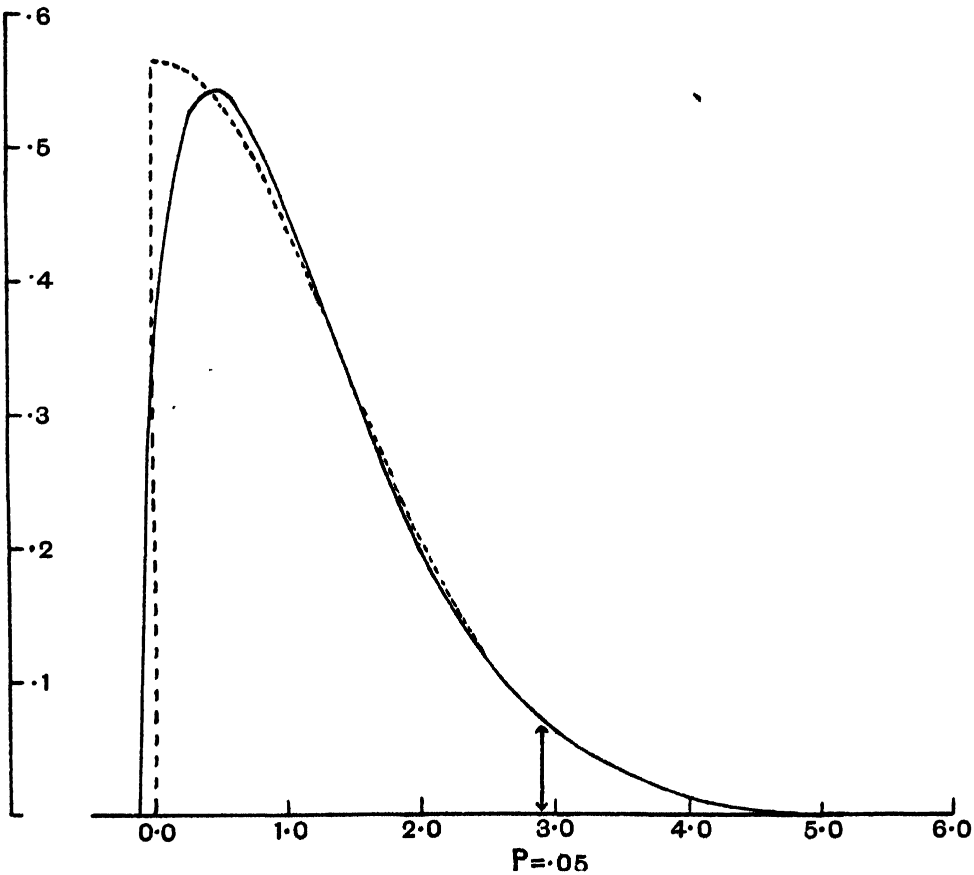


Fig. 1. Curve showing Probability of Occurrence of Various Ranges in Samples of two drawn from a series Distributed Normally with S.D. $=\sigma$.

—— Pearson's Approximation : Type I Curve

$$y = \cdot 543\left(1+\frac{x}{\cdot 574}\right)^{\cdot 569}\left(1-\frac{x}{6\cdot 623}\right)^{6\cdot 569}.$$

- - - - - Actual Curve for Comparison : Half of Normal Curve

$$y = \frac{1}{2\sqrt{\pi}\,.\,\sigma}\,.\,e^{-\frac{x^2}{4\sigma^2}}.$$

Routine Analysis. The difference between research and routine is fundamental to the scope of the present paper, and it lies in the relation between the analyst and his work rather than in the actual process of the analysis. Thus, what is at one time a research involving concentrated thought and watchfulness on the part

of the analyst may, later on, become the merest routine; every step known and prepared for in advance, and requiring not the resourcefulness of the chemist of high degree but the machine-like accuracy of the well-trained assistant.

This is not to say either that research may not make use of routine processes, as it frequently does, or that routine processes may not form the subject of research, as they constantly should; yet broadly speaking, we are not concerned with the distinguished chemist who determines the atomic weight of an element to n places of decimals, and has theories about the value of the $(n+1)$th—it would be impertinence to talk of errors in such a connection.

No, we are going to deal with the chemist who has to make similar analyses day after day and year after year; with for example, the public analyst who provides evidence to convict the milkman of watering his milk, and the grocer of sanding his sugar; with the works chemist, who maybe spends his whole life in determining the acidity or alkalinity of solutions; or again, with the assayer, who must find out which of innumerable samples of ore are payable.

There is often enough little or no scientific interest in such determinations, yet their practical value is in the aggregate enormous; the application of science to industry would without them be all but impossible.

These people are not so much troubling themselves about the nth place of decimals; their problem is to get results as quickly and as cheaply as possible; quickly, because events may be waiting upon them, and cheaply for reasons that need hardly be elaborated.

They must, however, attain sufficient accuracy for the purpose in hand, which is generally concerned with the third figure rather than the fourth, and is often enough satisfied with the second. Nevertheless without this minimum of accuracy the analysis is worthless, so that the chemist in charge of the laboratory has to make himself very sure that it is reached.

Obviously he cannot be sure, unless he has made some determinations of the error, and he can only reduce his error if he has a working knowledge of the sources of error.

Sources of Error. The first of these, very often the chief of them, is not strictly a laboratory error; it arises from the difficulty of obtaining a sample in a bottle which shall represent perhaps some tons, or even hundreds of tons, of material. This difficulty of sampling provides a convenient excuse for discordant results, but the wise chemist will see to it that the sample is drawn in a manner which will rob this excuse of any appreciable validity. And that is by no means easy: but the errors of commercial sampling do not fall within the scope of this paper, so I do not propose to say more about them here.

Nor do I propose to deal with the allied problem of sub-sampling the sample which has been received for analysis. This may be in the case of solids quite a difficult matter, and can lead to appreciable error unless a suitable technique is employed.

After this, each operation of the analysis contributes its error; I am told that the standard error of weighing on a balance is about one in two thousand; all analyses involve at least two weighings and there are often more. Then we have such things as titration, generally contributing quite a small error; transfer of material from one vessel to another; digestion at a uniform temperature; filtration, incineration, and so forth; all these add their quota.

These errors are not necessarily symmetrical, some of them involve loss of material, and for this reason a chemist will sometimes prefer the higher of two results.

Perhaps a description of a very simple analysis may illustrate the kind of thing that happens. Let us suppose that it is required to estimate the percentage of moisture in a sample of grain, not as part of a research but for the commercial valuation of a large bulk in a ship or warehouse; it will very likely be one of a number of analyses the results of which will be required by the next day.

First, the sample is sub-sampled and a weighed portion of the ground-up material is put into an oven on a small tray. The oven is kept at a constant temperature for a fixed number of hours, the tray is then removed, cooled over concentrated sulphuric acid and quickly weighed. The loss of weight is taken to be the moisture present in the weighed quantity which was put into the oven.

Here we have the errors of sub-sampling, grinding, two weighings, and of driving off moisture by heat; hardly any one of these operations is as simple as it sounds. The grinding for example, whether done in a mill or with a pestle and mortar, leaves material on the grinding surfaces; this material is not the same as the bulk but is composed of the finer or more adhesive part of it. It is, therefore, necessary to grind and throw away a small quantity before dealing with the portion which is to be weighed. Then we have the fact that organic matter exposed to the atmosphere, generally if not always, tends to get into equilibrium with the moisture in the air, hence both grinding and weighing must be done rapidly.

When in the oven the loss of weight will depend not only on the exact temperature and time, but on the ventilation of the oven and the number of samples in it. Nor is all the loss necessarily moisture, carbon dioxide may either be formed and lost by oxidation, or be lost by splitting off from some already oxidised compound. We may even get the estimation too low owing to an increase of weight due to absorption of oxygen.

Of course, in a research one would work in an atmosphere of nitrogen and weigh the moisture absorbed by phosphorus pentoxide, determining one sample at a time and weighing at intervals until the weight became constant, but routine analysis has neither time nor money for this: it has to rely on keeping the conditions constant. The most it can do is to check an occasional result by the more lengthy method.

All this sounds as if the results would be very inaccurate, yet it is not so. The moisture of grain, lying between 10–20 %, can be determined with a standard

deviation measured in percentage moisture of about ·2, or one part in 500. Naturally, different laboratories, using different ovens set up under different conditions, do not necessarily agree with one another, but they will probably agree to this order of accuracy in their relative estimates when comparing different samples, and that is usually what is required.

We now come to a phenomenon which will be familiar to those who have had astronomical experience, namely that analyses made alongside one another tend to have similar errors; not only so but such errors, which I may call semi-constant, tend to persist throughout the day and some of them throughout the week or the month.

Why this is so is often quite obscure, though a statistical examination may enable the head of the laboratory to clear up large sources of error of this kind: it is not likely that he will eliminate all such errors.

The chemist who wishes to impress his clients will therefore arrange to do repetition analyses as nearly as possible at the same time, but if he wishes to diminish his real error he will separate them by as wide an interval of time as possible. Here are some examples:

In 1905 a quantity of material was taken, mixed as well as possible and stored in Winchester bottles. Samples were taken from these and analysed daily between the beginning of April and the end of August—100 in all. This, though statistically speaking a small sample, represents an amount of work which a routine chemist will not easily be persuaded to undertake.

At each analysis seven items were determined and of these I have now examined five: all are troubled to a greater or less extent by semi-constant errors as is most easily shown by a comparison of twice the variance of a single analysis with once that of the difference between consecutive observations: if the arrangement were random they would of course be the same within the error of random sampling.

TABLE I.

Item	Twice Variance	Variance of Difference	Correlation between consecutive analyses
1	2·20	1·60	+·27
2	·625	·434	+·31
3	·0748	·0606	+·19
4	·171	·157	+·09
5	5·42	4·68	+·09

Of course, not all of these correlation coefficients are individually significant, but they are illustrative of a general phenomenon. I do not recollect having met with a case where the correlation was negative.

The two top lines of dots in Figure 2 give the individual analyses of items 2

and 3, the latter of which gives the percentage of moisture in the samples. The lines across the diagrams show the mean values of these.

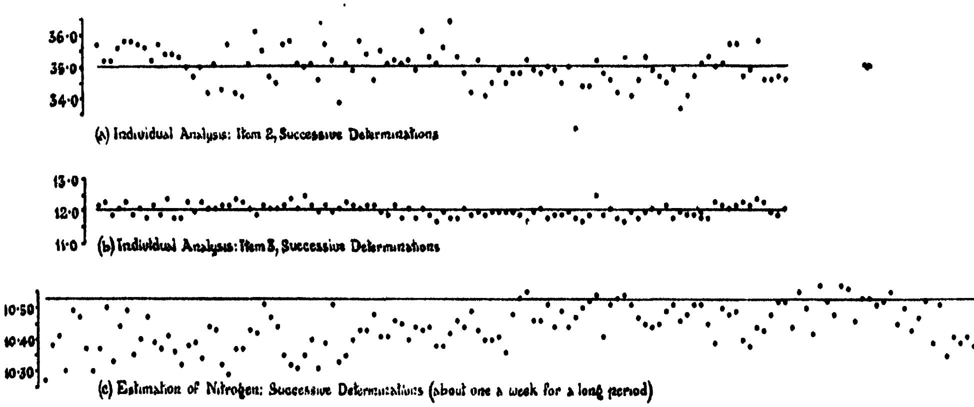


Fig. 2.

I will now give another case of a routine analysis repeated in a time series. Here, as a check on the accuracy of the estimation of nitrogen by the Kjeldahl method, a determination of the nitrogen in pure crystalline aspartic acid was made about once a week from 1903 to the present time. The method is a standard one for the determination of "amino" nitrogen in organic matter. A weighed quantity of the substance to be analysed is digested in strong sulphuric acid which destroys the organic matter and converts the nitrogen into ammonium sulphate. Excess of alkali is then added and the nitrogen distils over in the form of ammonia and is caught in a measured quantity of acid, where it is estimated by titration of the excessive acid with deci-normal soda.

Of course the amount of nitrogen in a crystalline substance can be calculated within narrow limits and the third row in Figure 2 gives the calculated (as a straight line) and the actual (as spots) since April 29th 1924 up to the end of 1926.

At first the results were all too low, but the details of the process were under examination and the later estimates have risen and the variance has decreased owing to improvements which have been effected: simultaneously, the time taken has been reduced by half. For about six months before the beginning of last November the results were remarkably good; one could have calculated the atomic weight of nitrogen from the mean with an accuracy which would hardly have disgraced research, but there has since been a falling off, the average of the last seven being rather over 1 °/$_{o}$ too low. This illustrates the sort of difficulty which arises in routine analysis for no one is conscious of any alteration in method, nor has a close search revealed the cause of the change.

The error statistics which I have cited up to the present have all been obtained by the laboratory in the course of investigation into, and control of, its error. I am now going to give some figures taken from some published analyses which seem to me to show that similar "semi-constant" errors probably exist in another

laboratory; it would surprise me to find any laboratory without them, but it is only by chance that they become apparent unless they are deliberately sought for.

The analyses are published in the Report on the Sugar Beet Experiments 1925, issued and distributed without charge by the Department of Agriculture of the Irish Free State.

These experiments were conducted at 424 farms, all the 26 counties being represented, and the complete programme, consisting of two plots of each of four varieties, one top dressed with nitrate of soda and one not, was successfully carried out in 163 cases, in another 190 it was found necessary to top dress all the plots, and the remaining 71 cases fell through for one reason or another. It is the 163 complete results with which I propose to deal.

It will be seen that each farm produced eight different lots of beet and as *each* of these was analysed to find the percentage of sugar we can average the figures to get the percentage of sugar for the farm. Further, the date on which the analyses of each farm were carried out is given in the report, and in Figure 3 are given the averages of analyses made on the same day as a central point with a line extending upwards and downwards showing the extent of twice the Standard Deviation of the mean of the number of analyses, ranging from 8 (one farm) to 96 (twelve farms), which were made on that day.

It is obvious from an inspection of the figure that there was a distinct rise of sugar between the beginning and end of November, which is doubtless due to the gradual maturing of the roots, but it is not easy to account for the marked dip shown by the analyses carried out on the 5th, 7th and 8th of December, on any other supposition except that of laboratory error.

The thirteen farms, the produce of which was analysed on those dates, were in five counties, so that the roots were sent up by five different men and may be considered a random sample of the material to be analysed in the early part of December. It has been suggested that the loss of sugar was due to the action of frost on the roots before they were drawn from the ground or whilst in transit from the farms to the State Laboratory. From enquiries which I have made I am satisfied that the lower sugar content is not attributable to such action for such frosts as were experienced did not apparently affect the leaves, let alone the roots, and the packing of the beet to ensure its arrival in fresh condition at the laboratory obviated any possibility of freezing in transit.

I have also been informed that beetroots lose sugar when they are clamped. I am assured, however, that none of the samples to which the report relates was pitted or clamped but that each sample of roots was washed, topped or crowned and despatched to the laboratory immediately after being taken out of the ground. The roots were forwarded by passenger train so as to secure quick transit, were unpacked immediately before the analysis was commenced and, as a rule, the analysis of a sample was completed within twenty-four hours of its receipt in the laboratory. It seems likely, therefore, that the low results were due to errors of a similar nature to those which were observed in the other laboratory.

To embark on a long series of analyses in order to determine error is always a considerable undertaking and is often impossible owing to the tendency of organic substances to change with time: added to this, unless special precautions are taken, such as were taken in 1905, the operators may, in spite of themselves, be more careful when analysing special samples of this kind, so that the series may not represent a random sample of analytical errors.

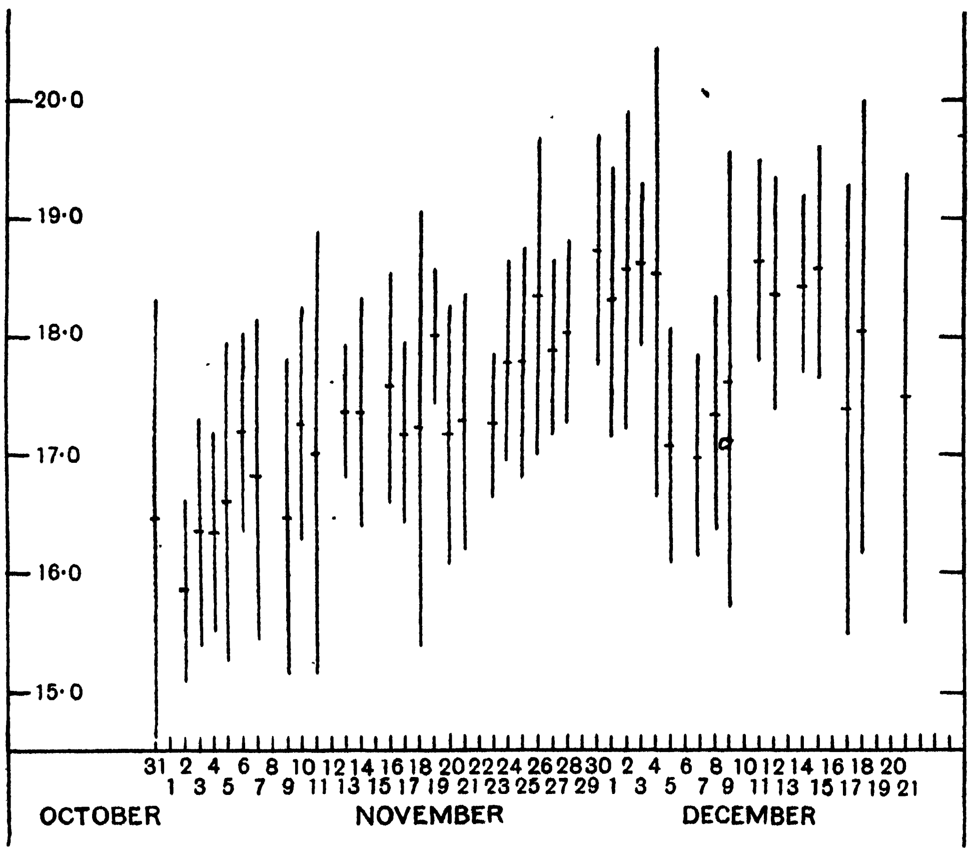


Fig. 8. Means of Daily Analyses with lines showing on each side of the Mean twice the S.D. appropriate to the Number of Analyses made on any given day. The S.D. is derived from the total observations by the formula

$$\sigma = \frac{1}{\sqrt{n}} \cdot \sqrt{\frac{S(a-\bar{a})^2}{S(n-1)}},$$

where a = Average of a Farm,
$\bar{a}$ = Mean of a Day's Analyses,
n = Number of Farms analysed in the Day.

It is convenient, therefore, to take advantage of the fact that important analyses are often repeated as part of the routine and to calculate the Standard Deviation of the error from the differences between pairs by simply dividing the variance of the differences by 2 and taking the square root.

I give here a Table of Standard Deviations of errors of the items 1 to 5, the variance of which I gave before, but having in addition further determinations made from the differences between 100 pairs analysed in 1925 and in 1926.

TABLE II.

Item	S.D. 1905	Error Differences between pairs		
		1905	1925	1926
1	1·048	·895	·731	·660
2	·559	·466	·386	·523
3	·193	·174	·138	·152
4	·293	·280	·326	·272
5	1·640	1·570	2·810	2·120

The Standard Error arrived at in this way is that of analyses made within a comparatively short period of time and does not take account of the variation of the "instantaneous mean" which we have just been observing. It is therefore the correct measure of the error if we wish to compare such analyses with each other but is too small if the analyses were separated by a wide interval. On the other hand, the Standard Error derived from 100 analyses spread over three months is too large when we are dealing with the differences between consecutive analyses. The difficulty can only be removed by reducing the secular variation to negligible limits.

Perhaps it would be well to illustrate this point in some further detail. Suppose a merchant to be offered two samples of grain at the same price: as far as he can judge they are of equal value but he is uncertain whether the moisture is the same. He gets them analysed and is returned the figure 14 °/₀ for sample A, and 15 °/₀ for sample B. If the Standard Deviation of the error is ·2 °/₀ clearly he should purchase A; if the error were 4 °/₀ it would not much matter which he bought. But observe in this case, as in many others, he is really only concerned with the difference between A and B, and if he controls the analysis he will get them done alongside each other so as to avoid their being affected by semi-constant error, and the error of the analysis will be about that found from 100 differences.

On the other hand, suppose he has bought a cargo of grain and an analysis tells him that the moisture is 17 °/₀ while it is common knowledge that 17·5 °/₀ is the highest moisture at which grain will keep. Here he is not concerned with relative but with absolute values and the error now includes the semi-constant error, so that the value deduced from the 100 analyses spread over a long time is the better.

As a sort of corollary of the existence of semi-constant errors in the same laboratory, we find that different laboratories have different constant errors, and a wise man will always consult the same analyst and not be troubled overmuch if a second analyst does not exactly agree with him.

I have now, I hope, shown that routine analyses are subject to errors of which

it behoves the head of the laboratory to be well aware. He may then judge whether his analyses are sufficiently accurate to bear the weight of any actions which it may be proposed to base upon them, and if not, how many repetitions will suffice to make them so; he will realise that an analysis made elsewhere is not necessarily less valuable than his own because it does not agree absolutely with it, and he will be in a better position to set about improving the details of his methods than if he were ignorant of the magnitude of his errors.

I now turn to the particular point raised by Dr Pearson's paper. It will be realised from what has gone before that important analyses may have to be repeated and the same applies of course to those which have given results at variance with *à priori* expectation. Very important results may even have to be repeated more than once, and it is only natural to regard these pairs—triplets or quartets—with suspicion if the results are not "concordant," i.e. have a wide range.

The result is that there is a tendency to make further repetition in such cases, to reject discordant results, and to accept the mean of the remaining observations: all the same this instinctive distrust of width of range needs some justification, and, if justified, some rules for repetitions.

For if the error were normally distributed there would be no advantage in rejection; this follows from the fact that in normal distributions there is no correlation between the square of the mean and the variance: similarly, in platykurtic* distributions those samples with large variance even tend to have the *more* accurate means. Actually, however, many if not most routine analyses have a leptokurtic error system, possibly because the standard deviation as well as the mean is subject to variation with time, and in such cases rejection of outlying observations improves the accuracy of the mean; apart from this we are all fallible and the procedure takes account of blunders.

* In case any of my readers may be unfamiliar with the term "kurtosis" we may define mesokurtic as "having β_2 equal to 3," while platykurtic curves have $\beta_2 < 3$ and leptokurtic > 3. The important property which follows from this is that platykurtic curves have shorter "tails" than the

normal curve of error and leptokurtic longer "tails." I myself bear in mind the meaning of the words by the above *memoria technica*, where the first figure represents platypus, and the second kangaroos, noted for "lepping," though, perhaps, with equal reason they should be hares!

The following table gives the values of β_2 for samples of the five items of analysis which I have given before:

TABLE III.

Item	100 analyses of 1905	Differences between consecutive analyses of 1905	Differences between 100 pairs in 1925	Differences between 100 pairs in 1926
1	3·1	2·7	4·8	5·1
2	3·5	2·9	8·2	7·4
3	2·3	2·6	2·9	3·2
4	2·9	2·7	10·4	16·2
5	10·0	5·5	5·0	7·1

In this Table the differences between the β_2's of twenty years ago and those of the present day are rather remarkable, and though with small samples such as these the standard deviation of β_2 is enormous I should hesitate to assert that they are due to random sampling; I am inclined to think that there has probably been a two-fold change, (1) that the error of the great majority has decreased, and (2) that possibly owing to work being carried on at higher pressure there is a rather greater liability to blunders. In this way the S.D. has remained much the same but the kurtosis has increased. Be that as it may, the tendency to leptokurtosis is apparent and repetitions justified except in the case of No. 3, which, as I mentioned before, indicates moisture. Here the kurtosis of the difference between pairs is approximately "meso" while that of the 100 analyses appears to be distinctly platykurtic; this is in accordance with another distribution of moisture determinations which I have examined.

Why this should be I have no idea, but obviously if a normal error were superposed on an instantaneous mean which moves to and fro on, let us say, a sine curve, the resulting distribution would be platykurtic: something of this sort may have happened.

Assuming however that discordant observations are to be repeated and if necessary rejected, it is obviously of advantage to work on a regular system, and since we do not know where the mean is I propose to use the range as follows:

Let W_n be the limit at which with a sample of n, the chance of obtaining a greater range than W_n is p (say ·05), then if w_n the actual range of a sample be greater than W_n repetition should be made. Let w_{n+1} be the range of the new sample including the repetition, then if $w_{n+1} < W_{n+1}$ the mean of the $n+1$ results should be accepted. If, on the other hand, $w_{n+1} > W_{n+1}$ the most outlying observation should be rejected, and if then the resulting $w_n < W_n$ the mean of these n should be accepted, but if not, a further repetition should be made and the whole $n+2$ observations examined afresh, and so on until a sample of at least n is

obtained lying within the required limits. For example we may have a quartet of analyses

$$\left.\begin{array}{l} 22{\cdot}8 \\ 23{\cdot}5 \\ 26{\cdot}0 \\ 26{\cdot}6 \end{array}\right\} \text{ the values of } W_n \text{ for this analysis (S.D. ·675) being as follows } \begin{array}{l} W_4 = 2{\cdot}4. \\ W_5 = 2{\cdot}6. \\ W_6 = 2{\cdot}7. \\ W_7 = 2{\cdot}8. \end{array}$$

Here $w_4 = 3{\cdot}8$ so we repeat and get 23·9. Then $w_5 = 3{\cdot}8$ and we reject 22·8 leaving $w_4 = 3{\cdot}1$. We therefore repeat again getting 25·5. Then we have $w_6 = 3{\cdot}8$, $w_5 = 3{\cdot}1$ (rejecting 22·8) and $w_4 = 2{\cdot}5$ (rejecting 26·6). Still another repetition gives 25·0 and we reject in turn 22·8, 26·6, and 26·0, leaving 23·5, 23·9, 25·0, and 25·5, with a range of 2·0 and an average of 24·5 which we accept.

To obtain W_n, the curves giving the frequency distributions of range for samples taken from a normal population were drawn from Pearson's constants and the limits at which p is ·1, ·05 and ·02 were determined. This gives us limits for samples of 2, 3, 4, 5, 6, and between 6–10 we can interpolate with the aid of Tippett's values for 10, 20, and 60. W_n is of course given in terms of the standard error calculated from samples of analyses such as I have instanced above.

TABLE IV.

	$p = {\cdot}1$	$p = {\cdot}05$	$p = {\cdot}02$
W_2	2·3	2·9	3·3
W_3	2·9	3·4	3·8
W_4	3·2	3·6	4·1
W_5	3·4	3·8	4·3
W_6	3·7	4·0	4·5
W_7	3·7	4·1	4·5
W_8	3·8	4·2	4·6
W_9	3·9	4·3	4·7
W_{10}	4·1	4·5	4·9

Figure 4 gives a comparison of the distribution of range in samples of 4 calculated from Pearson's constants with the actual distribution in samples of 4 which occurred in the ordinary course of business when an important series of analyses was being made: the item was that which I have indicated as (1) in the Tables of this paper.

It will be seen that while the general shape of the curve gives a fairly good fit ($P = {\cdot}13$ for 5 groups) there is excess at the tail end showing the leptokurtic nature of the distribution and the advantage of repetition.

To recapitulate, routine analyses are subject to errors of which an estimate can be made either by a special analysis of a comparatively large number of samples of the same material, or by considering the differences between pairs which occur in the ordinary course of business. Owing to the fact that there is usually a secular variation in the error these will not in general give the same result and care must be exercised in the use of the standard deviation obtained. From such determinations of error combined with certain factors obtained from Dr Pearson's

paper on the range of small samples, we have derived limits at which repetitions should be made and beyond which outlying observations should be rejected. A rule is given for the application of this procedure, but it should always be remembered that such rules are to be regarded as aids to and not as substitutes for common sense.

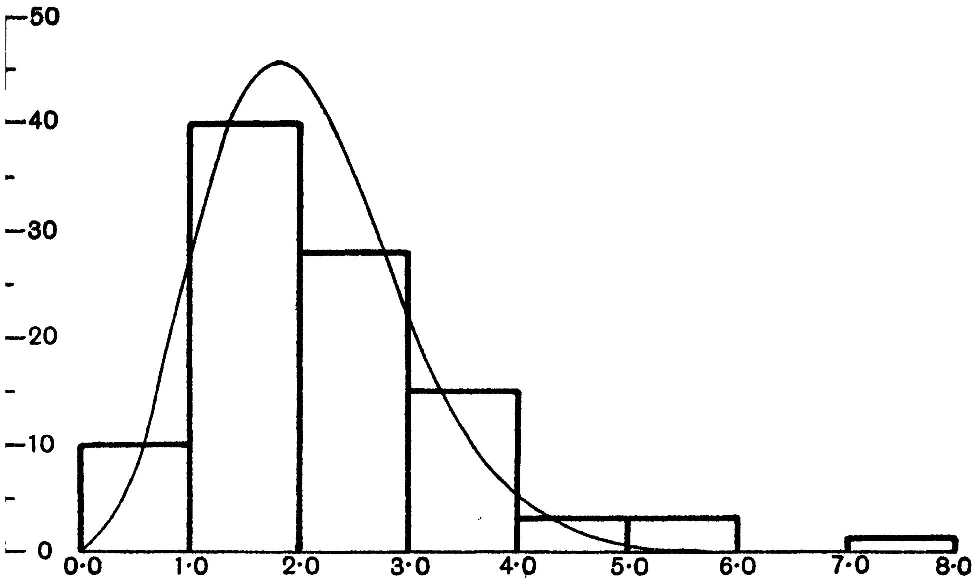


Fig. 4. Frequency Curve showing Expected and Histogram showing Actual Number of Ranges of given size in Samples of 4 for 100 trials.

Equation to the curve is:

$$y = 45{\cdot}694\left(1+\frac{x}{2{\cdot}101}\right)^{4{\cdot}499}\left(1-\frac{x}{9{\cdot}055}\right)^{19{\cdot}395}.$$

Equations to the Curves for Distribution of Ranges in Samples of Various Sizes.

n	Type of Curve	Equation (to give $N=1000$)	
2	I	$y = 543{\cdot}062\left(1+\frac{x}{{\cdot}574}\right)^{{\cdot}569}\left(1-\frac{x}{6{\cdot}623}\right)^{6{\cdot}569}$.	
3	I	$y = 462{\cdot}017\left(1+\frac{x}{1{\cdot}547}\right)^{2{\cdot}456}\left(1-\frac{x}{7{\cdot}590}\right)^{12{\cdot}050}$.	
4	I	$y = 456{\cdot}941\left(1+\frac{x}{2{\cdot}101}\right)^{4{\cdot}499}\left(1-\frac{x}{9{\cdot}055}\right)^{19{\cdot}395}$.	
5	I	$y = 465{\cdot}255\left(1+\frac{x}{2{\cdot}478}\right)^{6{\cdot}601}\left(1-\frac{x}{10{\cdot}892}\right)^{29{\cdot}009}$.	
6	I	$y = 475{\cdot}268\left(1+\frac{x}{2{\cdot}754}\right)^{8{\cdot}688}\left(1-\frac{x}{13{\cdot}134}\right)^{41{\cdot}433}$.	
10	I	$y = 508{\cdot}412\left(1+\frac{x}{3{\cdot}659}\right)^{20{\cdot}772}\left(1-\frac{x}{73{\cdot}905}\right)^{419{\cdot}514}$.	
20	VI	$y = y_0(x-41{\cdot}567)^{31{\cdot}440}\,x^{-373{\cdot}818}$	[$\log y_0 = 603{\cdot}835\ 959\ 3$].
60	VI	$y = y_0(x-11{\cdot}673)^{39{\cdot}324}\,x^{-174{\cdot}252}$	[$\log y_0 = 187{\cdot}208\ 715\ 5$].

I should like to thank the authorities in charge of the laboratory who have allowed me to use their figures and several friends who have helped in the preparation of the paper, particularly "Mathetes" who has computed the equations and drawn the figures for me. I am further indebted to Dr Hinchcliff of the Free State Department of Agriculture who supplied me with information about the sugar beet.

DEXTRALITY AND SINISTRALITY OF HAND AND EYE*.

By T. L. WOO and KARL PEARSON.

Keine Erklärung ist aber doch besser als eine oder mehrere irrthümliche.
KARL VON BARDELEBEN. *Verhandl. der anatom. Gesellschaft*, 1909, S. 56.

(1) *Introductory.* The object of the present paper is to investigate what light can be thrown on dextrality and sinistrality of Hand and Eye by the reduction of a portion of Francis Galton's copious data stored in the Galton Laboratory and placed at our disposal for this purpose. In the present paper we shall deal only with the males of Galton's *first* Anthropometric Laboratory, that of the Health Exhibition of 1884. These comprise some 7000 cases in which records were taken of (i) the visual acuity of right and left eyes, measured by the distance in inches at which diamond type could be read; (ii) the grip in lbs. of right and left hands. What we are measuring in the latter case is clearly the relative muscular strengths of right and left hands. It is less easy in the former case to apportion our measure between muscular and sensory factors, but we certainly have a measure of the superiority for a particular purpose of the right or left eye.

The very contradictory evidence at present available for the association of ocular and manual dextrality or sinistrality as the case may be appears largely to overlook the fact that different observers have measured very different characteristics. It does not follow without experiment *ad hoc*, although *à priori* it may seem to some probable, that manual sinistrality for muscular power necessarily is associated with manual sinistrality for sensitivity, or that ocular sinistrality for acuity of vision is necessarily associated with ocular sinistrality for "sighting." In discussing the divergent results of different observers this point is occasionally overlooked, and neglecting it we are forced to the conclusion either that the data were statistically inadequate to solve the problem (which indeed is often the case), or that the experimental method was itself faulty.

Conclusions so definitely contradictory as those of the present writers and those of M. van Biervliet† can only be accounted for by a fallacious method of selection

* A study of the literature on this subject will soon convince the reader that a clear-cut terminology is very desirable. The term "dexterity" has gained a new atmosphere, but "dextrality" is sufficiently unusual to be specialised in the present scientific sense with its opposite "sinistrality." We are then able to speak of ocular dextrality, ocular sinistrality and manual dextrality, manual sinistrality with consistent terminology. We may also adopt the terms ocular neutrality and manual neutrality, when both eyes or both hands are used indifferently or equally. The term "ambidextrality" is a misnomer and should be replaced by ambilaterality or neutrality.

† "L'asymétrie sensorielle," *Bulletins de l'Académie royale de Belgique*, 3me série, Tom. XXXIV. pp. 326—366, Bruxelles, 1897.

or by a recognition that muscular sinistrality is independent of sensorial sinistrality. Professor van Biervliet is quite definite on the subject:

"Il existe une asymétrie qui paraît s'étendre à tous les organes de sens. Le côté droit chez la majorité des sujets, le côté gauche chez la minorité est plus sensible de $\frac{1}{8}$ environ que le côté opposé."

"Nous avons pu l'établir pour le sens musculaire, la vision, l'audition et le toucher." (p. 366.)

In van Biervliet's experiments the dynamometer was rejected as a means of measuring the muscularity of right and left hands on the ground that the right hand being almost always used, its practice plays a great part in the result. Instead of the dynamometer he suspended from the index finger of both hands really equal weights which the subject could not see, and then enquired of the subject which was the heavier. If the subject said that in the left hand, he was a dextralist (*droitier*), if that in the right hand, a sinistralist (*gaucher*). The experiment was repeated with several series of weights; no neutralist appeared in the 120 subjects examined. Presumably the support of the weights is muscular, and it is by no means clear why the muscular practice of the right hand should not be as influential here as in the case of the dynamometer. Having classified his *droitiers* and *gauchers* in this manner, a measure of handedness was obtained by putting four standard weights in succession on the dominant hand and loading up the other hand until the subject said the two weights were equal. In this manner a measure was reached of the degree of dextrality in dextralists, and of the degree of sinistrality in sinistralists*.

And now comes a very astonishing conclusion: there is no variation whatever in degree of dextrality or of sinistrality. If you are a dextralist, then whatever weight be suspended from your right hand the left will equate it to a weight $\frac{1}{8}$ less; if you are a sinistralist, then whatever weight be suspended from your left hand the right will equate it to a weight $\frac{1}{8}$ less. In the tables provided by van Biervliet there is only one out of 122 subjects who does not obey this rule, and van Biervliet throws him out as anomalous.

Our author next takes the case of intensity of sound, and finds that all his dextralists for muscular sense are dextralists for sound, and all his sinistralists for muscular sense are sinistralists for sound. And what is still more remarkable they require a sound in the non-dominant ear to be $\frac{1}{8}$ more intense to equate it with a sound of given intensity in the dominant ear. No exceptions appear to this rule beyond the one already thrown out for manual muscularity, who appears as a neutral practically here and is again discarded.

Turning to the visual acuity experiments they appear to have been made in a very thorough manner because both eyes were first *de novo* corrected and the appropriate glasses used in the observations. These consisted essentially in

* It does not appear to have occurred to M. van Biervliet that in this manner of experimenting he was introducing a new factor, that of judgment, as well as the mere measure of sensory intensity.

measuring the distances at which right or left eye respectively could read the smallest of Snellen's type letters. Unlike Galton's method the letters were not brought closer to the eye, but the subject starting at 9 metres distance was brought closer to the letters. Here the result was as startling as in the previous cases; whatever the visual acuity of the dominant eye might be, then that of the other eye was almost exactly $\frac{1}{9}$ less*. Again the sinistralist and dextralist for vision are the same as for muscularity and audition.

Lastly van Biervliet compared with the aesthesiometer the sensitivity at the back of the two hands. Once more the same phenomenon repeated itself. All the former dextralists were dextralists for tactile sensitivity, and all the former sinistralists were again sinistralists; on no occasion did the hand dominant muscularly show a less sensitivity than the non-dominant hand. And what is more the difference of the sensitivities of the dominant and non-dominant hands was again very nearly $\frac{1}{9}$ of the latter†.

We have entered very fully into van Biervliet's results because they provide a perfectly definite statement (accompanied by more careful experimental data than we have met with elsewhere) that a human being is either dextral or sinistral, never neutral, and what he is for one characteristic he is for all that admit of dextrality and sinistrality.

This conclusion of van Biervliet is taken as a dogmatic truth by a number of writers—chiefly American. The original idea, however, of perfect dextrality or perfect sinistrality does not appear to have originated with the Belgian investigator. We find Sir George Humphry as early as 1861 laying stress on the superiority of right hand, right eye and right foot‡. He apparently considered that the superiority of the right hand gave a slight superiority to the right leg and the right eye. He admits that none of the explanations of right-handedness are quite satisfactory; he does not consider that the slight difference in the disposition within the chest between the blood vessels which supply the right and left arms is an anatomical reason adequate to account for the disparity between the two limbs. He doubts if right-handedness is congenital, and inclines to the belief that it is merely acquired, but that the tendency to acquire it is congenital§. In other words the infant at birth has no preference but rapidly acquires one for the right hand. Humphry thinks the dominance of one hand has been a great advantage to man as thereby "we acquire a greater degree of skilfulness and dexterity than we should do if both hands were equally employed." But this does not explain why the right hand should be preferred to the left. The suggestion

* The accordance is slightly worse than in the cases of muscular sense and audition, but only slightly.

† "Donc, si l'on représente par 10 l'acuité des nerfs tactiles du côté le plus sensible, il faut représenter l'acuité du côté le moins sensible, par 9 environ. Exactement pour les droitiers par 9·06 avec une variation moyenne de 0·12 environ. Exactement pour les gauchers par 8·95 avec une variation de 0·17 environ," *loc. cit.* p. 366.

‡ *The Human Foot and the Human Hand*, Cambridge, 1861. See especially pp. 201 *et seq.*

§ Since the tendency to acquire is congenital, i.e. presumably hereditary, one may best interpret this as a post-natal ontogenetic repetition of human evolutionary history.

that the left hand must be relatively passive shielding the heart, while the right activates the spear or sword, seems to fail because it suggests that dextrality has arisen at a relatively late stage of human evolution*.

The percentage of left-handedness, very roughly a quarter†, has suggested to certain investigators that sinistrality is a Mendelian recessive. If we suppose man as other mammals to have been originally ambilateral, and then to have become unilateral, there must have been a time when dextrality and sinistrality were in a 1 : 1 ratio for the 25 °/。 to have been perpetuated. But how, and it must have been suddenly to obtain that 25 °/。, did sinistrality become recessive? If we have to assume an age in which dextrality and sinistrality were indifferent in order to start with a 1 to 1 ratio, how did sinistrality become recessive? The supporters of this view have not explained how this question is to be answered. That dextrality and sinistrality are Mendelian allelomorphs has been asserted on the basis of papers by Jordan‡ and Ramaley§. But if Jordan's pedigrees are to be trusted, sinistrality may be sometimes a dominant and sometimes a recessive character, and even this assumption scarcely satisfies all the variations in his pedigrees. It cannot be too often emphasised that if we start with a population involving $\sqrt{p}$ dominant and $\sqrt{q}$ recessive individuals the stable population after random mating will be in the proportion of p dominants, $2\sqrt{pq}$ heterozygotes and q recessives, and it is only when $p = q$ that we reach a Mendelian quarter for the recessives. What the distribution would be when a character can be occasionally recessive and occasionally dominant it is not possible to determine, but what is quite certain is that we cannot under such circumstances determine the percentage of sinistrality in the community or use the rough percentage accordance as an argument for Mendelian inheritance.

Following Humphry's idea of the advantage|| of unilaterality we can suppose

* There is small doubt that right-handedness was a characteristic of palaeolithic man, and von Bardeleben takes us back (although only on skeletal measurements) to the apes, finding the Macaque an ambilateralist, the Gibbon usually a dextralist, the Orang and Man dextralists, the Gorilla a sinistralist and the Chimpanzee usually so also. *Anatomischer Anzeiger, Ergänzungsheft*, Bd. XXXIV. (1909), SS. 48 u. f.

† 22 °/。 in van Biervliet's 100 cases, 28·8 °/。 in our 7000 cases, 29·7 °/。 in Parson's data of 865 cases, and 15·7 °/。 in Ramaley's 1130 cases.

‡ "The Inheritance of Lefthandedness," *American Breeders' Magazine*, Vol. II. 1911, pp. 19 and 113, *Journal of Genetics*, Vol. IV. 1915, pp. 67—81.

§ *American Naturalist*, Vol. XLVII. pp. 734—738.

|| There seems, however, to be quite a number of cases in which ambilaterality might be more advantageous to primitive man than unilaterality. Thus equality of both hands in ascending or descending a tree might be more valuable than increased adroitness in a single hand. Again it might be more advantageous to throw two stones, one from either hand, than one stone from a single hand, even with slightly greater weight and celerity. The "feeling" that we now recognise that we could not possibly do two things at one time—take two aims at once—even if it be not a result of our unilaterality—i.e. a measure of our loss of ambilaterality—throws us back on something of a psychological character, which may possibly be the source of unilaterality (but not of dextrality), namely that thought as a mental process is unique, thoughts may be rapidly successive, but cannot be contemporary. Yet an ambilateral man with two stones aimed in rapid succession might have certain advantages over a unilateral man who had to recharge, even if his other hand was his magazine.

a state of affairs in which man was either dextral or sinistral, and this in a chance ratio of equality. We might even go further and suppose—although we have no knowledge of the heredity mechanism by which dominance arises—that the sinistral men broke up into two groups, in the one sinistrality was dominant and in the other recessive; it would be reasonable to suggest that in like manner the dextral men were differentiated into two groups, one with dextrality dominant and the other with dextrality recessive*. Selection must then be called in to get rid of entirely or greatly reduce the dextral recessives. To achieve this we ought to find some *à priori* reason for the superiority of right-handedness which still remains undiscovered†. Yet even when these hypotheses are accepted, there will remain absolutely no reason for suggesting a Mendelian quarter for the sinistral men. We cannot follow numerically the selection process through the ages from a 1 to 1 ratio of sinistrals and dextrals and determine what their present ratio *ought* to be, we can only observe what it actually is. The transition from ambilaterality to unilaterality is difficult to explain, it is still harder to account for the change from unilaterality to dextrality. At any rate the assertion that sinistrality is a Mendelian recessive does not appear to illuminate the subject.

After Humphry a number of writers have discussed, almost always without adequate data, the problem of what we may term absolute unilaterality, which comprises two factors, namely, that for each possible character ambilaterality does not exist, and that for all such characters there is identical laterality.

P. N. Callan‡ in 1881 refers to the "finger pointing test" as demonstrating quite readily that right-handed people are right-eyed; more elaborate instruments than the finger have since been devised and will be considered later. Callan is clearly of opinion that absolute unilaterality is the rule, and that though we have binocular vision we actually use one eye, generally the right one, more than its fellow. He goes so far as to assert that of all eyes examined the right was longer in antero-posterior diameter than the left and that this was caused by the extra work done by that eye. A somewhat similar use-differentiation has been asserted on the basis of differences of length in the long bones of the human body, and we are even told that it is possible to determine from the skeleton whether the subject has been dextral or sinistral§.

G. M. Gould‖ went still further, starting with the proposition that in all the higher animals in which a visual function is developed purposive movements follow

* A recessive dextrality is connoted by a dominant sinistrality, but this might have arisen from the sinistral group only.

† Sir George Humphry holds that there was no sufficient anatomical reason for right-sidedness. He was an anatomist. Professor van Biervliet, as a physiologist, thinks there is no physiological reason for right-sidedness, but that there must be an anatomical reason yet to be determined (*loc. cit.* p. 366).

‡ "Are we Right-Eyed?" *Medical Record*, New York, April 2, 1881.

§ The method has indeed been applied to measurements on the living. Those who have measured the differences in length of long bones on the right and left sides will be extremely doubtful as to whether any instrument can read on the living to anything like the requisite degree of accuracy.

‖ Gould wrote many papers in medical and scientific journals, and his opinions and evidence were reissued finally in the book *Righthandedness and Lefthandedness*, Lippincott and Co., 1908.

as a consequence of *sight.* He considered that sight was the first source of lateral differentiation and that the very cleavage of the brain into the two independent halves was due to the unilaterality and independence of ocular function (!). Gould holds that the hand on the side of the more perfect eye will be the hand preferred for skilful acts. No real proof is provided of this second proposition. If the left hemisphere is the one which functions in the case of dextralists, it is not clear why the right hand rather than the left should be selected to obey a command, or that this hand is less easily directed by the left hemisphere than the right hand*. The centres, Gould asserts, of right-eyedness, right-handedness, speech and writing must be topographically in the left cerebral hemisphere to insure speed, accuracy and coordination of united sensation, thought, will and action.

Professor H. C. Stevens† has published some fairly effective criticisms of Gould's assertions. Stevens' arguments may be summarised here:

The dependence of movement upon vision must be accepted as a fact, but the ultimate reason for this dependence must not be lost sight of, the contraction of muscles is the final term in the sequence of events called the reflex act, of which the excitement of a sense organ is the first term. Sensation, in all conscious acts, must precede movement. That the right or left hand should come to be used exclusively for all highly specialised actions as a consequence of the right or left eye being more nearly emmetropic‡ than its fellow seems to be untenable for the following reasons: (*a*) In binocular vision it is impossible to distinguish the field of vision of one eye from that of the other; to all intents and purposes the two eyes function as one. Even if the right eye, for example, were vastly worse than the other, the right half of the field of vision would not be less clear than the opposite half; the whole field would suffer a uniformly distributed effect. (*b*) If, as Gould seems to hold, the field of vision of each eye remained distinct from the

* An order from the brain to move the little fingers of both hands, say, or of the big toes of both feet—without questioning which hemisphere the order proceeds from—seems to be *simultaneously* obeyed by both sides.

† *Science*, August 6, 1909, p. 182.

‡ In data for the General Refraction of 450 boys from 8 to 15 years of age the mean General Refraction of the Right Eye was found to be $+\cdot0975 \pm \cdot0482$ and of the Left Eye $+\cdot0950 \pm \cdot0515$. This seems to indicate that as far as General Refraction is concerned there is no superiority of the right eye. The correlation between the two eyes is indeed as high as ·92.

Further, in school inspection data not taking General Refraction, but visual acuity which is fairly highly correlated with it, we find for *circa* 1000 cases of each sex, mean value: Boys ·7342 R., ·7304 L. and Girls ·6747 R., ·6759 L.; the measurements being by Snellen's types. In other words our data show no such superiority of the Right Eye as is suggested by Gould and other writers who claim that ocular dextrality is the source of a dominant absolute dextrality. Gould states (*Medical Record*, Nov. 2, 1907) that in 96 °/₀ of infants the right eye is the better seeing eye and this compels the right hand to work with it. In other words manual follows ocular dominance. But how was the better eye in 96 °/₀ of infants ascertained and on what sized population? Was it an objective measurement of refraction, and if so with or without a mydriatic? Clearly 96 °/₀ of dextralists in infancy is out of all proportion to the numbers found in later life by many observers, and if Gould's percentage be correct there must either be a selective destruction of dextralists or dextralists must be converted into sinistralists by training or environment. Most absolute unilateralists assert on the other hand that early training converts a very large number of original or native sinistralists into artificial dextralists!

other even in binocular vision, and each eye retained potential control of the muscles of the corresponding side of the body, what would be the gain of binocular over monocular lateral vision? Why should the latter have been replaced by the former in human evolution? (c) As a matter of fact the macular region of each retina is connected with both hemispheres, and it is only the corresponding peripheral regions which are exclusively associated with one or other hemisphere*. The macular region of the right retina is connected with the right hemisphere by just as short and pervious a neurone path, as with the left hemisphere. The associative neurones between the visual and motor centres of the right, are just as short and pervious as those of the left hemisphere, and for objects situated in front of the infant, the left hand may be used as conveniently as the right. Under these circumstances, in which there are two possible paths with no advantage of one over the other, why should the nerve impulse traverse, as a matter of fact, one chain of neurones rather than the other?

Thus far Professor Stevens, and his criticisms would seem to be valid, not only as against Gould's views, but against those of later writers, who have asserted that the superiority of the right eye has been the source of human dextrality. Nor can we find any definition of what is meant by "superiority" of the right eye, still less adequate experimental evidence for its actual existence.

Le Conte† asserted that for both strength and accuracy of muscular action, and for direction and "intelligence" of sighting, he himself as a dextralist and "2 or 3" sinistralists whom he had examined possessed absolute unilaterality. He repeated Callan's method of finger pointing to a distant object to determine the dominant eye in direction-finding, a matter to which we shall return later.

Wharton criticised Le Conte's statements by the following argument: If the right side of the body shows more dexterity than the left, surely it is the left eye that should share this excellence, if we are to suppose that this difference in dexterity depends upon any central organ. A person paralysed on the left side of the body loses sight, if sight be lost at all, in the right eye and *vice versa*. The argument apparently is that an acquired inferiority of a portion of the side should be accompanied by an acquired inferiority of all organs on that side. The absolute

* In the great bulk of cases does a person pick up an object, say, without first focussing both eyes upon it? Is it picked up merely by peripheral vision? It is almost impossible to understand what some writers mean by the hand *nearer* the sighting eye. In what sense is "near" used? Thus B. S. Parson writes (*Lefthandedness*, 1924, p. 51):

"The whole matter hinges on our judgment of the direction of objects seen indirectly. This, in binocular vision, is merely approximate, while monocularly—from a fixed somatic reference point—it is absolute. It follows as a most obvious corollary that the hands, which are so dependent on vision for accurate movement, are forced to accommodate themselves to this radical deviation from binocularity and that one hand—the one nearer the sighting eye—is by every anatomical and physiological consideration invested with vastly more importance functionally than the hand which is farther away." The individual is often rendered conscious of an object by peripheral vision, but does he not when he needs to handle it automatically focus both eyes upon it before he picks it up? Under such circumstances which is the "nearer hand"? A definition of "nearer" is certainly required from those who assert that manual dextrality has arisen from ocular dominance of the right eye.

† *Nature*, March 13, 1884, p. 452 (Le Conte); March 20, 1884, p. 477 (Wharton) and May 22, 1884, p. 72 (Le Conte).

unilateralists assert that: "The right-handed individual is almost invariably also right-legged and possesses greater acuity to sight and hearing on the right side*," but apart from van Biervliet's experiments on hearing have we adequate statistical data to demonstrate that ocular dextrality is almost invariably associated with other dextralities, for example, with pedal dextrality? Where is the sufficient material for proving even that manual and pedal unilateralities are homostatic? How indeed has pedal laterality been defined and measured? It appears to us that the advocates of absolute unilaterality may be correct in their views, but that they have allowed their theories far to outrun their statistical evidence†.

A very striking example of this may be found in probably the latest work on the subject—B. S. Parson's *Lefthandedness, A New Interpretation* (New York, Macmillan Company, 1924). This book contains a restatement of the unity of ocular and manual lateralism, the author having definitely pledged himself to the dogma that we are all unilateral and that the dominant eye determines the dominant hand. The instrument he proposes to use for determining the handedness is not really novel, it is merely a modification of the simpler finger pointing indication of Callan and Le Conte. He at once binds himself to his dogma by terming his apparatus for testing the sighting eye a "manuscope," i.e. he holds that ocular lateralism involves *ipso facto* the corresponding manual lateralism. The "new interpretation" appears to be only more emphatic emphasis of the opinions of the earlier absolute lateralists. Quite a number of the earlier writers have spoken of the right eye as usually "superior," and, though they have been rather loose in the use of the term, most of them we think have been referring to "facility" of vision, i.e. to visual acuity; this is certainly the case with van Biervliet and Gould. Callan and Le Conte seem to pass from the idea of visual acuity to that of sighting or direction-finding. Le Conte, however, appears somewhat doubtful on the point for he refers not only to the strength but to the "dexterity" of a sense organ. He admits that the impression of his left eye may be as vivid as, perhaps even more vivid than, that of his right eye, yet he says he sees "more intelligently" with his right eye. As he directly afterwards refers to sighting, one must presume that sighting is the measure of relative visual "intelligence." If superiority of the right eye is to be defined by greater visual acuity or by its being less subject to errors of refraction, surely some evidence ought to be provided that the "superior" eye really does the sighting? Those who define ocular dextrality by the statement that the right eye usually does the sighting, seem to overlook the need for a demonstration that sighting (if indeed it be uniquely associated with one eye) is an index of superiority in all other ways‡. If it be not, can we be dextral in

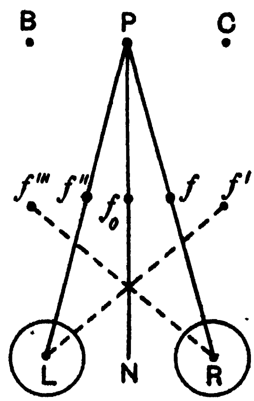
* For example: H. E. Jordan in his foreword to Parson's *Lefthandedness*, p. vi.

† A good illustration of the vague statistical statements of the absolute unilateralists may be taken from Gould's paper in the *Medical Record*, New York, Nov. 2, 1907: "The most amazing consequence of right-eyedness is lateral curvature of the spine. Approximately about 20,000,000 of the people of the United States have lateral curvature of the spine, and of these ¾ or about 15,000,000 indirectly owe their disease to right-eyedness"!

‡ "Even in adults oculists have found out that as a large rule the right eye is more nearly perfect than the left, is less subject to disease, accident, etc., and that the 'unconscious' wisdom of the

sighting and sinistral, say, in visual acuity? Unless it has been shown that greater facility of vision is synonymous with directional control then such a statement as the following "In about 96 °/₀ of infants the right eye is the better seeing eye, and this compels the right hand to work with it"* can give no support whatever to those who argue from directional control to absolute unilateralism. If, as we believe, there is evidence to show that the right eye has no substantial superiority in vision, to what must we attribute—if it really exists—its directional control?

Parson's manuscope is a very simple piece of apparatus, it is merely a modification of the Callan-Le Conte finger pointing test. Parson does not actually give an illustration of the instrument, but only diagrams, which indicate its mode of action, and he asserts definitely that it determines uniquely the sighting eye. He considers true binocular vision to be a very rare characteristic, and he apparently rarely met with it in the examination of 877 school children†—they all practically possessed ocular unilaterality as judged by sighting. On the other hand he holds that it can be deduced from the writings of Hering and Helmholtz that they themselves did possess impartial binocular vision, and that this resulted in their introduction of the "Cyclopean Eye" (*l.c.* p. 37). We can illustrate Parson's standpoint in the following diagram:



If L, R represent left and right eyes, NP the median plane, and we focus the eyes on a point P of this plane, then we believe if we hold a finger pointing to P, that it lies at f_0; really if we are ocularly dextral it lies at f, for when we close the right eye, the image moves to f', but if we close the left eye the image of the finger does not move at all. If we are ocularly sinistral the image of the finger is at f'' and passes on closing the left eye to f'''. If we are ocularly ambilateral the

organism will protect and cure it more certainly than the left, and when right-eyedness is once established nature will preserve it in despite of later oncoming amblyopia, ametropia or disease which then handicaps it much more than the left." Gould, *Long Island Medical Journal*, November, 1907. Presuming that ocular dominance is here determined by sighting, it would seem that that dominance—if we can accept Gould's statement—does not follow the better visual acuity, but is something *à priori* determined, i.e. "indirectly and partially by heredity."

* *Medical Record*, New York, Nov. 2, 1907.

† There were 12 ocular neutralists, 1·37°/₀, *loc. cit.* p. 105.

image at f_0 passes towards the right on closing the right eye and towards the left on closing the left eye. Now Parson asserts that either f and f' or f'' and f''' exist uniquely for every individual. And according as to which class an individual falls into he is not only ocularly dextral or sinistral, but also manually dextral or sinistral natively and without further question. Parson simplifies the scheme extremely; the subject uses a stereoscopic apparatus with nose notch and looks at a point P in the median plane, then objects at B and C are uncovered, and he is asked what he sees. According to Parson he will see either C or B (see figure). If he sees C he is right-eyed and right-handed; if B left-eyed and left-handed. If we are to accept the tests on 877 children, only 12 children said they saw both B and C*. If there are any exceptions to identical ocular and manual laterality Parson states that there are two possible explanations†:

"First, the original sighting eye may have become visually inferior to the other eye, and in consequence been supplanted in all sighting operations by its mate."

This statement is important as we may infer from it that Parson believes the sighting eye to be the visually superior.

"Second, the original manual bias may have been changed in greater or less degree as a result of injury to the preferred member‡, or else through reversed usage prescribed and enforced by parents or teachers."

Again: "In making tests with the manuscope the operator should ever remember that eyedness is *cause* and handedness *effect*."

For this dogma we can find in Parson's work no evidence beyond that conveyed in the following words§:

"It will be found that the hand which is on the same side of the body as the sighting eye operates easily and naturally along the near-by line of sight, while the hand brought over from the side of the quiescent eye finds itself farther from its native lateral position, in a somewhat strange and anomalous zone of activity, and therefore operates at a marked disadvantage."

No details are given of accurate experimental work demonstrating this "marked disadvantage," and we have not been able to discover it by closing one eye and picking up first with the "near-by" and then the far-off hand. The statement seems to reiterate the term "nearer hand" which we have previously found difficult of interpretation.

Parson traces unilaterality from an earlier ambilateral condition by supposing as other writers a great advantage in the former in the making and use of artefacts. But when he comes to consider why dextrality has become the usual rule he can put nothing forward but "the very obvious advantage that would accrue to the warrior who as he faced his opponent carried his spear or club in his right hand."

* *Loc. cit.* p. 80.

† These children Parson appears to disregard in the sweeping statements he makes as to absolute unilaterality.

‡ *Loc. cit.* p. 50.

§ This admission is again of importance because Parson never appears to have enquired about change of manual bias in the ocular dextralists who were manual dextralists, but only in the case of the ocular sinistralists who were manual dextralists.

But this seems to contradict his fundamental statement that eyedness is cause and handedness effect. Can we let him off with his own statement that the great problem is how man became unilateral, and that why he is chiefly dextral in his unilateralism is "a matter of secondary importance" (pp. 63–4)? His general conclusion is that "eyedness, accompanied by handedness, is a fixed characteristic of the race, pure binocular vision accompanied by ambidexterity* representing the ancient parent type, and right-eyedness accompanied by right-handedness and left-eyedness accompanied by left-handedness the two subsequent mutations" (p. 69).

Mutation is a blessed word but it does not carry us very far in a scientific account of the origin of absolute unilateralism or the reason for sinistralism being taken as recessive.

We must now pass to Parson's experimental results which seem to us singularly inadequate to demonstrate his theory. He seeks *à priori* to disarm criticism by saying that:

"Between innocent and wilful misrepresentation of various sorts the investigator's task is not always an easy one. The principal thing to remember in any case is that no matter what the present condition may be, handedness and eyedness were *originally* in strict accord" (p. 89).

But surely this is the point we want experimentally to determine? If we accept it as a truth beyond dispute why experiment at all; or if we do experiment, can we claim the impartial judgment which all scientific statistical observation demands? But let us consider Parson's experimental data.

He first took 20 random adult cases†, of course a quite inadequate number for an investigation of this importance. The manuscopic test applied to these adults (ages 18 to 64) gave 15 ocular dextralists and 5 ocular sinistralists, a quite satisfactory Mendelian quarter. Now we might have anticipated that he would have used some muscular test to determine handedness. On the contrary the only method of ascertaining handedness seems to have been enquiring which hand was most used. On this basis only one—one of the five ocular sinistralists—described himself as a manual sinistralist. Special enquiry was now made of the four remaining ocular sinistralists. In the first case, that of a lawyer, it was found that his right eye had poor visibility‡, it was therefore concluded that he was "natively" an ocular dextralist. In the second case it was found out from the man's mother§ that he had as a child been strongly left-handed although he called himself right-

* I.e. our ambilaterality.

† *Loc. cit.* p. 92.

‡ Here, as so frequently, change of native sighting is said to be associated with loss of visual acuity, or the latter is made the basis for preferring one eye as the sighting eye.

§ The absolute unilateralists admit that an infant is in accordance with the supposed ontogenesis originally an ambilateralist, and that *without training* preference for the right hand *begins* in about the seventh and eighth months. But there are no observations adequate to determine the variations in the period during which manual laterality is completely established. Mothers and grandmothers (especially the latter, who frequently have time to pay more attention to the early stages of infancy), particularly in the uneducated classes, know nothing of the physiological development of complete dextrality, and impressed by the unusualness of manual sinistrality are not unapt to emphasise this side of a normal infantile ambilaterality and mistake it for a sign of original sinistrality.

handed. In the third case the ocular sinistralist said he was right-handed, and had not been left-handed as a child; he was afterwards observed busily engaged setting type with his left hand, holding the "stick" in his right hand. In the fourth case the subject again said he was right-handed, but subsequently wrote to say that his grandmother had told him that he was left-handed until four years of age. *Of those who were ocular dextralists and said they were right-handed no questions appear to have been asked, nor were enquiries made of parents or grandparents as to whether their offspring had been as children manual dextralists or manual sinistralists.*

When we realise how many persons do not know that they have been left-handed as children, we see the importance of a test for manual laterality as apart from a mere statement that one hand is more used than the other! Parson's data—except for cases which disproved his theory—offer no security that his ocular and manual dextralists had not changed their sighting eye or their working hand. Such a test Parson does not think needful, because according to him the manuscope at once answers the question. But if four out of five ocular sinistralists can be in error as to their sight or their handedness, what right have we to assume scientifically that of 15 dextralists none will be in error either as to native sight or native hand? Over and above this the data of 20 persons observed are quite inadequate for answering a problem of this difficulty. We turned accordingly to the longer series of 877 children, observed in a public school at Elizabeth, New Jersey*. Here we find 608 ocular dextralists and 257 ocular sinistralists as found by the manuscope, while only 12 appeared to use the right and left visual lines impartially. No test of manual lateralism appears to have been given beyond asking the children if they were right or left-handed. Only *four* children whom the manuscope credited with ocular dextrality claimed to be left-handed. The first a boy said he had no eye trouble that he knew of; the second a girl said that her left eye was weak; the third a girl said that she had no eye trouble she knew of; the fourth a boy said that his right eye was the better of the two. Parson considers that these four left-handed cases are not exceptions to the rule but strong confirmations of it; they are cases "in which the visuo-manual harmony has been violated." We must again observe that there is no enquiry as to change of handedness in childhood, or whether the *left eye* is weaker than the right in any of the 604 cases, which obey the theory. We now turn to the 257 left-eyed children. Of these only 32 said they were left-handed, and 225 claimed to be right-handed. To these cross-examination was applied; only 20 of the children admitted they were once left-handed, 19 others said they did certain things with the left hand, three said they had a bad right eye, and three had undefined "eye trouble." And when these 45 children are further struck off the list there still remain 180 children out of 257 ocular sinistralists, who apparently were manual dextralists, that is about $\frac{4}{5}$ths. Is it not clear that, if the theory of absolute laterality be true†, it cannot be proved by

* *Loc. cit.* pp. 106 *et seq.*

† Parson makes no reply to the very strong arguments of von Bardeleben against the doctrine of absolute unilaterality.

a method which leaves 180 out of 257 cases exceptions to the rule? It is not possible to glide by the difficulty saying that they are cases of innocent or wilful misrepresentation (p. 89), or wherein the original tendency has been frustrated (p. 114). Parson quotes Bardeleben as saying that 6·8 °/₀ of children are admittedly left-handed and 26·0 °/₀ have been changed to right-handedness. But we find it very difficult to accept as convincing the anthropometric and anatomical examinations by which von Bardeleben determined that 26 °/₀ of his dextralists had been originally sinistralists*.

But if some 25 °/₀ of all individuals are converts to manual dextrality and yet manage to succeed decently in life, what becomes of the hypothesis, from which Parson and others start, that absolute unilaterality is essential to the efficiency of the individual†?

In conclusion our author appeals to Mendelism and Jordan to confirm his ratio of dextrality to sinistrality. If we take his 180 supposed converts to manual dextrality to be genuine dextralists then we fail entirely to obtain a Mendelian quarter. On the other hand if we suppose out of *observed* manual dextralists upwards of 70 °/₀ are natively manual sinistralists what becomes of the pedigrees of Jordan, which must be correspondingly modified, and which Parson supposes demonstrate as they stand the Mendelian character of handedness? He cannot have it both ways. If we take ocular lateralism as defined by Parson—although we know nothing at present about its inheritance—and assume that 257 out of 865 are sinistralists, we have an excess of 40·75 in our observations from the Mendelian quarter, which is 3·20 times the standard deviation (12·74); the odds are accordingly about 145 to 1 against a deviation as large or larger from the true Mendelian value occurring in 865 cases. It does not seem therefore possible to extract evidence in favour of Parson's numbers from their supposed accordance with Mendelism and, if it were possible, we should be still left with the unanswered problem as to how an indifferent unilateralism developed into a dominant dextrality and a recessive sinistrality. His major proposition that absolute unilaterality has arisen because of its "ease, rapidity, and sureness of movement, economy of muscular effort and consequent all-round effectiveness of action‡" still remains unproven for he has not tested his left-eyed converts to manual dextrality against left-eyed manual sinistralists and shown that the latter are all-round substantially more effective in action.

* They include measurements on the body, aspect of nose to right or left, feeling the cranium right and left for the size of Broca's organ of speech, etc., etc.

† "With eyedness firmly entrenched on one side of the organism and handedness struggling along on the other, there result an indirectness of action and a waste of effort that seriously interfere with the lateral harmony of function so necessary to bodily well-being and efficiency. The organism may be said to dissipate a valuable portion of its energies in antagonising *itself*, instead of conserving and concentrating its powers for the all-important struggle with its environment." Parson, *loc. cit.* p. 126.

Surely the writer might have established this fundamental basis of the absolute unilateral theory by pitting the 82 left-eyed, left-handed children against the 180 left-eyed converts to manual dextrality in a suitable series of manual tests to demonstrate that the latter were much inferior owing to their self-antagonisms!

‡ *Loc. cit.* p. 132.

We have devoted so much space to a discussion of the most recent work on our topic, not because we consider it an important contribution to our knowledge, but because it raises questions our data are able to some extent to answer. These questions are the following:

(*a*) Does the amount or frequency of ocular dextrality increase with age?

(*b*) Does the amount or frequency of manual dextrality increase with age?

Children are not trained to use the right eye rather than the left, but it is asserted that if left-handed they are taught to be right-handed.

(*c*) Is the right eye superior to the left and by what amount?

(*d*) Is the frequency of the superiority of the right eye in any way commensurable with the frequency of the superiority of the right hand?

(*e*) What if any is the association between superiority of the right eye and superiority of the right hand?

If there be no correlation, it is difficult to believe in absolute unilaterality.

(*f*) How far are van Biervliet's results—for constant superiority of one side over the other for all characters, a superiority stated by him to be $\frac{1}{9}$th—confirmed by our data for visual acuity and for muscularity of upper limbs?

Let us now turn to our own data and see their bearing on the above questions. The two characters we shall deal with in this paper are as we have stated in the opening words visual acuity as measured by the distance at which diamond type could be read respectively by right and left eyes, and muscularity of right and left hands as measured by a grip dynamometer. The grip dynamometer is not extensively used in England for any form of practice, and we believe that measurements with it, being free of the judgment factor, are just as trustworthy as suspended weights for determining native handedness. We are quite certain that these measurements are more effective than direct questioning of the subject, or of his relatives, for determining native handedness. The training which is given to youthful converts to dextralism—hand-shaking, knife using, hat-raising, handwriting, etc.—is not necessarily of an essentially muscular character. The expert dextral handwriter may still remain a powerful left-hand bowler. We do not think the discarding of the grip test justifiable, and should have had far more confidence in Parson's results had he controlled his mere statements as to handedness by such a test. In the next place the measure of visual acuity has been made without correction. Not only were spectacles far rarer in 1881 than to-day, but the big question is raised when the sight is corrected, whether we may not modify the natively "superior" eye. Had we to repeat visual acuity experiments to-day, it would require a great deal of consideration, and probably preliminary experiment, to settle whether it was advisable to use corrected or uncorrected sight. If the dominance of the eye has been determined at an early stage of development, it may well be better determined without than with correction.

In view of the fact that van Biervliet uses visual acuity, and that Parson and others who use dominance in sighting appear to attribute that dominance to

superior vision, we are satisfied that measurements of visual acuity can contribute to the problem of unilaterality as much as, if not more than, the determination of the sighting eye. The weak points of the latter method, as well as of the mere statement of "hand most used," lie (i) in their being subjective and not objective, and (ii) in their providing no relative measure of the degree of dominance of either eyedness or handedness.

Problems (*a—c*). *Has the amount or frequency of ocular dextrality any relation to the amount or frequency of manual dextrality and how are these characters affected by age?*

Taking first all our material in which no defect of the eye was reported, and where uncorrected the eyes could read diamond type at 4 to 5 inches*, we have for all ages 4948 cases. Of these we found no detectable visual acuity difference in no less than 2711 cases, the right eye was stronger in 1108 cases and the left eye in 1129 cases. In other words we have for percentages:

Ocular Dextralists	Ocular Ambilateralists	Ocular Sinistralists
22·82 °/₀ ± ·40 °/₀	54·79 °/₀ ± ·48 °/₀	22·39 °/₀ ± ·40 °/₀

The difference in the percentages of dextralists and sinistralists is insignificant having regard to the probable errors, and we must conclude on this material that so far from ocular ambilateralists being a vanishing minority in the community, they form more than half of it. Unilaterality, if we measure "superiority" of eye by visual acuity, is not the universal rule; it exists in slightly over 45 °/₀ of males. Further the superiority of the right eye, if we measure it by the percentage of ocular dextralists, is only slightly more frequent in occurrence than superiority of the left eye; ocular sinistralists are merely 0·43 °/₀ less frequent than dextralists; and this is insignificant. Accordingly we find it impossible to speak of a "master eye," or to credit the right eye with being the master eye in 75 °/₀ of individuals. We are thus driven to the conclusion either that laterality is not absolute, i.e. is not the same, if we measure different characters of the eye, or that there is some grave fault in the manner in which either Parson or Galton obtained his data, or possibly in both.

Before we examine conceivable sources of error let us give similar results for manual laterality as evidenced by grip. For the same 4948 cases there were only 333 ambilateralists (or ambidextrous persons as some prefer to call them), 3185 were manual dextralists and 1430 manual sinistralists, leading to the following percentages:

Manual Dextralists	Manual Ambilateralists	Manual Sinistralists
64·37 °/₀ ± ·46	6·73 °/₀ ± ·24	28·90 °/₀ ± ·43

* An eye that could not read diamond type at 4 to 5 inches may be assumed to be defective, and therefore can be discarded from consideration.

Here we are nearer to the usual views, there are somewhat under 30°/₀ of sinistralists, but the probable error of the percentage precludes its being identified with a Mendelian 25 °/₀. The ambilateralists are few in number, but their probable error precludes their being treated as non-existent. The dextralists are in a majority, but the probable error of their percentage is such that we cannot look upon them as three-quarters of the population.

If any considerable number of the manual dextralists are really converted sinistralists, and this is what Parson supposes in his own case, it is clear that the number of manual sinistralists would be still farther removed from the Mendelian quarter. We are inclined to think that the grip difference test really does closely indicate the total number of manual sinistralists in the population, and that no correction is needful here as when the handedness is determined by a verbal enquiry.

A comparison with the ocular distribution of the same population shows us that it is quite impossible for all manual dextralists to be really ocular dextralists unless we assume the great bulk of the ambilateralists to be, for some undiscovered reason, really ocular dextralists.

It may not be without interest to examine the combined table for ocular and manual lateralities.

TABLE I.

Percentage in each Category of Lateralism.

Ocular Laterality from Visual Acuity	Manual Laterality from Grip-Measurement			Totals
	Dextrality	Ambilaterality	Sinistrality	
Dextrality	721 (14·57 °/₀) [22·63 °/₀]	68 (1·37 °/₀) [20·36 °/₀]	319 (6·45 °/₀) [22·32 °/₀]	1108 (22·82 °/₀)
Ambilaterality ...	1751 (35·39 °/₀) [54·98 °/₀]	187 (3·78 °/₀) [56·16 °/₀]	773 (15·62 °/₀) [54·05 °/₀]	2711 (54·79 °/₀)
Sinistrality ...	713 (14·41 °/₀) [22·39 °/₀]	78 (1·58 °/₀) [23·48 °/₀]	338 (6·83 °/₀) [23·63 °/₀]	1129 (22·39 °/₀)
Totals	3185 (64·37 °/₀)	333 (6·73 °/₀)	1430 (28·90 °/₀)	4948 (100 °/₀)

Here the numbers in round brackets are the percentages of the total population and the numbers in square brackets are the percentages in the cells of the numbers at the foot of the columns. From the percentages in round brackets we see that only 25·18°/₀, i.e. one quarter of the total number, have the same laterality for eye and hand. From the percentages in square brackets we see that for each type of manual laterality, the percentages of eye laterality are sensibly the same as in the totals column. In other words the distribution of eye laterality is sensibly unaffected by the character of the manual laterality. We can place the matter in another light before the reader. The following table gives the percentages in each

category actually observed and the columns headed *expected* give the percentages which might be expected if there was no relation whatever between the manual and ocular lateralities.

TABLE II.

Percentages Observed and Expected for each Category of Manual and Ocular Lateralities.

Ocular Laterality	Manual Laterality						Totals
	Dextrality		Ambilaterality		Sinistrality		
	Observed	Expected	Observed	Expected	Observed	Expected	
Dextrality ...	14·57 °/o	14·69 °/o	1·37 °/o	1·53 °/o	6·45 °/o	6·59 °/o	22·82 °/o
Ambilaterality ...	35·39 °/o	35·27 °/o	3·78 °/o	3·69 °/o	15·62 °/o	15·83 °/o	54·79 °/o
Sinistrality ...	14·41 °/o	14·41 °/o	1·58 °/o	1·51 °/o	6·83 °/o	6·48 °/o	22·39 °/o
Totals	64·37 °/o		6·73 °/o		28·90 °/o		100 °/o

The differences between the percentages actually observed and those to be expected if the manual and ocular lateralities were absolutely independent are very slight, occurring only in the decimal places of the percentages, and are such as might easily arise from errors of random sampling, to say nothing of anomalies of observation. We must conclude from our data that *there is no evidence whatever of even a correlation between ocular and manual lateralities to say nothing of a master eye determining which is the master hand. Our data are wholly opposed to the theory of absolute laterality.*

Now the propounders of the theory of absolute laterality appeal repeatedly when their theory fails to the principle that the master or "superior" eye has lost its native superiority of vision, and they frequently speak of the right eye as having superior vision, and so being usually the master eye. They thus throw back even "sighting" on superior visual acuity in the right eye. Our data exhibit no such superiority of vision; but, if appeal has to be ultimately made in cases of what they consider anomalous sighting to superior vision, then it seems at least as rational to start from visual acuity as from sighting, indeed more so. If they now drop the visually superior eye as the source of sighting, then at least they must admit that for ocular characters absolute laterality does not hold; an individual may be dextral for one character of the eye and sinistral for a second. And if for the eye, why not for the hand? It does not follow that superior muscularity and superior sensitivity are both dextral or both sinistral.

If we cannot admit that sighting is a better character for judging ocular lateralism than visual acuity, still less can we allow that our other character—actual

measurement of muscularity by grip—is a worse test of laterality of the hand than asking children, or even adults, which hand is more used by them. Until more elaborate *measurement* on ampler numbers is forthcoming, we feel compelled to assert that the existence of absolute unilaterality in man is simply unproven.

We will now proceed to analyse our data a little more closely. So far we have only considered the laterality of individuals, clubbing together all ages. We next ask: Do the amounts of dextrality and sinistrality change with age? Have we possibly confused the issues by clubbing all ages together? Naturally in our investigations this question, although taken later here, was answered first. Table III gives the percentages for the various categories of manual and ocular laterality. Having regard to the probable errors, there is no evidence of any definite change of laterality with age. If we were to disregard the probable errors—which would be of course absurd—we might even assert that the younger the individual the greater the amount of manual dextrality; at any rate for our data, which are not very copious for the younger ages, there is no sign whatever of that great swing over—some 70 °/₀ of the sinistralists to dextrality—which some authorities seek to establish. Still disregarding the probable errors, we might even suppose at adolescence a swing over to sinistrality*! Anyhow there is evidence enough to show that 65 °/₀ of manual dextrality is very closely the rule at all ages, and that the corresponding value for ocular dextrality is 25 °/₀. No age changes will bring these values into identity.

Diagrams I and II give for each quinquenniad the percentages of each laterality for the hand and eyes respectively. Zones of 2·5 times the probable error determined for the frequency of each quinquenniad are given, and it is seen that there is small individual significance. But taking the sweep of the graphs as a whole, the following suggestions may be made:

(*a*) There is even up to age 45 a more or less continuous fall in manual dextrality and an increase in manual sinistrality.

(*b*) Up to age 33 there is a fall in ocular dextrality and an increase in ocular sinistrality.

(*c*) After age 40 there appears to be an increase of ocular ambilaterality, possibly induced by the increasing use of glasses†.

The next criticism of our measurements which has to be met is of the following character: If it be admitted that the percentages of the laterality categories are not markedly changed during the bulk of life, say from 18 to 65, still the intensities of dextrality and sinistrality may vary and invalidate any clubbing of different age groups. This criticism can hardly be raised by the absolute lateralists, who

* Taking all observations under 18 years of age the percentage of manual sinistrality is 25·86 ± ·99 °/₀, but for the entire population it is 28·79 ± ·37 °/₀, the difference is 2·93 ± 1·06 °/₀, which is 2·8 times its probable error, and therefore approaching significance. The imaginative investigator might even suggest that the encouragement of sport in English boyhood may tend to increase sinistrality at the expense of slight dextrality or ambilaterality!

† The reader must be reminded that the acuity is measured with uncorrected sight. Corrected sight may render the determination of dextral or sinistral superiority nugatory.

TABLE III. *Percentages of Dextrality, Sinistrality, and Ambilaterality of Eye and Hand for two or more years of age and all ages together.*

(*a*) Ocular Laterality.

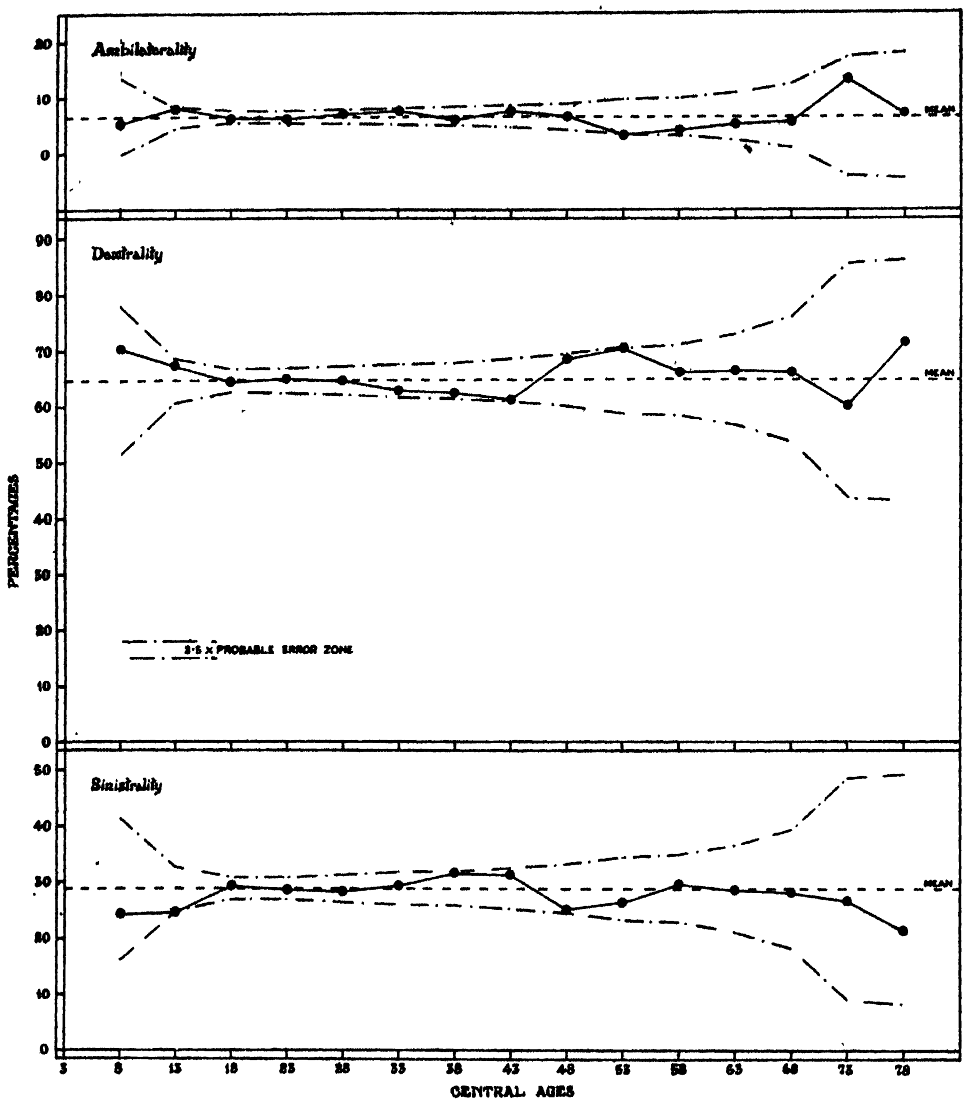
Age	Observations	Percentages		
		Dextrality	Sinistrality	Ambilaterality
6—10	32	37·5 ± 5·2	12·5 ± 5·1	50·0 ± 6·0
11—	64	32·8 ± 3·7	12·5 ± 3·6	54·7 ± 4·2
13—	173	27·2 ± 2·2	21·9 ± 2·2	50·9 ± 2·6
15—	365	28·2 ± 1·5	21·6 ± 1·5	50·2 ± 1·8
17—	570	26·0 ± 1·2	22·1 ± 1·2	51·9 ± 1·4
19—	659	24·3 ± 1·1	21·5 ± 1·1	54·2 ± 1·3
21—	565	25·0 ± 1·2	26·7 ± 1·2	48·3 ± 1·4
23—	542	22·7 ± 1·3	27·1 ± 1·2	50·2 ± 1·4
25—	406	23·2 ± 1·4	24·6 ± 1·4	52·2 ± 1·7
27—	348	23·0 ± 1·6	26·7 ± 1·5	50·3 ± 1·8
29—	353	23·2 ± 1·6	26·6 ± 1·5	50·2 ± 1·8
31—	281	20·3 ± 1·7	28·5 ± 1·7	51·2 ± 2·0
33—	254	26·8 ± 1·8	20·1 ± 1·8	53·1 ± 2·1
35—	244	23·3 ± 1·9	27·5 ± 1·8	49·2 ± 2·2
37—	233	28·8 ± 1·9	24·4 ± 1·9	46·8 ± 2·2
39—	200	30·0 ± 2·1	23·0 ± 2·0	47·0 ± 2·4
41—	167	20·4 ± 2·3	26·9 ± 2·2	52·7 ± 2·6
43—	143	25·9 ± 2·4	21·7 ± 2·4	52·4 ± 2·8
45—	109	23·9 ± 2·8	16·5 ± 2·8	59·6 ± 3·2
47—	96	29·2 ± 3·0	29·2 ± 2·9	41·6 ± 3·4
49—	56	25·0 ± 3·9	23·2 ± 3·9	51·8 ± 4·5
51—55	73	27·4 ± 3·4	19·2 ± 3·4	53·4 ± 3·9
56—60	36	38·9 ± 4·9	22·2 ± 4·8	38·9 ± 5·6
61—75	27	22·2 ± 5·6	25·9 ± 5·6	51·9 ± 6·5
All Ages	5996	25·00 ± ·38	24·13 ± ·37	50·87 ± ·44

(*b*) Manual Laterality.

Age	Observations	Percentages		
		Dextrality	Sinistrality	Ambilaterality
6—10	37	70·3 ± 5·3	24·3 ± 5·0	5·4 ± 2·8
11—	66	66·7 ± 4·0	22·7 ± 3·8	10·6 ± 2·1
13—	174	66·1 ± 2·4	24·1 ± 2·3	9·8 ± 1·3
15—	394	67·3 ± 1·6	26·4 ± 1·5	6·3 ± 0·8
17—	608	66·8 ± 1·3	28·0 ± 1·2	5·2 ± 0·7
19—	698	61·7 ± 1·2	31·4 ± 1·2	6·9 ± 0·6
21—	612	65·0 ± 1·3	28·8 ± 1·2	6·2 ± 0·7
23—	590	64·6 ± 1·3	28·3 ± 1·3	7·1 ± 0·7
25—	446	65·7 ± 1·5	27·4 ± 1·5	6·9 ± 0·8
27—	384	64·1 ± 1·6	28·1 ± 1·6	7·8 ± 0·9
29—	384	64·6 ± 1·6	30·2 ± 1·6	5·2 ± 0·9
31—	304	65·1 ± 1·8	28·3 ± 1·8	6·6 ± 1·0
33—	278	61·2 ± 1·9	30·9 ± 1·8	7·9 ± 1·0
35—	271	63·5 ± 2·0	28·8 ± 1·9	7·7 ± 1·0
37—	264	60·6 ± 2·0	34·8 ± 1·9	4·6 ± 1·0
39—	226	62·8 ± 2·1	29·7 ± 2·0	7·5 ± 1·1
41—	189	58·2 ± 2·3	34·4 ± 2·2	7·4 ± 1·2
43—	167	65·9 ± 2·5	26·9 ± 2·4	7·2 ± 1·3
45—	135	60·0 ± 2·8	28·9 ± 2·6	11·1 ± 1·4
47—	131	73·3 ± 2·8	22·9 ± 2·7	3·8 ± 1·5
49—	107	65·4 ± 3·1	28·1 ± 3·0	6·5 ± 1·6
51—	74	67·6 ± 3·7	29·7 ± 3·6	2·7 ± 1·9
53—	79	72·2 ± 3·6	24·0 ± 3·4	3·8 ± 1·9
55—	72	66·7 ± 3·8	25·0 ± 3·6	8·3 ± 2·0
57—	71	69·0 ± 3·8	29·6 ± 3·6	1·4 ± 2·0
59—	51	64·7 ± 4·5	33·3 ± 4·3	2·0 ± 2·3
61—65	98	66·3 ± 3·3	28·6 ± 3·1	5·1 ± 1·7
66—70	53	66·0 ± 4·4	28·3 ± 4·2	5·7 ± 2·3
71—81	29	65·5 ± 6·0	24·2 ± 5·7	10·3 ± 3·1
All Ages	6992	64·62 ± ·39	28·79 ± ·37	6·59 ± ·20

believe that each individual is natively an absolutely determined lateralist, and the only modification is a large transfer of native sinistralists (70 to 80 %) to

PERCENTAGES OF MANUAL LATERALITY FOR EVERY FIVE YEARS OF AGE



PERCENTAGES OF MANUAL LATERALITY FOR ALL AGES

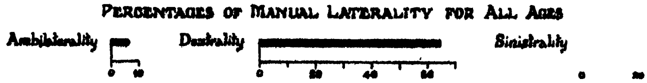
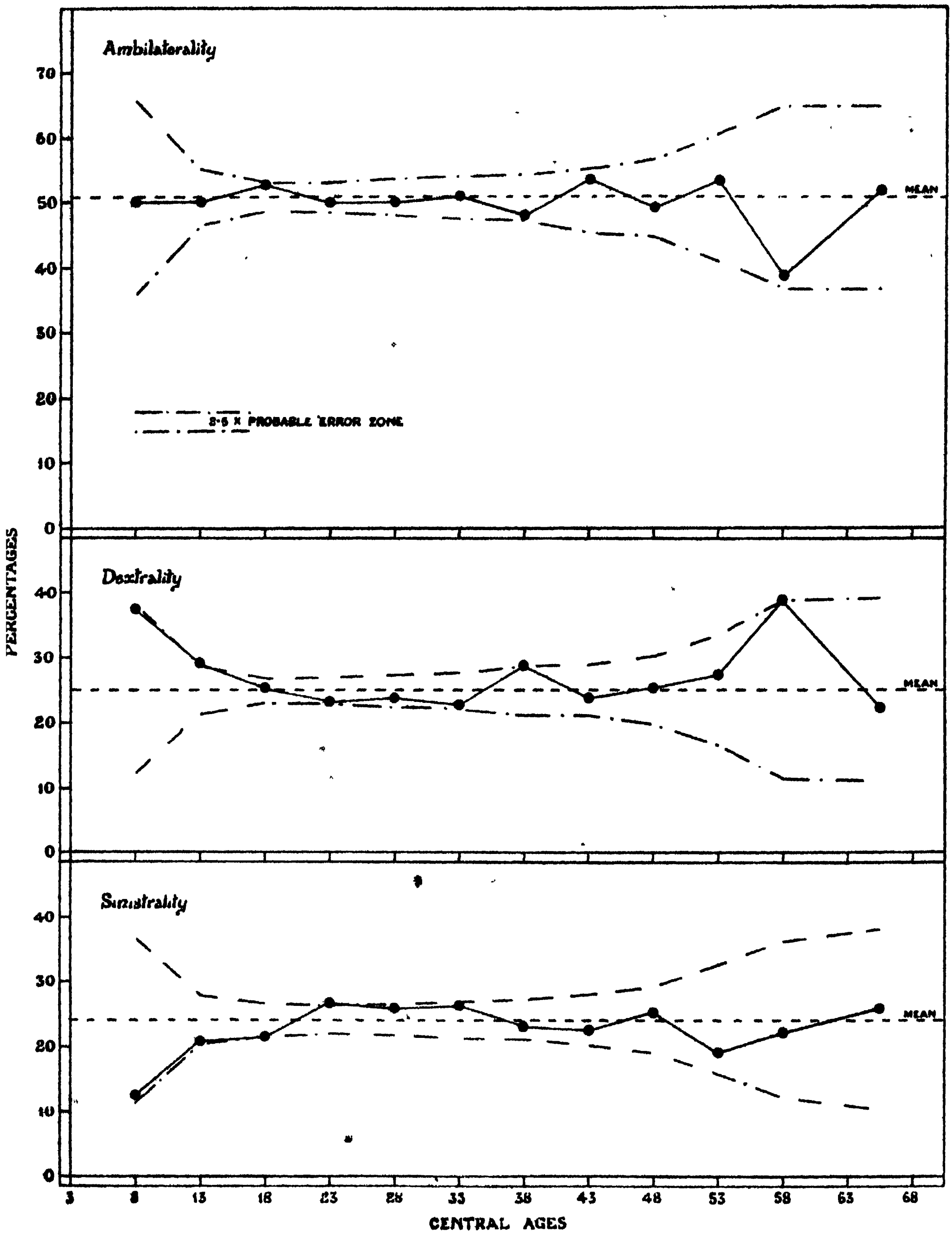


Diagram I.

trained dextralists in childhood. Our data, as indicated in (*a*) and (*b*) above, suggest a transfer in the opposite sense. Still the absolute lateralists are not necessarily the only or even the most trained statistical critics.

PERCENTAGES OF OCULAR LATERALITY FOR EVERY FIVE YEARS OF AGE



PERCENTAGES OF OCULAR LATERALITY FOR ALL AGES

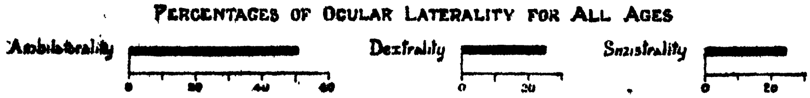


Diagram II.

Now we commenced our investigations by taking the difference of visual acuity in right and left eyes and the difference of muscular power in the right and left hands with a view to obtaining determinations of laterality. In a recent paper* our present data have been dealt with from the standpoint of growth or change with age, and the rapid rise to adolescence and later senescent fall are very manifest both in the case of grip and visual acuity. We expected that the *differences* of grip and visual acuity might require very considerable age corrections, and this not only with regard to their mean values, but also with regard to their standard deviations. We were somewhat startled to find how little change there was during the whole of life either in the mean manual or ocular differences or in their standard deviations.

Absolute grip of the stronger hand rises from about 11·6 lbs. at 6·5 years of age to about 84 lbs. at 32 years, and falls to 56 lbs. at 79 years. But the differences of right-handed and left-handed grips remain about 3·6 lbs. through all age groups.

Similarly visual acuity rises from about 20″ at 9·5 years to about 24″ at 19·5 years and falls to 4·5″ at 79·5 years. But the differences in right eye and left eye vary round ·03″ for all ages, and are of little significance having regard to the absolute visual acuities.

Diagram III illustrates these results graphically. It will be seen how constant are the mean differences and standard deviations as long as we have adequate data to determine them with any degree of accuracy.

The intensity of dextral manual superiority appears to fall from about age 9 to puberty, after which it rises rapidly to a maximum at adolescence and continues very steady during adult years. This maximum intensity of dextrality about 18 years is not due to an increased percentage of dextralists, for we have seen that they are slightly decreasing in numbers. It may be due to practice, in which case we should have the important result that practice and training can increase dextralism (i.e. the intensity of right-handed superiority) without increasing or even with decreasing dextrality (i.e. relative numbers of dextral individuals). On the other hand, we see ocular dextralism as measured by the intensity of right-eyed superiority decreasing from about 9 to 25 years, and in this case we have accordance with the decrease of ocular dextrality (as measured by the relative number of individuals) up to 28 years. Without being dogmatic, we may possibly throw out a tentative suggestion that practice with a master hand increases its relative efficiency, practice with a master eye decreases its relative efficiency. These changes will not adequately account for the slight changes in the numerical number of dextralists and sinistralists in the first thirty-five years of life, which we believe are definitely suggested by our data†.

* Ruger and Stoessiger: "Growth Curves of Certain Characters in Man (Males)," *Annals of Eugenics*, Vol. II. pp. 104, 106.

† Statistical slips often arise from confusing a greater frequency with a greater intensity. It is quite possible for the number of children with a given eye anomaly to increase with age, but the intensity of the anomaly in the population to decrease with age.

TABLE VI. Age and Difference of Visual Acuity between Two Eyes.

Difference of Visual Acuity in Inches (Left minus Right). Central Values.

Age in Years.	−24	−22	−20	−18	−16	−14	−12	−10	−8	−6	−4	−2	0	2	4	6	8	10	12	14	16	18	20	22	24	26	Totals
6—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	1
7—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	1
8—	—	—	—	—	—	—	—	—	—	—	2	—	3	2	—	—	—	—	—	—	—	—	—	—	—	—	7
9—	—	—	—	—	—	—	—	—	—	1	2	—	2	—	—	1	—	—	—	—	—	—	—	—	—	—	6
10—	—	—	—	—	—	—	—	—	1	—	1	5	9½	½	—	—	—	—	—	—	—	—	—	—	—	—	17
11—	—	—	—	—	—	—	—	—	—	—	2½	8½	13	2	—	1	—	—	—	—	—	—	—	—	—	—	27
12—	—	—	—	—	—	—	—	—	—	—	3	7	22	3	—	2	—	—	—	—	—	—	—	—	—	—	37
13—	—	—	—	—	—	—	—	—	—	4½	3½	11	32	10	—	—	2	—	1	—	—	—	—	—	—	—	64
14—	—	—	1	1	—	—	—	—	1	—	2	23	56	14	7	2	1	1	—	—	—	—	—	—	—	—	109
15—	—	—	—	—	1	—	—	—	1	1	11	29	69	23	7	1	—	—	1	—	—	—	—	—	—	—	144
16—	—	—	—	—	—	—	—	3½	1	4½	9	42	115½	27	10½	3	3	2	—	—	—	—	—	—	—	—	221
17—	—	—	—	2	—	—	1	—	4	4½	10½	45½	143½	46	6	5	4	1	2	—	—	—	—	—	—	—	275
18—	—	—	—	—	—	—	2	2½	2½	6	13½	52	156½	37½	15	2	2	2½	1	—	—	—	—	—	—	—	295
19—	—	—	—	—	—	—	—	1	3	5	17½	50½	154½	42	8½	2½	3	½	2	1	—	—	—	—	—	—	291
20—	—	—	—	—	1	—	—	2	4½	7	18	45½	211	53	13	8	5	—	—	—	—	—	—	—	—	—	368
21—	—	—	—	—	—	—	½	4½	4½	11	11½	42½	148½	45½	16½	6	4	—	1	—	—	2	—	—	—	—	298
22—	—	—	—	—	—	1	—	2	1	8	9½	42½	130½	52½	8½	7½	1	1	1	—	—	1	—	—	—	—	267
23—	—	—	—	½	½	2	1	½	5½	7½	12½	32½	157½	47	10	5	1	2	—	1	—	—	½	1½	1	—	289
24—	—	—	—	—	1	—	1	1	5½	2½	10	37	120½	51½	10	5	3	2	1	—	1	1	—	—	—	—	253
25—	—	—	—	—	1	—	½	½	3	3	3	27½	112½	28	12½	6½	4	1	—	—	—	—	—	—	—	—	203
26—	1	—	1	—	—	—	—	2	4	6	8½	31½	102	31½	8	4½	1	1	—	—	—	—	—	—	½	½	203
27—	—	—	—	1	1	1	—	—	2	6	7	21½	94	28	8½	4	2	1	—	—	2	—	—	—	—	—	179
28—	—	—	—	—	½	1	1½	—	½	1½	7½	26½	83	28½	6	2½	6	1	2	1	—	—	—	—	—	—	169
29—	—	—	—	—	1½	½	—	—	3½	3½	6½	19	93	23½	9	4	4	1	2	—	—	½	½	—	—	—	172
30—	—	—	—	—	—	—	1	1	2	4½	10½	25	90	30	11	3	—	—	1	—	1	1	—	—	—	—	181
31—	—	—	—	—	—	½	½	—	2½	4	4½	14½	70	16½	10	5	3	1	1½	½	—	—	—	—	—	—	134
32—	—	—	—	—	—	—	—	1	3	9½	5½	10	77½	21	9½	4	2	1	2	1	—	—	—	—	—	—	147
33—	—	—	—	½	½	—	—	2	1½	4½	4	22	72½	11	3½	5	—	2	—	1	—	—	—	—	—	—	130
34—	—	—	—	—	—	—	—	—	1½	2½	8	20½	64½	19	4	1½	1½	1	—	—	—	—	—	—	—	—	124
35—	—	—	—	—	1	—	—	1	—	—	9	13½	60	20½	12	3½	2½	1	½	½	—	—	—	—	—	—	125
36—	—	—	—	½	½	1	—	2½	3½	2	6½	14½	63	15	2½	4½	2	1	—	—	—	—	—	—	—	—	119
37—	—	—	—	1	1	—	—	2	3	3	6	24	53	18	4	3	—	—	—	1	—	1	—	—	—	—	120
38—	—	—	—	—	—	—	—	—	4	1	4½	16	59	16½	5	2	3	1	—	1	—	—	—	—	—	—	113
39—	—	—	—	—	—	—	—	1½	½	3½	9	17½	53½	10½	4	1	1	—	—	1	—	—	—	—	—	—	103
40—	—	—	—	—	—	—	—	4	1	4	3	15	42	12	4½	5½	½	1½	3	1	—	—	—	—	—	—	97
41—	—	—	—	—	1	1	—	—	1	—	4	9	46	11½	5½	2½	1½	—	—	—	—	—	—	—	—	—	83
42—	—	—	—	—	—	—	—	2	—	3	2	10½	43	15½	3½	1½	2	1	—	—	—	—	—	—	—	—	84
43—	—	—	—	—	—	—	½	½	—	1	—	9	35½	6½	2	5	—	1	—	—	—	—	—	—	—	—	61
44—	—	—	—	1	—	—	—	—	3	4	7	10½	41	5½	3	4	2	—	—	—	1	—	—	—	—	—	82
45—	—	—	—	—	—	—	1	—	—	3	5	7	33½	1½	2	—	—	1	1	—	1	—	—	—	—	—	56
46—	—	—	—	—	—	—	—	1	1	2	—	6	32	5	1	—	2½	1½	—	1	—	—	—	—	—	—	53
47—	—	—	—	—	—	—	1	—	1½	2½	3	5	20½	7	3½	—	4	—	—	—	—	—	—	—	—	—	48
48—	—	—	—	—	—	—	1	—	3	1	1	8½	20½	7	4	1	—	1	—	—	—	—	—	—	—	—	48
49—	—	—	—	—	—	—	—	—	—	1	5½	3½	12	3	—	3	—	—	—	—	—	—	—	—	—	—	28
50—	—	—	—	—	—	—	—	½	½	2	1	—	17	4	1	1	—	—	—	1	—	—	—	—	—	—	28
51—	—	—	—	—	—	—	—	—	—	1	1	1	10½	4½	—	1	—	—	—	—	—	—	—	—	—	—	19
52—	—	—	—	—	—	—	—	—	1	1	—	3	9½	½	—	—	—	—	—	—	—	—	—	—	—	—	15
53—	—	—	—	—	—	—	—	—	—	2	—	4	11	1	—	—	—	—	—	—	—	—	—	—	—	—	18
54—	—	—	—	—	—	—	—	—	—	2	—	1	4	2	1	1	—	—	—	—	—	—	—	—	—	—	11
55—	—	—	—	—	—	—	—	—	—	—	2½	½	5	2	—	—	—	—	—	—	—	—	—	—	—	—	10
56—	—	—	—	—	—	—	—	—	—	—	1	4	6	1	—	—	—	—	—	—	—	—	—	—	—	—	12
57—	—	—	—	—	—	—	—	—	—	—	½	½	4	—	—	—	1	—	1	—	—	—	—	—	—	—	7
58—	—	—	—	—	—	—	—	—	—	—	2	2½	½	—	1	—	—	—	—	—	—	—	—	—	—	—	6
59—	—	—	—	—	—	—	—	—	—	—	—	1½	3½	3	—	1	—	—	—	—	—	—	—	—	—	—	9
60—	—	—	—	—	—	—	—	—	—	—	—	½	1½	—	—	—	—	—	—	—	—	—	—	—	—	—	2
61—	—	—	—	—	—	—	—	1	—	—	—	1	2	—	—	1	1	—	½	½	—	—	—	—	—	—	7
62—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	2
63—	—	—	—	—	—	—	—	—	—	—	—	—	3	1	—	—	—	—	—	—	—	—	—	—	—	—	4
64—	—	—	—	—	—	—	—	—	—	—	—	1	2	—	—	—	—	—	—	—	—	—	—	—	—	—	3
65—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	1
66—	—	—	—	—	—	—	—	—	—	—	—	—	2	—	—	—	—	—	—	—	—	—	—	—	—	—	2
67—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	1
68—	—	—	—	—	—	—	—	—	—	—	—	—	2	1	—	—	—	—	—	—	—	—	—	—	—	—	3
69—	—	—	—	—	—	—	—	—	—	—	—	—	1	1	—	—	—	—	—	—	—	—	—	—	—	—	2
70—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	0
71—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	0
72—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
73—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	0
74—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	0
75—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	1
Totals	1	0	2	7½	11½	8	12½	39½	81	147½	278½	881½	3107½	869½	254½	132½	75½	32	24½	12½	6	6½	1	1½	1½	½	5996

To face p. 186

Tables VI and VII provide the data linking dextral superiority in eye and hand with age. We have the following results:

Ocular Dextral Superiority and Age:

Product-Moment Correlation Coefficient: $r = -\cdot 0131 \pm \cdot 0087$.

Correlation Ratio, Dextral Eye Superiority on Age:

$$\eta^2_{RE.A} = \cdot 003{,}932, \quad \bar{\eta}^2_{RE.A} = \cdot 007{,}505 \pm \cdot 001{,}063.$$

CHANGE IN DIFFERENCES OF VISUAL ACUITY AND MUSCULAR STRENGTH RIGHT AND LEFT SIDE WITH AGE

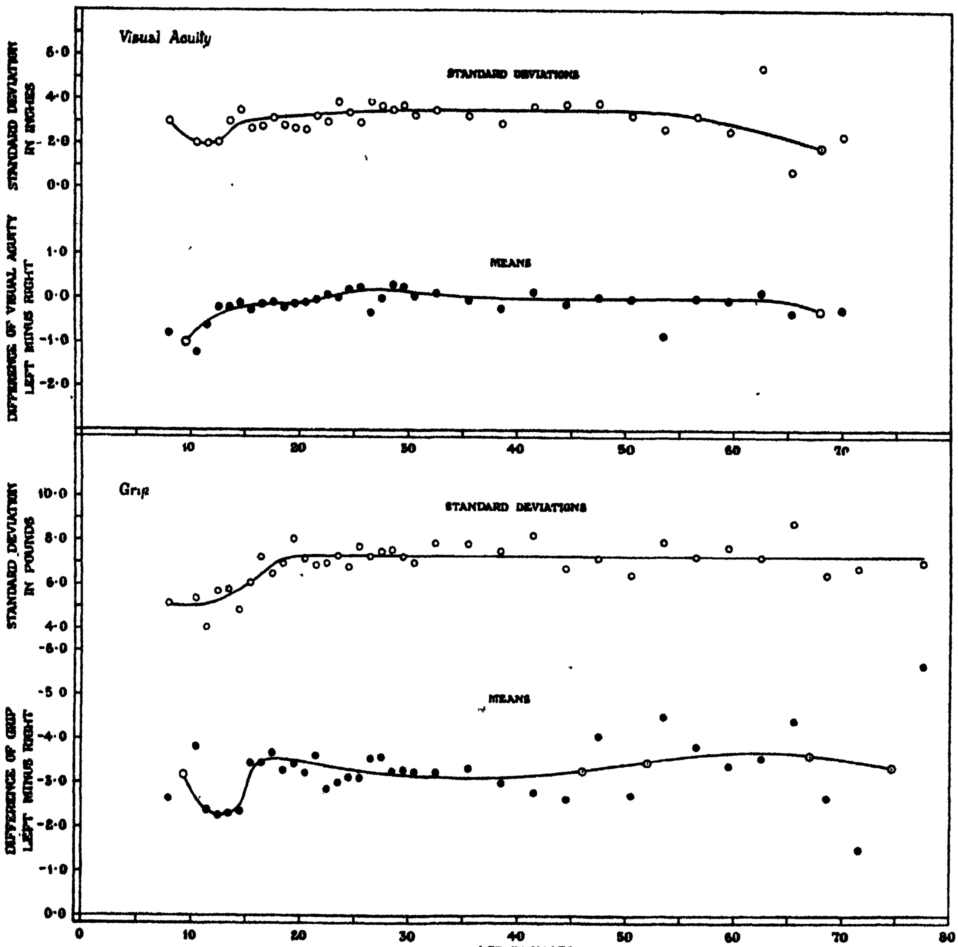


Diagram III.

Clearly there is no practical association between intensity of dextral superiority and age.

Manual Dextral Superiority and Age:

Product-Moment Correlation Coefficient: $r = + \cdot 0090 \pm \cdot 0081$.

Correlation Ratio, Dextral Hand Superiority on Age:

$$\eta^2_{RH.A} = \cdot 006{,}110, \quad \bar{\eta}^2_{RH.A} = \cdot 008{,}438 \pm \cdot 001{,}043.$$

TABLE IV. *Comparison of Actual Values of Constants with their Smooth Values for Each Age or Age Group.*

(*a*) Ocular Laterality. (Left minus Right.)

Age	Number of Observations	Means		Standard Deviations	
		Smooth Value	Actual Value	Smooth Value	Actual Value
6·5	15 (M.A. = 8·70)*		−0·80 ± ·52		2·99 ± ·37
7·5					
8·5					
9·5				2·40	
10·5	17	− ·88	−1·24 ± ·34	2·10	2·10 ± ·24
11·5	27	− ·65	−0·68 ± ·26	2·00	1·98 ± ·18
12·5	37	− ·49	−0·22 ± ·23	2·03	2·07 ± ·16
13·5	64	− ·35	−0·23 ± ·25	2·25	2·98 ± ·18
14·5	109	− ·26	−0·13 ± ·22	2·80	3·47 ± ·16
15·5	144	− ·21	−0·28 ± ·15	2·95	2·66 ± ·11
16·5	221	− ·16	−0·14 ± ·12	3·03	2·74 ± ·09
17·5	275	− ·14	−0·10 ± ·13	3·08	3·11 ± ·09
18·5	295	− ·13	−0·21 ± ·11	3·12	2·79 ± ·08
19·5	291	− ·11	−0·12 ± ·11	3·18	2·66 ± ·07
20·5	368	− ·11	−0·08 ± ·09	3·23	2·59 ± ·06
21·5	298	− ·06	−0·04 ± ·13	3·25	3·21 ± ·09
22·5	267	− ·01	+0·07 ± ·12	3·30	2·94 ± ·09
23·5	289	+ ·05	+0·02 ± ·15	3·35	3·86 ± ·11
24·5	253	+ ·11	+0·21 ± ·14	3·38	3·38 ± ·10
25·5	203	+ ·16	+0·25 ± ·14	3·40	2·91 ± ·10
26·5	203	+ ·19	−0·32 ± ·18	3·43	3·90 ± ·13
27·5	179	+ ·21	+0·01 ± ·19	3·48	3·70 ± ·13
28·5	169	+ ·20	+0·33 ± ·18	3·50	3·52 ± ·13
29·5	172	+ ·18	+0·28 ± ·19	3·55	3·72 ± ·14
30·5	181	+ ·16	+0·06 ± ·16	,,	3·26 ± ·12
31·5		+ ·13		,,	
32·5	411	+ ·11	+0·14 ± ·12	,,	3·49 ± ·08
33·5		+ ·09		,,	
34·5		+ ·07		,,	
35·5	368	+ ·06	−0·02 ± ·11	,,	3·23 ± ·08
36·5		+ ·04		,,	
37·5		+ ·03		,,	
38·5	336	+ ·02	−0·21 ± ·11	,,	2·89 ± ·08
39·5		+ ·01		,,	
40·5		+ ·01		,,	
41·5	264	+ ·01	+0·16 ± ·15	,,	3·67 ± ·11
42·5		·00		,,	
43·5		·00		,,	
44·5	199	·00	−0·12 ± ·18	,,	3·77 ± ·13
45·5		− ·01		,,	
46·5		− ·01		,,	
47·5	149	− ·01	+0·02 ± ·21	,,	3·83 ± ·15
48·5		− ·01		,,	
49·5		− ·01		3·50	
50·5	75	,,	−0·01 ± ·25	3·48	3·26 ± ·18
51·5		,,		3·45	
52·5		,,		3·43	
53·5	44	,,	−0·34 ± ·27	3·40	2·65 ± ·19
54·5		,,		3·38	
55·5		,,		3·33	
56·5	29	,,	0·00 ± ·40	3·28	3·23 ± ·29
57·5		,,		3·18	
58·5		,,		3·10	
59·5	17	− ·02	−0·06 ± ·42	2·98	2·55 ± ·30
60·5		− ·02		2·88	
61·5		− ·03		2·78	
62·5	13	− ·03	+1·31 ± 1·02	2·68	5·44 ± ·72
63·5		− ·04		2·53	
64·5		− ·05		2·43	
65·5	6	− ·08	−0·33 ± ·21	2·25	0·75 ± ·15
66·5		− ·10		2·10	
67·5		− ·18		1·98	
68·5		− ·28		1·85	
69·5					
70·5					
71·5	8 (M.A. = 70·0)*		−0·25 ± ·56		2·83 ± ·39
72·5					
73·5					
74·5					
75·5					
	5996				

* Denotes mean age of a group in years for actual means.

TABLE V.

(b) Manual Laterality. (Left minus Right.)

Age	Observations	Means: Smooth Value	Means: Actual Value	Standard Deviations: Smooth Value	Standard Deviations: Actual Value
6·5					
7·5	20 (M.A. = 9·39)*	−3·60	−2·65 ± ·77	5·10	5·18 ± ·55
8·5					
9·5					
10·5	17	−2·90	−3·82 ± ·88	5·05	5·37 ± ·62
11·5	28	−2·53	−2·39 ± ·52	5·03	4·06 ± ·37
12·5	38	−2·35	−2·26 ± ·62	5·08	5·71 ± ·44
13·5	64	−2·26	−2·31 ± ·49	5·15	5·79 ± ·35
14·5	110	−2·34	−2·35 ± ·31	5·33	4·84 ± ·22
15·5	155	−2·68	−3·46 ± ·33	5·55	6·10 ± ·23
16·5	289	−3·38	−3·45 ± ·32	5·90	7·25 ± ·22
17·5	288	−3·50	−3·68 ± ·26	6·25	6·50 ± ·18
18·5	320	−3·53	−3·29 ± ·26	6·65	6·93 ± ·19
19·5	309	−3·54	−3·44 ± ·31	6·98	8·09 ± ·22
20·5	339	−3·49	−3·22 ± ·24	7·20	7·15 ± ·17
21·5	321	−3·45	−3·61 ± ·26	7·25	6·88 ± ·18
22·5	291	−3·41	−2·87 ± ·28	7·28	6·98 ± ·20
23·5	315	−3·36	−3·03 ± ·28	7·30	7·30 ± ·20
24·5	275	−3·31	−3·18 ± ·28	,,	6·79 ± ·20
25·5	222	−3·28	−3·12 ± ·35	,,	7·72 ± ·25
26·5	224	−3·24	−3·56 ± ·33	,,	7·28 ± ·23
27·5	196	−3·22	−3·58 ± ·36	,,	7·49 ± ·26
28·5	188	−3·19	−3·25 ± ·37	,,	7·58 ± ·26
29·5	186	−3·18	−3·27 ± ·36	,,	7·24 ± ·25
30·5	198	−3·16	−3·25 ± ·34	,,	6·99 ± ·24
31·5		−3·14		,,	
32·5	442	−3·13	−3·25 ± ·25	,,	7·90 ± ·18
33·5		−3·13		,,	
34·5		−3·12		,,	
35·5	411	−3·11	−3·35 ± ·26	,,	7·85 ± ·18
36·5		−3·11		,,	
37·5		−3·12		,,	
38·5	377	−3·13	−3·02 ± ·26	,,	7·56 ± ·19
39·5		−3·13		,,	
40·5		−3·14		,,	
41·5	302	−3·16	−2·79 ± ·32	,,	8·25 ± ·23
42·5		−3·18		,,	
43·5		−3·20		,,	
44·5	237	−3·23	−2·65 ± ·30	,,	6·79 ± ·21
45·5		−3·27		,,	
46·5		−3·29		,,	
47·5	196	−3·33	−4·07 ± ·35	,,	7·21 ± ·25
48·5		−3·36		,,	
49·5		−3·39		,,	
50·5	150	−3·43	−2·73 ± ·36	,,	6·47 ± ·25
51·5		−3·46		7·28	
52·5		−3·48		,,	
53·5	110	−3·53	−4·54 ± ·51	,,	7·95 ± ·36
54·5		−3·56		,,	
55·5		−3·59		,,	
56·5	109	−3·62	−3·84 ± ·47	,,	7·27 ± ·32
57·5		−3·64		,,	
58·5		−3·66		,,	
59·5	85	−3·69	−3·40 ± ·56	,,	7·71 ± ·40
60·5		−3·72		,,	
61·5		−3·74		,,	
62·5	68	−3·73	−3·57 ± ·59	7·26	7·26 ± ·42
63·5		−3·71		,,	
64·5		−3·70		,,	
65·5	42	−3·69	−4·43 ± ·91	,,	8·80 ± ·65
66·5		−3·68		7·25	
67·5		−3·65		,,	
68·5	34	−3·60	−2·68 ± ·75	,,	6·45 ± ·53
69·5		−3·58		,,	
70·5		−3·55		,,	
71·5	20	−3·53	−1·50 ± 1·02	7·23	6·77 ± ·72
72·5		−3·48		,,	
73·5		−3·43		,,	
74·5					
75·5					
76·5					
77·5	16 (M.A. = 78·1)*	−3·39	−5·69 ± 1·18	7·21	7·01 ± ·84
78·5					
79·5					
80·5					
81·5					
	6992				

* Denotes mean age of a group in years for actual means.

Again there is no sensible association between the absolute differences of muscularity of right and left hands. *À priori* we expected the differences to be nearly proportional to the absolute values, but this does not seem to be the case. Not only are the product-moment correlations insignificant having regard to their probable errors, but the regression curves as judged by their η^2 are not significantly skew. We might have judged ourselves excused from making any age correction on the result of these Tables, actually we did allow for age in the earliest years observed.

The data thus slightly modified provided Table VIII. This is a correlation table between intensity of dextral superiority of hand and eye. The correlational constants of this table are the following:

Product-Moment Correlation Coefficient: $r = + \cdot 0113 \pm \cdot 0096$.

Correlation Ratios of Right Hand and Right Eye Superiority:

$$\eta^2_{RH:RE} = \cdot 004{,}860, \quad \bar{\eta}^2_{RH:RE} = \cdot 003{,}840 \pm \cdot 000{,}839,$$

$$\eta^2_{RE:RH} = \cdot 009{,}108, \quad \bar{\eta}^2_{RE:RH} = \cdot 007{,}680 \pm \cdot 001{,}184.$$

Not a single one of these results is significant, having regard to its probable error. There is no relation, either linear or skew, between a superiority of the eye and a superiority of the hand. Just as we have shown that there is no association between the percentages of ocular and manual dextralists, so now we complete the proof by indicating that ocular superiority of the right eye is in no way associated in its intensity with that of manual superiority in the right hand.

We have failed to find in ample data any relationship of ocular and manual dextralities, whether in coincidence of occurrence or in intensity. All our tests show them to be independent. Diagram IV (p. 191) indicates how completely the two variates are independent.

We have now answered as far as our data permit the problems we started to discuss (see p. 178), except those which are concerned with van Biervliet's conclusions. They have all been answered in a sense wholly opposed to absolute unilaterality. Many of the questions which that theory raises as to why the right eye is "superior," and why the right hand is the "nearer," cease to be of the slightest importance, if in the bulk of cases the visual acuity shows no preference for either eye and the number of manual dextralists is two to three times as great as the number of ocular dextralists.

Table IX provides the values of the constants as determined from the whole of our data. The only remarks to be made are that we have difference of grip in a thousand more cases than difference of visual acuity. In the case of grip cases were excluded only in which some injury to right or left hand was recorded. In the case of visual acuity cases were excluded in which (*a*) the sight was said to be defective, the eye injured, or it was stated that glasses had to be or were used; (*b*) the difference of sight in the two eyes amounted to as much as 15″, in such cases some eye trouble may be presumed to exist in the worse sighted eye (e.g. oncoming cataract).

Difference of Visual Acuity and Strength of Grip (Left minus Right)

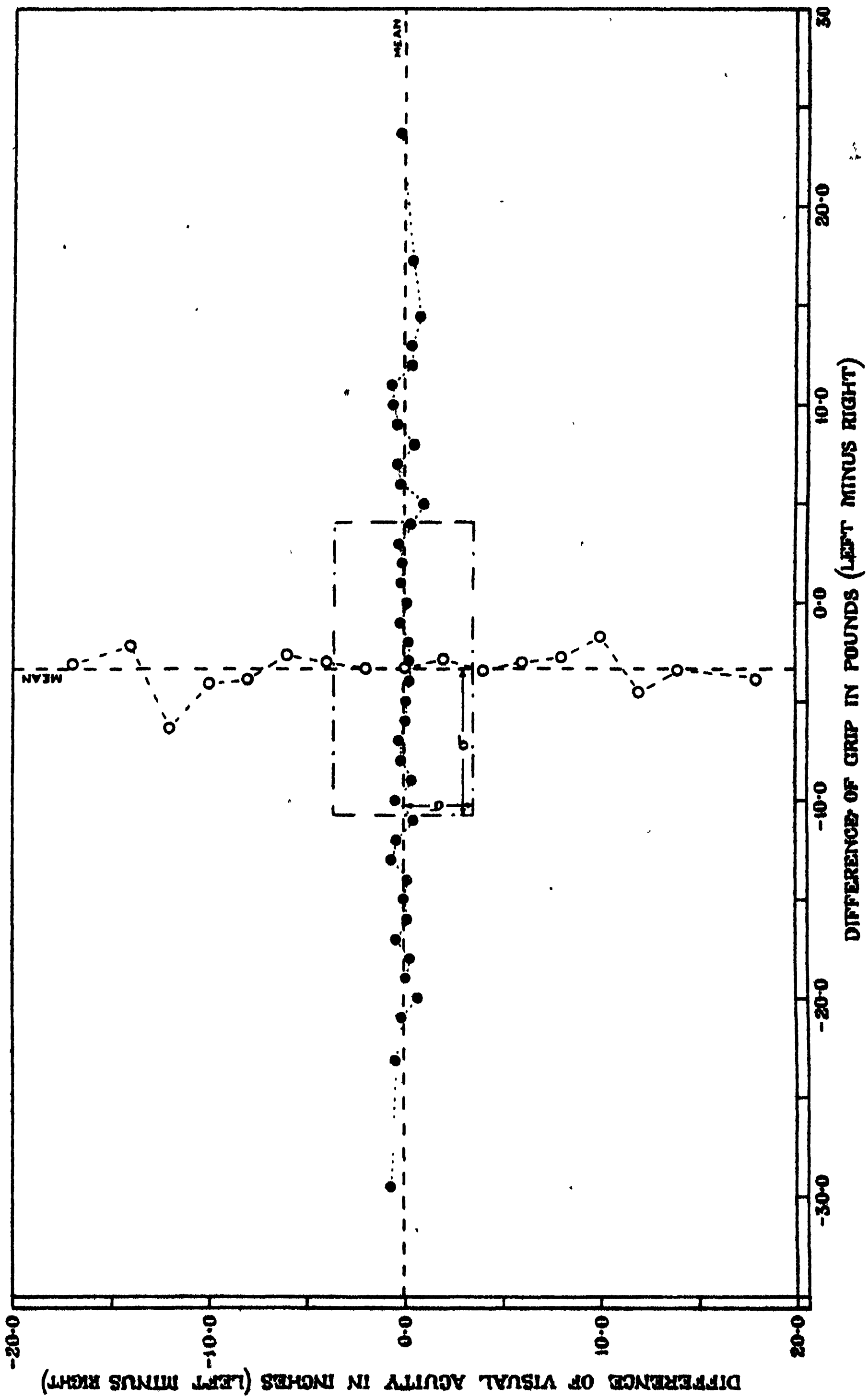


TABLE IX.

Constants of Data combined for all Ages.

Character	Mean	Standard Deviation
Difference: Right minus Left Visual Acuity		
(a) All cases (5996)	+0″·0317 ± ·0285	3″·2696 ± ·0201
(b) Right Eye Superiority (1499) ...	+3″·6065 ± ·0571	3″·2804 ± ·0404
(c) Left Eye Superiority (1447) ...	−3″·6351 ± ·0530	2″·9917 ± ·0375
Difference: Right minus Left Grip		
(a′) All cases (6992)	+3·2386 ± ·0586 lbs.	7·2652 ± ·0414 lbs.
(b′) Right Hand Superiority (4518)	+7·2371 ± ·0525 lbs.	5·2346 ± ·0371 lbs.
(c′) Left Hand Superiority (2013)	−4·9940 ± ·0603 lbs.	4·0082 ± ·0426 lbs.
Ages, corresponding to		
(a) All cases of Visual Acuity ...	27·6715 ± ·0880 yrs.	10·0980 ± ·0622 yrs.
(b) Right Eye Superiority ...	27·6554 ± ·1771 yrs.	10·1641 ± ·1253 yrs.
(c) Left Eye Superiority	27·9105 ± ·1748 yrs.	9·8598 ± ·1236 yrs.
(a′) All cases of Grip	29·8707 ± ·1010 yrs.	12·5257 ± ·0714 yrs.
(b′) Right Hand Superiority ...	29·8725 ± ·1271 yrs.	12·6670 ± ·0899 yrs.
(c′) Left Hand Superiority ...	29·9918 ± ·1852 yrs.	12·3195 ± ·1310 yrs.

In the case of Visual Acuity the superiorities of Right and Left Eyes when they exist are on the average equal, and their variabilities are sensibly equal. In the case of manual superiorities the Right Hand has an average superiority nearly 45 °/₀ greater than the superiority of the Left Hand, and its superiority is significantly more variable. The fact that the age means and their variabilities are significantly greater for grip than for visual acuity results from many of the old men being still able to exert muscular force, while they could no longer read diamond type at any distance without glasses.

We will first consider the two cases in which van Biervliet actually gives the absolute values for both sides, i.e. in the visual acuity of both eyes and the sensitivity of both hands. We will proceed exactly as we have done in the case of our own data and take the difference left minus right side character. The differences in acuity are measured in decimals of a metre, the differences in sensitivity in decimals of a centimetre. There are only 85 cases in all.

Diagram V (p. 194) is a "scatter diagram" of these observations. Now the diagram brings out at once most important relations. In the first place there is not a single case in which the visual acuities of left and right eyes are equal, and this notwithstanding that both eyes have been independently corrected for the test. The "gaucher" always sees after correction about half a metre farther with his left eye than his right, and the "droitier" with his right eye half a metre better than with his left. The "gaucher" is about 2 mm. more sensitive with his left hand than with his right, and the "droitier" with his right hand than his left. It is

not that one side is somewhat better than the other, but the diagram shows that "gauchers" and "droitiers" form two groups separated for both characters by a wide interval in space.

TABLE X.

Differences of Visual Acuity and Sensitivity of Hand based on van Biervliet's data.

"Gauchers" (19)		"Droitiers" (66)					
Visual Acuity	Sensitivity	Visual Acuity	Sensitivity	Visual Acuity	Sensitivity	Visual Acuity	Sensitivity
·52	·23	– ·43	– ·18	– ·73	– ·21	– ·48	– ·14
·76	·26	– ·34	– ·23	– ·52	– ·17	– ·61	– ·19
·45	·20	– ·60	– ·25	– ·65	– ·16	– ·71	– ·20
·47	·22	– ·60	– ·24	– ·58	– 22	– ·52	– ·13
·63	·29	– ·68	– ·19	– ·60	– ·25	– ·80	– ·19
·52	·22	– ·45	– ·35	– ·58	– ·23	– ·49	– ·19
·54	·18	– ·34	– ·19	– ·64	– ·17	– ·47	– ·19
·36	·25	– ·31	– ·34	– ·68	– ·17	– ·96	– ·16
·63	·16	– ·62	– ·20	– ·52	– ·16	– ·37	– ·20
·64	·23	– ·41	– ·20	– ·72	– ·21	– ·68	– ·20
·54	·19	– ·61	– ·23	– ·60	– ·17	– ·40	– ·22
·82	·23	– ·53	– ·20	– ·59	– ·19	– ·65	– ·22
·51	·29	– ·59	– ·22	– ·36	– ·17	– ·28	– ·19
·51	·30	– ·37	– ·20	– ·71	– ·23	– ·64	– ·17
·48	·18	– ·54	– ·20	– ·31	– ·16	– ·48	– ·15
·67	·21	– ·53	– ·25	– ·39	– ·16	– ·53	– ·23
·51	·22	– ·56	– ·20	– ·57	– ·20	– ·41	– ·22
·52	·28	– ·55	– ·14	– ·46	– ·25	– ·72	– ·11
·32	·27	– ·70	– ·21	– ·73	– ·23	– ·70	– ·14
—	—	– ·47	– ·21	– ·50	– ·18	– ·46	– ·19
—	—	– ·62	– ·22	– ·65	– ·17	– ·70	– ·14
—	—	– ·46	– ·27	– ·43	– ·20	– ·40	– ·16

There can be no compromise between our data with a continuous distribution of the lateral difference of visual acuity and those of van Biervliet. Our data give no association whatever between the muscular lateral difference and the ocular. His give a correlation of $+ \cdot 929 \pm \cdot 010$, but it is only because his material is heterogeneous. If we separate into his two groups of "gauchers" and "droitiers," we find:

	"Gauchers"	"Droitiers"	Total
Mean Visual Difference ($L - R$)	·547 ± ·018 m.	– ·550 ± ·011 m.	– ·305 ± ·013 m.
Standard Deviation	·119 ± ·013 m.	·134 ± ·008 m.	·475 ± ·025 m.
Mean Sensitivity ($L - R$) ...	·232 ± ·006 cm.	– ·199 ± ·003 cm.	– ·102 ± ·013 cm.
Standard Deviation	·040 ± ·004 cm.	·042 ± ·002 cm.	·184 ± ·010 cm.
Correlation of Visual with Sensitival Difference	– ·084 ± ·154	– ·139 ± ·072	+ ·929 ± ·010.

Clearly within either *component* group there is no significant correlation of visual difference with sensitival difference, or degree of ocular dominance is not associated with degree of manual dominance. This is exactly what we find for our *total* group, while van Biervliet gets what we may term a spurious correlation by

VAN BIERVLIET'S DATA FOR SENSITIVITY OF HANDS AND ACUITY OF EYES

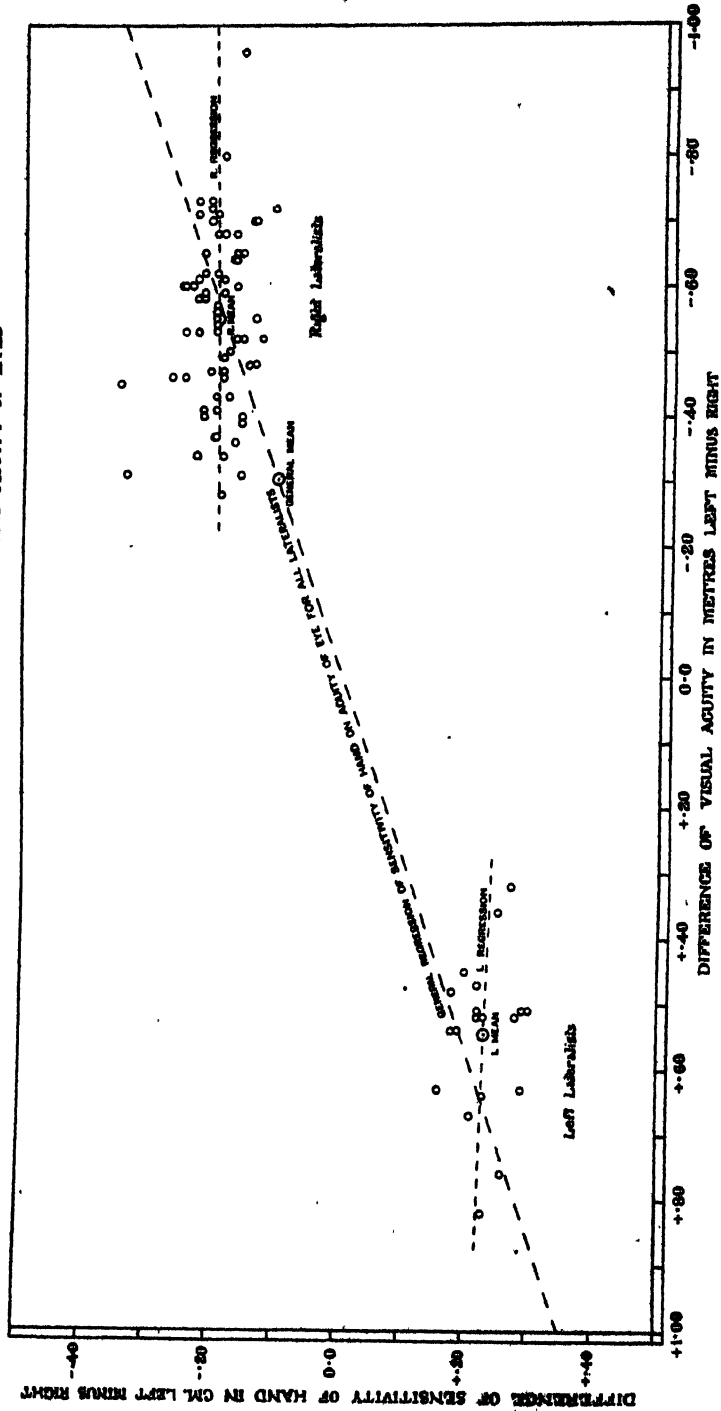


Diagram V.

putting together his two heterogeneous elements. Turning to our Table VIII, we see that a similar result could only be obtained had we selected for experiment two groups of dextral and sinistral unilateralists. It is difficult to understand how van Biervliet could, of course unconsciously, have done this, but our data and his are quite incompatible, and the question of which is representative narrows itself down to the point of asking whether the difference of visual acuity for the two eyes is a continuous variate, or whether there is a range in the portion round equality from about + ·30 m. to − ·30 m., in which there is never any frequency, or so little that it is not traceable in the examination of 100 individuals.

We will now approach the subject from a different standpoint. We took first all our cases, 3838, for Visual Acuity in males from age 20 to 40*, we corrected both eyes for age changes reducing to a standard age of 30, by aid of Ruger's Table C for growth changes in this material†. We then formed a correlation table (see Table XI) of dextral and sinistral visual acuity. In the same way we took Grip from age 20 to age 50. Here we had 4883 cases; we corrected them to age 35 and formed a correlation table (see Table XII) of dextral and sinistral muscularity.

The constants of these two tables are as follows, where V stands for Visual Acuity and M for Muscularity:

$$\text{Mean } V_R = 23{\cdot}2600 \pm {\cdot}0710 \text{ inches}, \quad \sigma_{V_R} = 6{\cdot}5211 \pm {\cdot}0502 \text{ inches}.$$

$$\text{Mean } V_L = 23{\cdot}2546 \pm {\cdot}0710 \text{ inches}, \quad \sigma_{V_L} = 6{\cdot}5255 \pm {\cdot}0502 \text{ inches}.$$

$$\text{Correlation Coefficient } r = + {\cdot}9324 \pm {\cdot}0014.$$

$$\text{Regression Lines: } \begin{cases} \text{Probable } V_R = {\cdot}9318\, V_L + 1{\cdot}5914. \\ \text{Probable } V_L = {\cdot}9330\, V_R + 1{\cdot}5530. \end{cases}$$

Again:

$$\text{Mean } M_R = 81{\cdot}8895 \pm {\cdot}1076 \text{ lbs.}, \quad \sigma_{M_R} = 11{\cdot}1498 \pm {\cdot}0761 \text{ lbs.}$$

$$\text{Mean } M_L = 78{\cdot}8724 \pm {\cdot}1097 \text{ lbs.}, \quad \sigma_{M_L} = 11{\cdot}3652 \pm {\cdot}0776 \text{ lbs.}$$

$$\text{Correlation Coefficient } r = + {\cdot}7909 \pm {\cdot}0036.$$

$$\text{Regression Lines: } \begin{cases} \text{Probable } M_R = {\cdot}7759 M_L + 20{\cdot}6924. \\ \text{Probable } M_L = {\cdot}8062 M_R + 12{\cdot}8531. \end{cases}$$

Now let the reader follow the diagonals from left-hand top to right-hand bottom corner of Tables XI and XII; if van Biervliet's sample is a typical one the cells encountered by these diagonals ought to contain no frequency. Indeed not only these cells; in both correlation tables there should be a broad empty belt, a high-road with no frequency cutting diagonally across these tables. All we can say is that there is not a sign in our data for either of these two characters of non-continuous alternatives—dextrality and sinistrality.

Let us now consider how far these tables will enable us to reach the mysterious ratio 9/10 of van Biervliet's experiments. We suppose in the first place the reader

* These ages were selected so that the standard deviation of the arrays might for the given ages be treated as constant.

† *Annals of Eugenics*, Vol. II. pp. 84—85.

TABLE XI.

Visual Acuity of Left Eye and Right Eye, corrected for Age.

Visual Acuity of Left Eye in Inches (Central Values).

Visual Acuity of Right Eye in Inches (Central Values).	*3·5*	*5·5*	*7·5*	*9·5*	*11·5*	*13·5*	*15·5*	*17·5*	*19·5*	*21·5*	*23·5*	*25·5*	*27·5*	*29·5*	*31·5*	*33·5*	*35·5*	*37·5*	*39·5*	Totals	Ratio Q'
3·5	1¾	2¼	½	—	¼	¼	—	—	—	—	—	—	—	—	—	—	—	—	—	5	
5·5	3	**8½**	4¼	¾	½	½	¼	¼	1½	¼	¾	—	—	½	½	—	—	—	—	21½	·464
7·5	¾	5	**10½**	6½	4½	¾	2	½	½	¼	¾	¾	¼	—	—	—	—	—	—	33	
9·5	—	¾	4¾	**33**	21¾	7½	8	6¾	1	1½	1	3	1	½	¾	1¼	—	—	—	92½	·601
11·5	—	—	1	25½	**70**	24¼	9¾	9¾	5¾	4½	2¼	2½	2	1¼	1¼	¼	—	—	—	160½	·657
13·5	—	½	1	2¾	19½	**61½**	22¾	18	5½	3¼	1½	2½	¾	½	—	—	—	—	—	140	·755
15·5	—	—	1	4	13½	28	**71**	29¼	15¾	8½	4¾	5½	1¼	2½	—	1	—	—	—	186	·761
17·5	—	—	1¾	6¾	9	8½	27	**85¼**	54¾	17¼	7¾	3½	2½	3½	1	—	—	—	—	228½	·825
19·5	¼	¾	¾	3	7½	13½	19½	50½	**141¼**	60¾	24¾	7¼	8½	4¼	2	—	—	—	—	345	·840
21·5	¼	¾	—	1¾	5¾	8¾	9¾	20¾	59¼	**139½**	83	30	16½	3¼	1¼	—	—	—	—	380½	·872
23·5	—	—	—	1¾	3	1	6½	11¼	31½	65	**151¾**	71½	27¼	13¼	3½	—	—	—	—	387½	·883
25·5	—	—	—	1¼	1¾	¾	2¼	13	13¼	27¾	71¾	**204½**	106	28½	7¼	1¾	¼	—	—	480	·905
27·5	—	—	—	1	2½	1¾	2½	4¼	8	13	25¼	87¼	**214**	118½	13	3½	½	—	—	495	·922
29·5	—	—	—	¾	1¼	3½	1¼	½	8¼	5¾	9	26½	86	**182**	70¼	7¼	¾	—	—	403	·930
31·5	—	—	½	¾	¼	—	¾	¼	¼	1¾	5½	5¼	21	52¼	**144¾**	40¼	1½	¼	—	275¼	·938
33·5	—	—	—	—	—	—	¾	¼	—	—	2½	5	3¾	10¾	37	**86½**	12	½	½	159½	·938
35·5	—	—	—	—	—	—	—	—	—	—	¼	½	¼	¼	3¼	16	**19**	¼	¼	40	
37·5	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	¾	½	½	½	2¼	·949
39·5	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	½	2½	3	
Totals	6	18½	26	89½	161	161	184	250½	346½	349	392½	456	491	422	285¾	158½	34½	2	3¾	3838	—
Ratio Q		·514		·621	·652	·723	·763	·811	·832	·864	·878	·899	·916	·926	·936	·941		·949		—	—

TABLE XII. *Strength of Left Hand and Right Hand, corrected for Age.*

Strength of Left Hand in Pounds (Central Values).

Strength of Right Hand in Pounds (Central Values).	*39*	*42*	*45*	*48*	*51*	*54*	*57*	*60*	*63*	*66*	*69*	*72*	*75*	*78*	*81*	*84*	*87*	*90*	*93*	*96*	*99*	*102*	*105*	*108*	*111*	*114*	Totals	Ratio Q′
39	**1**	2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	3	·854
42	1	**2**	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	4	
45	1	—	**2**	1	—	—	—	1	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	6	
48	—	1	—	**3**	1	1	—	1	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	8	
51	—	1	1	2	**2**	—	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	8	
54	1	—	—	1	4	**4**	7	5	2	—	2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	26	·897
57	—	1	1	2	4	7	**7**	5	4	5	—	1	—	—	1	—	—	—	—	—	—	—	—	—	—	—	38	·881
60	—	—	2	2	3	2	9	**16**	7	11	4	4	2	1	—	2	—	—	—	—	—	—	—	—	—	—	65	·874
63	—	—	1	3	—	4	12	19	**30**	22	13	10	5	3	1	1	—	—	—	—	—	—	—	—	—	—	124	·902
66	—	—	2	1	3	8	12	24	27	**45**	22	13	12	5	4	3	1	—	—	—	—	—	—	—	—	—	182	·899
69	—	—	1	1	2	3	9	34	30	43	**50**	45	19	12	10	—	1	—	—	—	—	—	—	—	—	—	260	·924
72	—	1	—	1	1	3	9	24	20	44	67	**66**	53	22	24	11	7	3	—	—	—	—	—	—	—	—	356	·915
75	—	—	1	—	1	1	2	16	10	41	59	89	**82**	49	49	28	12	5	1	—	—	—	—	—	—	—	446	·921
78	—	—	—	—	—	2	3	7	14	29	49	65	92	**88**	83	39	18	8	5	3	1	1	—	—	—	—	507	·931
81	—	—	—	—	1	2	—	4	10	20	25	58	78	92	**111**	71	34	23	7	8	1	—	—	—	—	—	545	·932
84	—	—	—	—	—	—	1	1	2	11	23	34	68	68	92	**96**	68	36	16	7	2	—	—	—	1	—	526	·939
87	—	—	—	—	—	—	—	1	1	3	9	14	33	57	80	71	**71**	37	21	14	9	3	—	—	—	—	424	·934
90	—	—	—	—	—	—	—	—	1	2	6	11	11	32	53	81	56	**53**	33	19	8	3	3	—	—	—	372	·942
93	—	—	—	—	—	—	—	—	1	1	1	6	21	13	37	33	45	50	**39**	20	16	4	2	—	1	—	290	·945
96	—	—	—	—	—	—	—	1	—	1	1	4	8	11	16	34	37	46	46	**51**	19	9	4	—	—	—	288	·954
99	—	—	—	—	—	—	—	—	—	1	—	—	3	6	11	12	16	31	26	37	**20**	9	4	6	—	—	182	·947
102	—	—	—	—	—	—	—	—	—	—	—	—	—	5	4	6	9	14	11	14	17	**16**	7	3	3	—	109	·953
105	—	—	—	—	—	—	—	—	—	—	1	—	—	1	1	3	—	3	5	7	11	14	**9**	3	1	—	59	·966
108	—	—	—	—	—	—	—	—	—	—	1	—	1	—	1	2	1	1	3	7	6	3	6	**3**	—	—	35	·974
111	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2	—	—	1	2	2	1	—	3	2	**1**	2	16	
114	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	2	**1**	4	
Totals	4	8	12	17	22	37	72	160	159	281	333	420	488	465	580	493	376	311	215	189	112	62	38	17	9	3	4883	
Ratio Q	·726			·794	·816	·817	·849	·855	·872	·877	·890	·901	·902	·908	·912	·919	·927	·932	·942	·945	·947	·947	·963	·974			—	

has Table XII now before him; he will mark that of those with a right-hand grip of 78 lbs., there were 88 with an equal left-hand grip. These 88 stand on the diagonal of the Table, emphasised by heavy type in the Table. All the individuals to the right of 88 in the 78 lbs. row have a greater strength in their left hand than their right, there are 158 of them, and they are "Gauchers" or muscular sinistralists. Their mean strength with the left hand is 83·791 lbs., 5·791 lbs. greater than their right-hand strength. The ratio of 78 to 83·791 is ·931. The corresponding ratio, Q', for each array of sinistralists is placed in the last right-hand column; it varies from ·85 to ·97.

Now turn to the men with a left-hand strength of 78 lbs., i.e. the array or column headed 78. All the men below the diagonal cell 88* are dextralists, for their right hand is stronger than their left. There are 285 of them and their grip of the right hand averages 85·884 lbs. or 7·884 lbs. more than the left. The ratio of their left-hand to their right-hand grip is ·908. Similar ratios, Q, have been computed for all arrays of dextralists and are placed at the foot of each column. They run from about ·73 to ·97 with great smoothness. But suppose that instead of 4883 tested individuals, we had only 85 tested individuals, some 19 "Gauchers" and 66 "Droitiers," like van Biervliet. The 19 "Gauchers" would congregate round the 81 lbs. row and the 66 "Droitiers" round the 81 lbs. column. We should thus lose track of the fact that there was a continuous change in the ratio of left-hand strength to right-hand strength for dextralists and of right-hand strength to left-hand strength for sinistralists, and noticing the appearance of a ratio about 9/10, might assume this to be a physiological law.

Precisely similar results occur in the case of visual acuity; the modal value of acuity is reading diamond type at 27·5″. Here the Q ratio ·916 and the Q' ratio ·922 and the numbers about and around these are what a small sample would give rise to. The impression would be again formed that the ocular dextralist had vision of the right eye to vision of the left as 10 to 9, roughly; and the ocular sinistralist had vision of the left eye to vision of the right as 10 to 9, roughly again. Actually the ratios Q and Q' have a greater variation for visual acuity than the corresponding ratios have for muscularity, and this is what van Biervliet indeed found. If this view be correct van Biervliet's 9/10 was not a universal physiological law, he had merely run up against a result of any correlation table of high coefficient, such as one meets with when one correlates any homologous characters in the two sides of the body. Assuming the distribution to be approximately normal we can indeed compute the Q and Q' ratios as soon as we know the means, the standard deviations and the correlation coefficient of the two variates.

Let us consider, for example, the Q ratio for the dextralists, who have a left hand of strength Y lbs. The array of individuals corresponding to a left-hand strength

* We leave out of account these 88 ambilateralists, because van Biervliet would assert they do not exist. It is perfectly true that ambilateralism, if we could measure with infinite accuracy, would only be a dividing line, and therefore exhibit no frequency. Actually we have given it a range of $\pm 1·5$ lbs. We believe that a man, who was equally efficient to the extent of $\pm 1·5$ lbs. in both hands, would be for all practical purposes ambilateral. Even if we knew how to divide up the diagonal groups, their division would make but little change in our Q, Q' ratios.

of Y lbs. will have a mean $\tilde{X}$ given by the equation: $\tilde{X} = \cdot 7759 Y + 20 \cdot 6924$ and a standard deviation $\sigma_1 \sqrt{1 - r^2} = \Sigma_1 = 6 \cdot 8230$. We have to find first the area of this distribution (supposed a normal curve) cut off by the ordinate at $\tilde{X} - Y, = N_Y \frac{1}{2}(1 + \alpha)$, and secondly the abscissa from its centre to its mean, i.e. $\bar{\xi} = \frac{Z_{\tilde{X}-Y}}{\frac{1}{2}(1+\alpha)} \Sigma_1$, by a well-known property of the normal curve.

Then $$Q = Y/(\tilde{X} + \bar{\xi}).$$

Now owing to the general symmetry of the body $\tilde{X}$ will differ only by a small quantity from Y and $\bar{\xi}$ will be a small quantity.

Accordingly $Q = Y/(Y + \text{small quantity})$ and will not differ greatly from 9/10.

For example if $Y = 78$ lbs., $\tilde{X} = 81 \cdot 2126$ lbs., $\tilde{X} - Y = 3 \cdot 2126$, and

$$(\tilde{X} - Y)/\Sigma_1 = 3 \cdot 2126 / 6 \cdot 8230 = \cdot 47084.$$

We have accordingly to find from the Tables of the probability integral and the ordinates of the normal curve the values of $\frac{1}{2}(1 + \alpha)$ and z corresponding to an $x = \cdot 47084$. We reach

$$\tfrac{1}{2}(1 + \alpha) = \cdot 68112, \quad z = \cdot 35708,$$

leading to $$\bar{\xi} = \frac{\cdot 35708}{\cdot 68112} \times 6 \cdot 8230 = 3 \cdot 5770.$$

Accordingly the mean strength of dextralists whose left hand has a grip of 78 lbs. is

$$\tilde{X} + \bar{\xi} = 81 \cdot 2126 + 3 \cdot 5770 = 84 \cdot 7896 \text{ lbs.},$$

and $$Q = 78/84 \cdot 7896 = \cdot 920.$$

The value of Q we found by not breaking up the ambilateralists, but leaving them out of account, and by not assuming the frequency distribution normal was ·908. The accordance is quite as close as might reasonably be anticipated.

We have reached thus an explanation of van Biervliet's 9/10ths; it is not a universal law of absolute unilateralism; it is a rough approximation to the values of Q which flow from any approximately normal correlation table of high coefficient.

While thus able to throw light on the 9/10th "law," we can provide no explanation of why van Biervliet's frequency distribution of visual acuity and sensitivity differences should show an absolute absence of ambilateralism.

As we have previously stated van Biervliet's data, slender as they are, still form the best available evidence for the theory of absolute unilateralism. Our much more extensive material is wholly opposed to that theory, and in view of this we feel it safe to assert that the theory is at present unproven. But we go further and suggest that a dominant ocular dextrality in some as yet unexplained and mysterious manner* enforcing a dominant manual dextrality is unprovable as well as unproven.

It appears to us that lateralism whether ocular or manual is a continuous variate, and that dextrality and sinistrality are not opposed alternatives, but quantities capable of taking values of continuous intensity and passing one into the other.

* The "near-by" hypothesis.

DETERMINATION OF THE CRANIAL CAPACITY OF THE NEGRO FROM MEASUREMENTS ON THE SKULL OR THE LIVING HEAD.

By MIRIAM L. TILDESLEY.

(1) *Introductory.*

In 1901 Dr Alice Lee and Professor Karl Pearson presented formulae for predicting, on the basis of their various diametral measurements, the probable cranial capacity of German, Aino and Egyptian skulls and heads*. The trail thus blazed was followed in 1914 by Dr Isserlis, who computed the constants of formulae for estimating the capacity of the Negro skull, ♂ and ♀, using for his work the product of its length, breadth and total height†.

For the Negro skull these formulae serve, but for the head of the living Negro there was still no formula available, since no measurement corresponding to the basio-bregmatic height can be taken on the living. The height above the ear passages which would have given a formula suitable for both skull and head was not available in the data at Dr Isserlis' disposal, his calculations being based upon measurements taken by Dr Crewdson Benington on three series of negro skulls, two of which, brought from the Gaboon in 1864 and 1880 by Du Chaillu, were presented to the British Museum (Natural History), and in measuring these Benington had to dispense with the craniophor, and accordingly with measurements of the auricular height also. The third series was of Batetela crania brought from the Congo and deposited in the Royal College of Surgeons Museum by Emil Torday in 1909; and in measuring these Benington was able to include auricular height. It seemed however not impossible now to supply the data missing for the Gaboon series, as they were presumably still at South Kensington. I found on enquiry of Mr W. P. Pycraft that the bulk of them were still there, and have by his kind permission been able to place these on the craniophor, and so obtain the remaining values necessary for working out a capacity-formula applicable to the living head, as well as adding to Dr Isserlis' formula for the skull an alternative one using different characters.

As indicated above, not quite all the specimens measured by Benington are still within reach, for six Gaboon skulls of the 1864 series have since been presented by the Natural History Museum to the Smithsonian Institute, Washington (viz. Nos. 80 and 69 A, ♂, and Nos. 4, 9, 39 and 74, ♀); also one ♂ skull had no number and was represented only by "?" in Benington's records, and this specimen I have failed to identify by its measurements alone.

* *Phil. Trans.* Vol. 196 A, 1901, pp. 225—264. † *Biometrika*, Vol. x. 1914—15, pp. 188—193.

TABLE I.

Gaboon Skulls, Du Chaillu Collection *. British Museum (Natural History).

Males								Females							
No.	Auric. height in mm.	No.	Auric. height in mm.	No.	Auric. height in mm.	No.	Auric. height in mm.	No.	Auric. height in mm.	No.	Auric. height in mm.	No.	Auric. height in mm.	No.	Auric. height in mm.
1864 Series		36	110	71	115	4	119	*1864 Series*		38	111·5	75	108·5	17	105
1	110·5	40	117	73	115·5	5	112	2	103§	39	—†	78	112·5	18	111·5
3	111	43	112·5	76	117	14	108	4	—†	41	100·5	81	111	19	108
7	116·5	44	110	77	113	22	111	5	110	42	107·5	82	109	21	114·5
8	112	45	114	80	—†	23	115·5	9	—†	46	107	84	122	33	114
11	113·5	47	117	87	119·5	24	112	10	106·5	48	107	86	118	34	106·5
13	108·5	49	108	89	113	26	113·5	12	111·5	51	112·5	88	116·5	36	107
14	109·5	50	109	?	—‡	27	113	16	113	52	115·5	91	111·5	37	119·5
15	108	53	117	11 A	114·5	28	113	17	109	54	111·5	90	109·5	38	110·5
18	115·5	55	116·5	40 A	109·5	29	107	19	108	56	107·5	*1880 Series*		13	110
25	111	57	105	69 A	—†	30	113·5	20	111·5	58	101·5	2	108	39	109
27	116·5	59	115	37	108	31	110·5	21	114·5	60	106·5	6	107·5		
28	114·5	61	121·5	79	119	32	114·5	22	112	64	114	7	104		
30	109·5	62	107	83	116·5	35	119·5	23	108	66	106	8	103§		
31	110·5	63	126·5	85	114·5	12 A	111·5	24	114	68	108·5	9	106·5§		
32	114	65	112	*1880 Series*		20	114	26	101·5	70	107·5	10	105·5		
33	121	67	107·5	1	110			29	105	72	102·5§	12 B	109		
34	114	69	111	3	113			35	111	74	—†	15	116		

* For measurements taken by Dr Crewdson Benington on these crania see *Biometrika*, Vol. VIII. 1911, Tables III*a*, III*b*, and IV, after p. 336.

† Sent to Smithsonian Institute, Washington. ‡ Specimen not identified. § Child under 15.

It will be noted that the specimens in Table I include four children under 15, which must be withdrawn from the series in using it for calculations concerning the characters of the adult skull; and of those that remain three males and one female (No. 3 ♂, *1864 Series*; and Nos. 12A ♂, 20 ♂, and 18 ♀, *1880 Series*) were too imperfect for Benington to measure their capacity. Our available Gaboon material is thus reduced by missing, imperfect, and non-adult skulls to 62 males and 54 females.

(2) *Basis and Scope of Proposed Formulae.*

To the Gaboon measurements we will add, as Dr Isserlis did, those taken by Benington* on the Batetela skulls brought by Torday from the Congo, of which 45 ♂ and 21 ♀ adults were sufficently complete to admit of length, breadth, auricular height and cranial capacity being measured. It is not claimed that the Batetela, living some thousand miles to the west of the Gaboon, are closely related to tribes within the Gaboon area; nor indeed is there any ground for assuming that the various Gaboon tribes from which Du Chaillu's skulls were obtained are all closely related among themselves, or that there are not physical differences between them which would make it possible, or even easy, to distinguish the members of one tribe from those of another. The only assumption made is that the Negro section of the human race evolved within the continent of Africa not only possesses, in its many different tribes and racial groups, certain common physical traits which enable their fellow-men at once to recognise them as "Negro," but have also in common sufficient average resemblance in the conformation of the cranial box for a prediction of skull-capacity for known length, breadth and auricular height based upon the average shape of a miscellaneous collection of skulls from the Congo and Gaboon, to be correct within reasonable limits when applied to individual negroes from these or other districts in Africa. Although the area to which it is proposed to apply our formulae is very large, its population is on the whole much more homogeneous as regards proportion of head-length and head-breadth than that, for example, of our own small island. Though there are districts, e.g. in Nigeria, where a mesocephalic type of skull has been evolved with occasional examples of brachycephaly occurring, the large majority of Negroes throughout the continent fall into the dolichocephalic category. If our prediction should prove to be of the same order of accuracy as that for an English skull based upon the formula obtained from the series of 17th century Londoners, we shall be satisfied that we have remedied the total lack of a capacity-formula applicable to the Negro head, and at the same time can look forward to the time when the formulae now worked out will be some day superseded by a number of others, each based upon adequate material from a group of closely related Negro tribes, and covering between them all sections of the race.

The coefficients of correlation in the above total series, ♂ and ♀, between measured capacity and the product of the three diameters are:

$$♂ \quad r_{c:L.B.OH} = +\cdot 8078 \pm \cdot 022,$$

$$♀ \quad r_{c:L.B.OH} = +\cdot 8101 \pm \cdot 027,$$

* *Loc. cit.* Tables I and II.

(3) *Capacity Formulae for Skulls worked out.*

TABLE II.

Constants required for Correlation of Capacity and Diametral Product.

	Mean capacity in cm.³	S.D. of capacity in cm.³	Mean product of $L \times B \times OH$ in cm.³	S.D. of product of $L \times B \times OH$ in cm.³
♂ crania				
45 Congo	1342·69 ± 12·90	128·29 ± 9·12	2806·98 ± 26·64	264·99 ± 18·84
46 Gaboon (1864)	1385·93 ± 10·47	105·30 ± 7·40	2762·70 ± 20·74	208·53 ± 14·66
16 Gaboon (1880)	1447·44 ± 18·48	109·62 ± 13·07	2848·44 ± 28·33	168·00 ± 20·03
107 Negro males	1376·94 ± 7·92	121·47 ± 5·60	2794·14 ± 15·07	231·04 ± 10·65
♀ crania				
21 Congo	1205·88 ± 15·85	107·68 ± 11·21	2434·76 ± 33·03	224·42 ± 23·36
38 Gaboon (1864)	1243·16 ± 12·75	116·54 ± 9·02	2492·05 ± 20·22	184·78 ± 14·30
16 Gaboon (1880)	1245·125 ± 16·59	98·41 ± 11·73	2472·69 ± 39·53	234·44 ± 27·95
75 Negro females	1233·14 ± 8·70	111·73 ± 6·15	2471·88 ± 16·28	209·09 ± 11·51

and the corresponding regression formulae are

$$♂ \text{ Cap.} = \cdot 000{,}421\, L.B.OH + 190 \cdot 16 \pm \frac{48}{\sqrt{n}},$$

$$♀ \text{ Cap.} = \cdot 000{,}433\, L.B.OH + 163 \cdot 125 \pm \frac{44}{\sqrt{n}},$$

where n is the number of individuals in a group whose mean cranial capacity is estimated by the corresponding formula from its mean length, breadth and auricular height.

(4) *Testing of the Formulae.*

The above formulae having been obtained from our Gaboon and Congo data, they were then applied to the length, breadth and auricular height of six male and six female crania from widely separated regions in Africa, now in the Museum of the Royal College of Surgeons of England, and the capacities thus estimated were compared with the values as ascertained by direct measurement.

In no case does the discrepancy amount to twice the probable error, the limit which should include ten-elevenths of the population to which the formulae apply, and if we wish to compare the mean deviation of our few samples from their predicted value (for ♂'s 63 cm.³, for ♀'s 26 cm.³) with the mean deviation that theory expects to find in a large population, we have:

$$\begin{aligned} \text{Mean deviation in cm.}^3 &= \cdot 7978 \text{ S.D.} \\ &= \cdot 7978 \times \text{Prob. Error}/\cdot 67449 \\ &= 1 \cdot 18 \times \text{P.E.} \end{aligned}$$

TABLE III.

Comparison between estimated and measured Capacities.

Description	No. in Roy. Coll. Surg. Hum. Osteol. Cat. 1907	Max. length in mm. L	Max. breadth in mm. B	Auricular height in mm. OH	Measured capacity in cm.^{3}	Capacity in cm.^{3} estimated by Negro formula	Excess or defect of capacity-prediction, in cm.^{3}
♂ crania							
Kaffir	1287	193	139	114·5	1513	1483 ± 48	−30
From Niam Niam country	1257^{6}	189	130	113	1411	1359 ± 48	−52
Wa Kamba from U. Kambani	1282^{7}	182·5	132	108·5	1361	1291 ± 48	−70
Yoruba, Egba Dist., S. Nigeria	1247^{2}	173·5	135	111	1374	1285 ± 48	−89
Zulu, Natal	1284	188·5	132	112	1417	1363 ± 48	−54
From Quiloa (slave from interior)	1261	185·5	138	109·5	1288	1370 ± 48	+82
♀ crania							
Kaffir	1286	176	128	110·5	1232	1241 ± 44	+9
From Niam Niam country	1257^{7}	168	129	106·5	1166	1163 ± 44	−3
Wa Kamba from U. Kambani	1282^{8}	167	125	105·5	1169	1117 ± 44	−52
Yoruba, from Epé near Lagos, S. Nigeria	1247·33 (addn.)	173	125·5	102	1143	1122 ± 44	−21
Zulu, Natal	1285	176·5	131	110	1199	1264 ± 44	+65
From Quiloa (slave from interior)	1260	175	134	110·5	1289	1285 ± 44	−4

That is $1{\cdot}18 \times 48$ (= 57) cm.^{3} for males, which is closely approximated by our actual mean deviation of 63 cm.^{3}; and $1{\cdot}18 \times 44$ (= 52) cm.^{3} for females, that for our samples being merely 26 cm.^{3}, much less than we should have been prepared to allow. In so far therefore as these six ♂ and six ♀ crania can supply a test, our formulae may be considered to have passed it*, and we may therefore hopefully recommend them for application to such other Negro tribes as may require an estimate of cranial capacity where no measurement can be taken direct.

(5) *Adaptation of the Formulae for use on the Living Head.*

In making their estimate of the thickness of flesh for which allowance must be made in modifying head measurements to the corresponding measurements on skulls, Lee and Pearson† found no exact measurements available taken on a long series; but by combining the indications given by such figures as could then be collected (all based upon short series, and all European) an allowance of 11 mm. was decided upon as approximately correct for all the three diameters we are using.

In adapting our Negro formulae we are, perhaps, more fortunate in that actual measurements of the thickness of flesh on the vertex and on the frontal, tem-

* That five out of the six crania of males give values below the predicted and four of these exceeding the probable error (odds 255 to 1) is possibly a sign that we have more than a random small sample.

† *Loc. cit.* p. 252.

poral and occipital terminals have been taken on 513 ♂ negro crania, three-fifths of which are those of natives of Mozambique, and the remainder belong to nine other tribes from various parts of Africa. The values, which Professor Karl Pearson very kindly permits me to use, are taken from a work shortly to be published by the Biometric Laboratory, and are as follows:

Mean thickness of Flesh on 513 ♂ Negro Skulls.

On frontal	3·29 mm.
„ occipital	4·06 „
„ temporal	3·56 „
„ vertex	5·58 „

The deduction to be made from measurements of maximum length on the living Negro head is thus seen to be approximately 7·3 mm., and from head-breadth 7·1 mm. It is possible, however, that these thicknesses, obtained post-mortem on subjects who had died in hospital, may be less than on the healthy population. Most of the Negro mine-workers (which these were) die of (1) pneumonia, (2) malaria, (3) tuberculosis, (4) dysentery. The two latter diseases would certainly lead to wasting of the scalp as of the rest of the body; as regards malaria also, natives who are ill with fevers are apt to lose flesh very rapidly, partly on account of the diet given during treatment. Pneumonia would affect less directly the tissues of the scalp but might, perhaps, select the less robust subjects*. It is difficult to know what allowance to make to equate our figures with those for the normal population in view of these considerations, but probably 2 to 3 mm. would be as much as it would be safe to add. We might perhaps make our scalp allowance on head-length and head-breadth 10 mm. for the normal Negro with greater accuracy than by adopting the corresponding figures in the hospital data.

As regards auricular height, to render this measurement on the head comparable with that on the skull, allowance must be made for the thickness of flesh at the apex (5·6 mm.) and for the distance between the "centre" of the ear-passage (i.e. where the centre of the ear-plug rests when the head-spanner is applied to the living subject) and that part of the bone on which the skull rests on the craniophor. It is probable that the position of the ear-plug on the living head is not exactly the same as on the skull† but it cannot be very far from it. The distance between plug-centre and "auricular point" in the skull is usually round about 6 mm.; and if we are to add to this the scalp thickness on the top of the head as measured on the hospital cases, the allowance becomes about 11·5 mm. If we increase this for the healthy population, at about the same rate as for length and breadth of head, we arrive at 13 mm. as our estimate. For female Negroes we have no such

* The exact points at which the thickness of scalp was measured are not defined in the manuscript, and the writer is now dead, but as during his researches he was in correspondence with Professor Pearson on the question of measurements, it is probable that the points used were the terminals of *l*, *b* and *oh*. In this case his "vertex" is what we term "apex," the point on the median, vertically above the auricular axis.

† Detailed dissecting room measurements are at present being undertaken in Manchester to determine their relative positions in English subjects.

measurements of flesh-thickness but may hazard a guess that this will be rather less than in males, perhaps to the average extent of 1 mm. on all measurements.

(6) *Conclusions.*

Taking as our basis a series of Congo (Batetela) and Gaboon (probably mixed) crania, we obtained the following formulae for estimating cranial capacity from measurements of maximum length and breadth, and auricular height, in millimetres, on the skull:

$$♂ \quad \text{Capacity in cubic centimetres} = {\cdot}000{,}421\ L.B.OH + 190{\cdot}16 \pm \frac{48}{\sqrt{n}},$$

$$♀ \quad \text{Capacity} \quad ,, \quad = {\cdot}000{,}433\ L.B.OH + 163{\cdot}125 \pm \frac{44}{\sqrt{n}}.$$

In adapting the first formula for use on the living head we have the choice of two alternatives:

(1) We may use measurements of scalp-thickness taken on 513 ♂ mine-workers who died in hospital, and so deduct 7·3 mm. from head-length, 7·1 mm. from head-breadth and approximately 11 mm. from auricular height, thus arriving at the following formula:

$$♂ \quad \text{Cap. in cm.}^3 = {\cdot}000{,}421\,(l - 7{\cdot}3)(b - 7{\cdot}1)(oh - 11{\cdot}5) + 190{\cdot}16 \pm \frac{48}{\sqrt{n}}.$$

(2) We may modify these thicknesses with the hope of making them correspond more nearly to those existing in a normal population and adopt the following estimate:

$$♂ \quad \text{Cap. in cm.}^3 = {\cdot}000{,}421\,(l - 10)(b - 10)(oh - 13) + 190{\cdot}6 \pm \frac{48}{\sqrt{n}}.$$

For females we have no exact measurements, but suggest as a probable approximation:

$$♀ \quad \text{Cap. in cm.}^3 = {\cdot}000{,}433\,(l - 9)(b - 9)(oh - 12) + 163{\cdot}125 \pm \frac{44}{\sqrt{n}}.$$

These formulae should be applicable to African Negroes of any district, to be superseded later, one may hope, by formulae giving closer predictions, based upon data obtained from within more homogeneous racial groups.

ANOTHER "HISTORICAL NOTE ON THE PROBLEM OF SMALL SAMPLES."

(EDITORIAL.)

I DO not propose to reply to the paper by Drs Greenwood and Isserlis* at great length. But the writers accuse two young workers in my Laboratory and indirectly the Editor of this Journal of unduly depreciating or ignoring the work of the late Professor Tchouproff† "who" they say "is no longer able to speak for himself." They refer to a paper by the Russian mathematician published in *Metron* for 1923. One of these gentlemen is a conjoint editor of *Metron* and must know full well that it does *not* appear on the date it is stated to be issued. Its appearance is now nearly 10 months behind the date at which to use the terms employed by Drs Greenwood and Isserlis "it is stated to have been issued." In 1923 *Metron* was three to four months behind the time it was stated to have been issued. Dr Splawa-Neyman's paper was written in 1922 and sent to the Editor of the Polish Journal *La Revue mensuelle de Statistique,* who published it in Tome VI., issued in 1923‡. It was thus impossible for Neyman to have seen Tchouproff's paper in *Metron*§. It would be just as reasonable to accuse Tchouproff of disregarding Neyman, as Neyman of disregarding Tchouproff!

Why was Neyman's paper republished by *Biometrika*? The answer is fairly obvious. While there are results in it, with which I was many years ago familiar, there are others—for example the correlations—which as far as I know are original. The whole are deduced by a straightforward method, which can be easily followed. The weakness of my own method, which proceeded from summing the products of variations in the powers of the frequencies, i.e. such expressions as $(\delta n_a)^\alpha (\delta n_b)^\beta (\delta n_c)^\gamma \ldots\ldots$, has been recognised, and Dr Neyman's paper gave an extended illustration of the application of "Student's" rival method. One result of Drs Greenwood and Isserlis' attack will be to convince any one who examines the subject that Dr Neyman's work was perfectly independent of Tchouproff's. To reach a result by an independent investigation, if only contemporaneously with (possibly anteriorly to) a man of made name, can only be described as "depreciating" the latter when a partizan standpoint is adopted. Those who adopt

* *Journal of the Royal Statistical Society*, Vol. XC. pp. 347—352.

† I prefer to adopt the form of spelling Professor Tchouproff himself used, when signing his name with Latin letters.

‡ A letter from Dr K. Bessalik, Professor of the University of Warsaw, who in 1922 was Director of the State Institute of Agricultural Research in Bydgoner, certifies that Dr Neyman's paper was written in Bydgoner in 1922, and that no English Journals were accessible to him.

§ Actually the part of Tchouproff's paper, where Neyman's result for ${}_2M_2$ is given, was issued in September 1923!

the bludgeon rather than the rapier will no doubt prefer the break-through algebra of the Russian to the simpler analysis of the Pole. But the editor who has to pay for printing and has to interest as well a circle of readers primarily occupied not with mathematics but with practical statistics selects the *brevitas ingenii* rather than the *moles opere magnifico sed longissimo exstructa.*

We now turn to the memoir by Dr Church. He had been engaged for several years on practical statistical problems, namely the testing of the influence of skewness and the method of random sampling on the distribution of the means and standard deviations of small samples. He started his work in my Laboratory in 1922. He was not concerned in the first place with algebraic work, but as a practical statistician with such algebra only as was necessary for his purpose. When he came to small samples from a limited population I referred him to Tchouproff's results in Volume XII. of *Biometrika.* He found that they did not fit observation, and wasted many weeks in retesting and trying to see where his sampling methods were at fault. No doubt Dr Greenwood, had he been in like case, would at once have put his finger on the spot where Tchouproff had slipped; but the great algebraic facility of his colleague Dr Isserlis had failed conspicuously to correct the error when he prepared and edited Tchouproff's paper for this Journal.

However with admittedly less algebraic facility Dr Church and I ultimately concluded that experiment was right and theory was wrong, and on my advice Dr Church extended "Student's" method to Tchouproff's case, and provided a new and in the opinion of both of us a shorter proof of the required formulae, together with the location of Tchouproff's slip. No doubt when localised it is easy to rediscover, but it is worthy of note that it required experimental work to detect a slip, which had apparently escaped all the algebraists and mathematicians, who no doubt for seven years had been studying Tchouproff's paper.

When Dr Church was working at small samples from finite populations Dr Neyman came to work in the Biometric Laboratory, and as we have seen Dr Church was under no obligation to cite Tchouproff rather than Neyman, who had found the ${}_2M_2$ formula independently and at least contemporaneously. Dr Church was not interested in algebra for algebra's own sake, he wanted to throw light on his own experimental sampling. According to our critics he might have cited Isserlis' work of 1915 or 1918, but they do not ask why Isserlis in 1915 did not cite Pearson of 1905. The fact is that any fairly competent mathematician could work out in a few hours the moments of the *means* of samples, once the moments of the hypergeometrical are known or indeed without them*. Nobody can lay claim to

* Church says of the moments of the means of samples that "their ultimate origin is doubtful, for I have been unable to ascertain who first deduced them." Tchouproff himself says they were "given almost simultaneously by L. Isserlis, Prof. G. Mortara and myself without that one of us had known (*sic*!) the investigations of the others" (*Metron*, Vol. II. p. 471). Isserlis and Greenwood state that one of them was given by Pearson in 1905, and a still more general form has been attributed to Markoff (1907)! Markoff's formula seems to amount to this:

If $z = x_1 + x_2 + \dots + x_n$, then

$$\sigma_z^2 = \sigma_{x_1}^2 + \sigma_{x_2}^2 + \dots + \sigma_{x_n}^2 + 2\sigma_{x_1}\sigma_{x_2}r_{x_1x_2} + 2\sigma_{x_1}\sigma_{x_3}r_{x_1x_3} + \dots + 2\sigma_{x_2}\sigma_{x_3}r_{x_2x_3} + \dots,$$

originality for simple algebraical processes, which provide results which any teacher of advanced statistics has probably worked out for himself and given in his lectures for many years.

The matter is otherwise with the third and fourth moments of the second moments of samples. They do not indeed require any mathematical originality, but they do require a prodigious amount of laborious elementary algebra, and are of such a character that no lecturer could easily reproduce them on the blackboard. Well, Tchouproff did not give these formulae, as our critics somewhat grudgingly admit, but they say he gave formulae from which they might have been deduced. Not having been provided, why should not Dr Church provide them in his own way? Why was he bound to follow up Tchouproff's lengthy analysis *, and not the shorter route which Dr Neyman and he had already adopted?

Does Dr Church claim these formulae as important results, for which he is to receive great credit? On the contrary he says—and I agree with him—that they are absolutely valueless to the practical statistician. They throw back on experimental investigation the problem of determining the limits within which it is legitimate "not to return to the bag," and Church's reason for publication of the formulae was not to demand kudos for a valuable contribution to science, but to indicate the practical futility of the theoretical formulae.

Any one reading hastily Drs Greenwood and Isserlis' "Historical Note" would

a formula which would not have struck any biometrician as original in 1907. See *Phil. Trans.* Vol. 192 A, p. 260, 1899. If we make all the standard deviations and correlations equal and write $u = z/n$, then

$$^2 = \frac{1}{n^2}\left(n\sigma^2 + 2\frac{n(n-1)}{2}r\sigma^2\right) = \frac{\sigma^2}{n}(1 + \overline{n-1}\,r).$$

In the particular case of $x_1, x_2, \dots x_n$ being selected from the series $X_1, X_2, \dots X_N$, $N > n$, then one readily sees that $S(X_1 - \bar{X}) = 0$, and accordingly $S(X_1 - \bar{X})^2 = -2S'(X_1 - \bar{X})(X_2 - \bar{X})$,

or
$$N\sigma_X{}^2 = -2\frac{N(N-1)}{2}r\sigma_X{}^2,$$

i.e. we have
$$r = -\frac{1}{N-1},$$

or
$$\sigma_u{}^2 = \frac{\sigma^2}{n}\frac{N-n}{N-1},$$

the formula apparently claimed by half-a-dozen statisticians! It requires very little additional labour to deduce in the same way the 3rd and 4th moment coefficients of u.

Under the circumstances Dr Church's remark that he did not know who really originated the formulae seems more than justified! Tchouproff (*Metron*, Vol. II. p. 471) claims that his values had been in use by his pupils before their publication in 1918. If priority is to be claimed on the basis of lecture notes, I would remark that several of Tchouproff's other formulae in *Biometrika*, Vol. XII. 1918, p. 186 (7) (up to $\nu_4(N)$) and p. 193 (23) were given by me in lectures as early as 1914—15, and possibly earlier. Actually questions of priority for simple algebraical results, which can be readily reached by any mathematically trained statistician, are absurd. The claim will depend on whether A or B thought them worth putting into print. In America the process of calculating moments about an arbitrary origin and then transferring to the mean is attributed to an American writer in 1917!

* Indeed may I not say as Andreas Tacquet of Clavius: "cujus ea est prolixitas, schemata verò tam implexa ac intricata, vt à nullo mortalium hactenus perlectum fuisse totum existimem." *Opticae* Liber III. fol. 178.

believe that: (i) Church's formulae for ${}_2M_3$ and ${}_2M_4$ were to be found earlier in Tchouproff, (ii) that he had committed a grave injustice to Tchouproff by citing ${}_2M_2$ from Neyman, the fact being that Church's formulae for ${}_2M_3$ and ${}_2M_4$ are *not* in Tchouproff's paper and that Neyman had worked out his results for ${}_2M_2$ independently and at the very least contemporaneously with Tchouproff. I by no means regret republishing Neyman's paper; Drs Greenwood and Isserlis have indeed proved how desirable its republication was.

Only the future can demonstrate whether it is not more profitable for the progress of science to encourage the young who have worked and are working under very disadvantageous conditions, than to apply the bludgeon to promising scientific recruits on the ground—in this case illusory—that they are wanting in respect to the hierarchs. One page of creative work I personally hold is worth fifty of such journalistic criticism.

K. P.

MISCELLANEA.

On Further Formulae for the Reconstruction of Cranial Capacity from External Measurements of the Skull.

BY K. PEARSON AND BRENDA N. STOESSIGER, Crewdson-Benington Student.

A NUMBER of formulae for such reconstruction have been provided, and, we think, two points have been adequately demonstrated, namely that:

(*a*) Product formulae, whether they be diametral or arcual, give the more satisfactory results.

(*b*) Product formulae differ very considerably with race and sex.

To what extent this latter difference depends on the race and sex, or is due to the relatively small number of crania in each series hitherto dealt with, has not yet been determined. It occurred to us therefore that it would be worth while determining product formulae for a really long series of crania, and of such only one case was at our disposal, namely the "*E* Series" of Egyptian Skulls from Gizeh of the XXVIth—XXXth Dynasties.

But the task of computing the products for 14 or 15 hundred crania and then forming correlation tables appeared too laborious to be carried out, and accordingly we were compelled to devise a more rapid method of reaching a reasonably accurate result.

If P be the product of any three variates X, Y, Z, and the coefficients of variation of X, Y, Z represented by $100v_X$, $100v_Y$ and $100v_Z$ be small, then it is possible to find with very considerable accuracy the mean ($\bar{P}$), the standard deviation (σ_P), the coefficient of variation ($100v_P$) and the correlation of the product with another variate (C), by aid of the formulae:

$$\bar{P}=\bar{X}\bar{Y}\bar{Z}(1+v_Y v_Z r_{YZ}+v_Z v_X r_{ZX}+v_X v_Y r_{XY}) \quad \text{(i)},$$

$$(\sigma_P/\bar{P})^2=v_P^2=(v_X^2+v_Y^2+v_Z^2+2v_Y v_Z r_{YZ}+2v_Z v_X r_{ZX}+2v_X v_Y r_{XY}) \quad \text{(ii)},$$

$$r_{PC}=(v_X r_{XC}+v_Y r_{YC}+v_Z r_{ZC})/v_P \quad \text{(iii)}.$$

We shall then have for the regression equation with the probable error for the individual skull:

$$C=\bar{C}+\frac{r_{PC}\sigma_C}{\sigma_P}(P-\bar{P})\pm \cdot 67449\sigma_C\sqrt{1-r_{PC}^2} \quad \text{(iv)}.$$

In the case of the "*E* Series" of skulls, all the correlations and coefficients of variation required are provided in the paper by Davin and Pearson*. Equation (i) is really given to provide a measure of how nearly we may take $\bar{P}=\bar{X}\bar{Y}\bar{Z}$.

For the "*E* Series" we have:

For Arcual Product, ♂'s: $\bar{P}_a=\bar{X}\bar{Y}\bar{Z}(1+\cdot 001,865)$,

" " " ♀'s: $\bar{P}_a=\bar{X}\bar{Y}\bar{Z}(1+\cdot 001,329)$,

For Diametral Product, ♂'s: $\bar{P}_a=\bar{X}\bar{Y}\bar{Z}(1+\cdot 001,2948)$,

" " " ♀'s: $\bar{P}_d=\bar{X}\bar{Y}\bar{Z}(1+\cdot 000,8885)$,

* *Biometrika*, Vol. xvi. pp. 337—339 and Table V, p. 348.

giving for the means:

	$\overline{P}_a$ = Arcual Product, cm.^{3}		$\overline{P}_d$ = Diametral Product, cm.^{3}	
	Actual	$\overline{X}\overline{Y}\overline{Z}$	Actual	$\overline{X}\overline{Y}\overline{Z}$
♂	59,408	59,298	3455·69	3451·222
♀	53,685	53,614	3103·62	3100·86

Extracting from the paper referred to the requisite values, we have:

Mean C: ♂'s 1438·86 cm.^{3}; ♀'s 1301·08 cm.^{3}

σ_C: ♂'s 113·51 cm.^{3}; ♀'s 92·06 cm.^{3}

For	♂'s	$\overline{U}$ = 518·66 mm.,	$\overline{Q}$* = 307·45 mm.,	$\overline{S}$ = 371·86 mm.
„	♀'s	$\overline{U}$ = 499·69 mm.,	$\overline{Q}$* = 298·25 mm.,	$\overline{S}$ = 359·75 mm.
„	♂'s	L = 185·34 mm.,	B = 138·88 mm.,	H† = 134·08 mm.
„	♀'s	L = 177·06 mm.,	B = 135·55 mm.,	H† = 129·20 mm.
„	♂'s	v_U = ·0265,	v_Q = ·0317,	v_S = ·0336,
„	♀'s	v_U = ·0235,	v_Q = ·0292,	v_S = ·0292,
„	♂'s	r_{CU} = ·7834,	r_{CQ} = ·7214,	r_{CS} = ·7
„	♀'s	r_{CU} = ·7325,	r_{CQ} = ·6914,	r_{CS} = ·7115,
„	♂'s	r_{QS} = ·6536,	r_{SU} = ·7547,	r_{QU} = ·5916,
„	♀'s	r_{QS} = ·5930,	r_{SU} = ·6797,	r_{QU} = ·5207,
„	♂'s	r_{CL} = ·6661,	r_{CB} = ·6817,	r_{CH} = ·5357,
„	♀'s	r_{CL} = ·5983,	r_{CB} = ·6798,	r_{CH} = ·4943,
„	♂'s	v_L = ·0309,	v_B = ·0343,	v_H = ·0375,
„	♀'s	v_L = ·0266,	v_B = ·0334,	v_H = ·0339,
„	♂'s	r_{BH} = ·3344,	r_{HL} = ·3830,	r_{LB} = ·3971,
„	♀'s	r_{BH} = ·3389,	r_{HL} = ·1896,	r_{LB} = ·3763.

Hence from (ii) we find:

v_{P_a} = ·081,03295 for ♂'s; = ·070,1155 for ♀'s,
σ_{P_a} = 4814·01 cm.^{3} „ ♂'s; = 3764·15 cm.^{3} „ ♀'s,
v_{P_d} = ·078,2760 „ ♂'s; = ·068,9228 „ ♀'s,
σ_{P_d} = 270·5166 cm.^{3} „ ♂'s; = 213·9105 cm.^{3} „ ♀'s.

Thus if C_a = probable C from arcual product, C_d = probable C from diametral product, we deduce from (iii):

r_{CP_a} = ·854,613 for ♂'s; = ·829,750 for ♀'s,
r_{CP_d} = ·798,470 „ ♂'s; = ·803,460 „ ♀'s;

and finally from (iv):

$C_a = 241{\cdot}73 + {\cdot}020{,}151 P_a \pm 39{\cdot}76/\sqrt{n}$ for ♂'s .... (α),
$C_a = 211{\cdot}64 + {\cdot}020{,}293 P_a \pm 34{\cdot}66/\sqrt{n}$ „ ♀'s .... (β),
$C_d = 280{\cdot}98 + {\cdot}335{,}042 P_d \pm 46{\cdot}79/\sqrt{n}$ „ ♂'s .... (γ),
$C_d = 227{\cdot}90 + {\cdot}345{,}783 P_d \pm 36{\cdot}97/\sqrt{n}$ „ ♀'s .... (δ).

* Transverse arc through apex from upper surfaces of ear-plugs.

† Basio-apical height. By an oversight, *Biometrika*, Vol. XVI. p. 331, it was said to be the basio-bregmatic height.

Measured and Computed Values of the Capacity of Egyptian Crania. (Males.)

Series / Measured by	Predynastic Naqada / Fawcett	1st Dynasty Royal Tombs Abydos / Motley	Early & Middle Dynasties El Kubaniah / N. Toldt	Middle Dynasties El Kubaniah / N. Toldt	18th to 19th Dynasties Abydos / Schmidt	18th to 21st Dynasties Thebes / Schmidt	18th to 20th Dynasties Thebes / Stahr	26th to 30th Dynasties Gizeh / Davin & Pearson	Modern Copts
No. (*circa*)	120	30	70	35	33	168	60	890	60
Length L ...	185·13	184·9	182·7	182·2	183·1	181·94	183·5	185·3	177·03
Breadth B ...	134·87	137·7	135·9	134·2	139·2	136·63	137·3	138·9	136·63
Height H ...	135·21	133·8	134·4*	137·4*	132·4	136·05	135·2*	134·1	136·05
Horizontal Arc U	511·02	513·5	506·2	505·6	517·1	510·76	512·6	518·7	501·76
Sagittal Arc S ...	373·02	369·0	371·8	372·7	374·5	372·44	372·1	371·9	365·95
Transverse Arc Q	304·22	304·9	[307·45]	[307·45]	308·1	306·07	[307·45]	307·45	311·81
Diametral Product P_d ...	3376	3407	3337	3360	3374·5	3332	3406	3451·5	3291
Arcual Product P_a ...	57991	57773	57864	57935	59665	58227	58642·5	593085	57254
Capacity									
Formula (α) ...	1410	1406	1408	1409	1444	1415	1423	1437	1395
" (γ) ...	1412	1422	1399	1407	1412	1393	1422	1437	1384
" (ϵ) ...	1409	1409	1404	1407	1435	1409	1422	1437	1390
Observed ...	1381 (88)	1364 (24)	1366 (67)	1355 (27)	1414 (25)	1388 (164)	1451 (50)	1439 (753)	1356 (59)

* Value of H obtained from H' by addition of 1·5 mm. Values of Q in square brackets are those for the population from which the formula was deduced as no transverse arc was measured in these cases.

The probable errors indicate that determinations from the arcual product are for both sexes better than determinations from the diametral product. The reader must bear in mind that both P_a and P_d must be read in cubic centimetres, or the arcs and diameters in centimetres, not in millimetres.

It might be supposed that an improved formula for reconstruction might be obtained by correlating with a bivariate regression, C with both P_a and P_d.

To determine $r_{P_aP_d}$ we require to use:

$$r_{P_aP_d} = \frac{\left[\begin{array}{l} v_U v_L r_{UL} + v_U v_B r_{UB} + v_U v_H r_{UH} \\ + v_Q v_L r_{QL} + v_Q v_B r_{QB} + v_Q v_H r_{QH} \\ + v_S v_L r_{SL} + v_S v_B r_{SB} + v_S v_H r_{SH} \end{array}\right]}{v_{P_a} v_{P_d}} \quad (\nu).$$

We find: $r_{P_aP_d} = \cdot 873{,}183$ for ♂'s; $= \cdot 886{,}909$ for ♀'s.

These lead to the bivariate regression formulae:

For ♂'s: $C = 191 \cdot 84 + \cdot 015{,}6236\, P_a + \cdot 092{,}2692\, P_d \pm 38 \cdot 90/\sqrt{n}$ .... (ϵ),

" ♀'s: $C = 157 \cdot 50 + \cdot 013{,}4271\, P_a + \cdot 136{,}2116\, P_d \pm 33 \cdot 45/\sqrt{n}$ .... (ζ).

It is clear from the probable errors that (ϵ) and (ζ) only better (α) and (β) to the extent of about a cubic centimetre. This might be foreseen from the high values of $r_{P_aP_d}$.

We have tested the male formulae (α), (γ) and (ϵ) against observation on a variety of Egyptian crania measured by different craniometricians. The results are given in the table on p. 213.

It must be admitted that the results are disappointing. While there is extraordinarily little variation in the six measurements taken of the eight series of these ancient crania, and further in their diametral and arcual products, the observed mean capacities vary widely. Thus the computed means vary from 1393 to 1437, or a range of 44 cm.³ But the observed means vary from 1355 to 1451, or a range of 96 cm.³ While the computed capacities place the series roughly—i.e. allowing for probable error—in historical order, the observed capacities tend to obscure this order, placing the Middle Dynasties at the head of the list. Of course something of this may be due to the fact that the computed capacities are based on a larger number of crania than the observed capacities, but we fear this cannot be the real origin of this discrepancy. The main source of the divergency, we regret to say, seems to us to lie in personal equation. The relative tightness of packing into skull and measuring glass has varied from observer to observer, and we get a series of observed capacities which provide means practically worthless for comparative purposes. Cranial capacities—except for the most rough judgments—are really valueless until we have some standardisation of observers. We therefore suggest that 10 or 12 standard skulls should be prepared as moulds taken from endocranial castings, and that these should be used to reduce the workers in craniometric laboratories to a common standard. It is feasible, of course, with any *crâne étalon* to test the capacity against water volume of an individual observer's packing. But a single *crâne étalon* really does not suffice as the effect of the packing depends to a considerable extent on the conformation of the inner table, and we need a diversity of forms to properly test the relative personal equation of observers. By the combined effort of craniometricians, it ought to be feasible to procure and distribute such a series of *crânes étalons**.

* Suggestions from craniologists willing to take part in such a scheme would be very welcome, and should be addressed to Professor Karl Pearson, Galton Laboratory, University College, London, W.C. 1.

Note on the relation of the (P, χ^2) Goodness of Fit Test to the Distribution of Standard Deviations in samples from a normal Population.

BY KARL PEARSON.

IF we have an indefinitely large population with classes $C_0, C_1, C_2, \dots C_n$, and we take samples of size M, those that contain m individuals belonging to the classes $C_1, C_2, \dots C_n$ will be distributed in the same proportions as would occur in samples of m extracted from an indefinitely large population containing only those n classes in the same proportions. Further it has been shown by a method suggested by H. E. Soper, that if we take the total frequency of class C_0 to be very much larger than the class frequencies of C_1 to C_n, then the distributions of the latter frequencies approach closer and closer to Poisson-Series and cease to be intercorrelated*.

Now let us apply this to a series of frequencies $m_1, m_2, \dots m_n$ arising from such a sample; each of these may be considered independent and to vary round its mean $\bar{m}_s$ with a standard deviation $\sqrt{\bar{m}_s}$ as given by the Poisson-Series. But

$$\frac{m_1-\bar{m}_1}{\sqrt{\bar{m}_1}}, \quad \frac{m_2-\bar{m}_2}{\sqrt{\bar{m}_2}}, \dots, \quad \frac{m_s-\bar{m}_s}{\sqrt{\bar{m}_s}}, \dots, \quad \frac{m_n-\bar{m}_n}{\sqrt{\bar{m}_n}}$$

are a series of n-variate deviations each divided by their standard deviation. If we assume for the moment that a Poisson-Series is adequately expressed by a normal curve, then, since these deviates are all divided by their standard deviations, they may be looked upon as a sample from the same normal curve. Accordingly if Σ^2 be given by

$$\Sigma^2 = \frac{1}{n} \overset{n}{\underset{1}{S}} \frac{(m_s-\bar{m}_s)^2}{\bar{m}_s} = \frac{\chi^2}{n} \text{ say},$$

then Σ is the standard deviation of a sample of n, drawn from variates following a normal distribution of standard deviation equal to unity. But the frequency distribution of the standard deviations Σ of samples of size n from a normal distribution of standard deviation σ is known to be

$$y = y_0 \left(\frac{\sqrt{n}\Sigma}{\sigma}\right)^{n-2} e^{-\frac{n\Sigma^2}{2\sigma^2}}.$$

Putting $\sigma = 1$ and replacing Σ^2 by χ^2/n, we have

$$y = y_0 \chi^{n-2} e^{-\frac{1}{2}\chi^2},$$

the familiar distribution of the χ^2's.

It will be seen that the whole proof depends on the legitimacy of the assumption that a Poisson-Series can be replaced by a normal curve. Now the values of β_1 and $\beta_2 - 3$ for a Poisson-Series, mean $\bar{m}$, are given by:

$$\beta_1 = \frac{1}{\bar{m}}, \quad \beta_2 - 3 = \frac{1}{\bar{m}},$$

or:

	$\bar{m}=10$	$=20$	$=30$	$=50$
β_1	·1	·05	·0$\dot{3}$	·02
β_2	3·1	3·05	3·0$\dot{3}$	3·02.

It is reasonably clear accordingly that the approach to normality is fairly rapid. The approach to a normal distribution is indeed of the same order of rapidity as that of the distribution of means of $\bar{m}$ individuals drawn from a population with its $\beta_1 = 1$ and $\beta_2 = 4$ respectively.

* *Biometrika*, Vol. XI. pp. 150—152.

The Application of the Theory of Differential Equations to the Solution of Problems connected with the Interdependence of Species.

By EGON S. PEARSON, D.Sc.

In *The Elements of Physical Biology**, A. J. Lotka has discussed how what he terms the Fundamental Equations of Kinetics of Evolving Systems,

$$\frac{dN_r}{dt}=f_r(N_1,\ N_2,\ \ldots\ N_n)\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)},$$

may be employed in describing many of the processes of evolution. In particular if N_r be the number of individuals at instant t in the species S_r, then if a suitable form be found for the functions f_r, the solutions of (i) will describe the interaction of n species living in contact with one another. If the solution of the equations

$$f_1=f_2=\ldots=f_n=0\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}$$

provides a set of values $N_1=q_1$, $N_2=q_2$, ... $N_n=q_n$, then the q's may be termed equilibrium values of the system. Taking $x_r=N_r-q_r$, Lotka assumes that f_r may be expanded by Taylor's Theorem in ascending powers of the x's, and indicates how the general solution of (i) can be expressed as the sum of a number of exponential terms, containing n quantities λ, which are the roots of a certain determinant $\Delta(\lambda)=0$. The form of the solution will depend upon whether these roots are positive, negative, or complex. In a later chapter Lotka has considered the special cases in which one species S_2 feeds on another S_1 or on both of two species S_1 and S_3.

More recently Professor Vito Volterra has published a long paper among the *Memorie della R. Accademia Nazionale dei Lincei*†, in which he considers in a similar manner—namely by the solution of differential equations—the laws which on certain assumptions may describe the variations in the numbers of the individuals of species living together. Volterra's interest in the subject was aroused by Dott. Umberto D'Ancona, who was seeking an explanation of certain problems suggested by the fishery statistics of the Upper Adriatic before, during and after the War. Volterra has approached the subject from a somewhat different standpoint from Lotka, and as he has proceeded further with the solution and interpretation of the equations both in the special and general cases, it seems of some interest to give a brief account of his work, and to discuss finally how far his results are confirmed by the statistics of D'Ancona.

In any problem of this nature it is of course impossible to expect that the complex play of forces can be adequately described by a mathematical formula; it is necessary to start from certain simplifying assumptions, to study only the bearing of the chief factors in the problem, and to consider the results obtained as at best only a first approximation to be tested against the statistical records of the biologist. The hypothesis from which it is necessary to start is that the assumption that the number of individuals, N, in a species increases or decreases continuously with the time is not so crude as to invalidate conclusions drawn therefrom. If, for example, we consider the unrestricted increase in a species, and can measure the excess under such conditions of the "birth rate" over the "death rate" by a constant ϵ, then it is necessary to assume that the increment δN in the size of the species, in *any* interval of time δt, is given by

$$\frac{\delta N}{N}=\epsilon\delta t\ \text{ or }\ \frac{dN}{dt}=\epsilon N\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(iii)}$$

leading to

$$N=N_0e^{\epsilon t}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(iv)}.$$

In practice we know that the coefficient ϵ is nearly always periodic, varying with the seasons of

* *The Elements of Physical Biology*, Baltimore, 1925.

† "Variazioni e fluttuazioni del numero d' individui in specie animali conviventi," 1926, Serie sesta, Vol. II. Fascicolo III.

the year*, but this would not prevent (iv) representing approximately the law of unrestricted increase taken over a long period of time, where ϵ is given an average value. If n species are living in contact, the problem is to consider the variation in N_r ($r=1, 2, \dots n$) arising from, (1) the "unrestricted" increase or *accrescimento bruto* of form (iii), and (2) the reciprocal actions of the members of one species on those of its own or another species, an interference which it is supposed will generally take the form of one individual preying upon another. To express the variation of N_r arising from (2), it is necessary to make a further assumption, namely that the interference of one species with another is proportional to the number of random encounters which can take place between them. Thus in unit time there will be on the average $m_{rs} N_r N_s$ encounters between members of S_r and S_s and $m_{rr} N_r (N_r - 1)$, or closely $m_{rr} N_r^2$, between the different members of S_r. That is to say the increment of N_r in time δt may be thought of as consisting of three portions,

$$\begin{cases} \delta N_r' = \epsilon_r N_r \delta t & \text{(v)}, \\ \delta N_r'' = -p_{rr} N_r^2 \delta t & \text{(vi)}, \\ \delta N_r''' = -\Sigma_s' (p_{rs} N_r N_s) \delta t \dagger & \text{(vii)}. \end{cases}$$

The p's are constants, which are either positive or negative, measuring the average gain or loss to the size of a species resulting from an encounter between two individuals. These three equations lead to the fundamental differential equation

$$\frac{dN_r}{dt} = \left(\epsilon_r - \sum_{s=1}^{n} (p_{rs} N_s)\right) N_r \quad \text{(viii)}.$$

Volterra does not attempt to minimise the limitation of this theoretical equation when applied to real problems, where the number and effect of encounters will vary according to the season of the year and the age of the individuals, but I think that we can agree that the relations (v), (vi) and (vii) combined in equation (viii) may lead to certain general laws which will throw light on the observed fluctuations in the size of species. The investigation is at any rate an interesting one. It will be seen that he does not, as Lotka, reach (viii) as an approximation derived from a Taylor expansion of the general equation (i), but considers it as a reasonable form of relation in itself, based on the hypothesis of encounters.

If attention is confined to a single species or group which is unaffected by any others, we have

$$\frac{dN_r}{dt} = (\epsilon_r - p_{rr} N_r) N_r \quad \text{(ix)}.$$

This equation represents a case in which the increasing numbers in themselves put a check upon the natural rate of increase, not necessarily because the individuals prey on each other, but because they struggle for a limited food supply. (ix) leads on integration to

$$N_r = \frac{C\epsilon_r}{Cp_{rr} + e^{-\epsilon_r t}} \quad \text{(x)},$$

which as Lotka points out is Pearl's "logistic" growth curve in its simplest form ‡.

Let us consider now the general solution of (viii); suppose $q_1, q_2, \dots q_n$ are the values of $N_1, N_2, \dots N_n$ found by solving the n linear equations

$$\epsilon_r - \sum_{s=1}^{n} (p_{rs} N_s) = 0, \qquad (r = 1, 2, \dots n) \quad \text{(xi)}.$$

* In his § 12 Volterra considers very briefly the effect of a simple periodicity in ϵ.

† Σ_s' implies the summation for all values of s from 1 to n, except r.

‡ In connection with the present theory, an interesting case is that in which Pearl shows that curve (x) fits very well the growth in a population of *Drosophila melanogaster* isolated in a glass jar and provided with a constant but limited food supply. See Raymond Pearl, *The Biology of Population Growth*, 1925.

If these q's are all positive, then the interactions in the biological system are such that a stationary state, or one of equilibrium, is possible, for which $\frac{dN_r}{dt}=0\;(r=1, 2, \ldots n)$, and the numbers of individuals in the species remain constant. Now write

$$F(N_1, N_2, \ldots N_n)=\sum_{r=1}^{n}\sum_{s=1}^{n}(\alpha_r p_{rs} N_r N_s) \quad \text{.........................(xii)},$$

where the α's are n undetermined positive constants. Then if

$$x_r=N_r-q_r=(n_r-1)\,q_r \quad \text{..................................(xiii)},$$

it can be shown that the solutions of (viii) satisfy

$$\left(\frac{e^{n_1}}{n_1}\right)^{\alpha_1 q_1}\left(\frac{e^{n_2}}{n_2}\right)^{\alpha_2 q_2}\cdots\left(\frac{e^{n_n}}{n_n}\right)^{\alpha_n q_n}=Ce^{-\int_0^t F(x_1, x_2, \ldots x_n)\,dt} \quad \text{.................(xiv)}.$$

If a stationary state is impossible, that is if some of the q's found from (xi) are negative, we cannot proceed very far with the solution in general terms. If however the q's are all positive, the two following cases appear to lead to solutions of special interest*.

1. When it is possible to choose a series of α's such that $F(x_1, x_2, \ldots x_n)$ is always definitely positive. In this case it can be shown that the numbers N_r tend asymptotically or in a series of damped oscillations towards the values q_r of the stationary state. That is to say under these conditions none of the species become exterminated and none increase indefinitely in size, but the whole system tends to one of biological equilibrium.

2. When it is possible to choose a series of α's such that $F(x_1, x_2, \ldots x_n)\equiv 0$ or vanishes identically. In this case the right-hand side of (xiv) is a constant, C.

It is worth while considering this solution in some detail since it is of wider applicability than appears at first sight, and leads us to some suggestive results. For $F\equiv 0$, it can be shown that $\frac{p_{rs}}{p_{sr}}=-\frac{\alpha_s}{\alpha_r}$ and $p_{rr}=0$, so that we may write

$$\alpha_r p_{rs}=a_{rs}=-a_{sr}=-\alpha_s p_{sr}, \quad a_{rr}=0.$$

Hence (viii) may be written

$$\frac{dN_r}{dt}=\left(\epsilon_r-\sum_{s=1}^{n}\left(\frac{a_{rs}}{\alpha_r}N_s\right)\right)N_r, \qquad (r=1, 2, \ldots n) \quad \text{....................(xv)}.$$

From this series of equations we see that in time δt the encounters between individuals of S_r and S_s result (if a_{rs} be positive) unfavourably to S_r by causing a decrement in numbers,

$$\delta N_r=a_{rs}/\alpha_r\, N_s N_r\,\delta t,$$

and favourably to S_s by an increment

$$\delta N_s=a_{rs}/\alpha_s\, N_s N_r\,\delta t.$$

a_{rs} is a constant for the two species depending upon the chance of mutual encounters and the proportion of these which are fatal to S_r, while the numbers $1/\alpha_r$ and $1/\alpha_s$ represent what may be termed the "equivalents" of individuals in the different species. Thus the destruction of M/α_r individuals of S_r by members of S_s means a "gain"† of M/α_s individuals to S_s, and this is to hold for all pairs r and s. It would no doubt be impossible to obtain direct evidence of the validity of this hypothesis of equivalents, which is the necessary and sufficient condition for $F\equiv 0$, but it does not appear unreasonable as an approximation to be tested indirectly by the results which follow from it. As Volterra shows, if we suppose the α's to be the mean weights of individuals in each species, then the hypothesis simply represents the crude idea that the

* In terms of energy concepts both Lotka and Volterra speak of "conservative," "dissipative," "reversible" systems, etc.

† That is to say a "gain" resulting from the increasing vigour and possibility of longer survival which the additional food gives to members of S_s.

destruction of M individuals in S_r corresponds to a transfer of total weight Ma_r to S_s, or to an average increase of its numbers by Ma_r/a_s.

In the simple case when $n=2$ and we consider only the relation between two species, no hypothesis of equivalents is necessary and the differential equations (xv) can be written

$$\frac{dN_1}{dt}=(\epsilon_1-p_1N_2)N_1;\quad \frac{dN_2}{dt}=(\epsilon_2-p_2N_1)N_2 \quad \text{.......................(xvi)},$$

the solutions of which satisfy

$$N_1^{\epsilon_2}e^{-p_2N_1}=CN_2^{\epsilon_1}e^{-p_1N_2} \quad \text{.................................(xvii)}.$$

For this there are ten different forms of solution depending on the combinations of different signs of the four constants ϵ_1, ϵ_2, p_1 and p_2. These solutions are explained very neatly by Volterra by graphical methods. In all cases but two* either N_1 or N_2 or both $\to\infty$ or $\to 0$ as $t\to\infty$, of which the first solutions are only of theoretical interest as it is clear that other restrictive factors would come increasingly into play as soon as the size of a species began to exceed a certain limit. The only stable case in which, on the assumptions made, two species can live together permanently is that for which

$$\epsilon_1>0,\quad \epsilon_2<0,\quad p_1>0,\quad p_2<0,$$

that is to say the case in which S_1 if left to itself would increase in numbers, while S_2 would become extinct were it not for the presence of S_1 on whose members it preys and feeds. In this case the following results can be deduced †.

(*a*) The numbers of individuals in the two species, N_1 and N_2, traverse a closed periodic cycle, the period of completing which depends only on ϵ_1, ϵ_2 and the initial conditions represented by the constant C.

(*b*) The mean values of N_1 and N_2 during the cycle are ϵ_2/p_2 and ϵ_1/p_1 respectively, and do not depend on the initial values.

(*c*) If at any moment individuals of both species are destroyed in proportion to their numbers at that moment, this is equivalent to a sudden decrease in ϵ_1 and an increase in ϵ_2, or to an increase in the mean N_1 and a decrease in the mean N_2. If this common destruction is so great that the modified ϵ_1 becomes negative, then we pass over from the system of stable cycles to one in which both N_1 and $N_2\to 0$.

Before considering the application of these laws to a practical case, let us see how far they can be extended to the general problem of many cohabiting species. Here Volterra shows that a solution of (xi) with q's which are positive and one for which $F\equiv 0$ is only possible theoretically if the number of species, n, is even. If these conditions are satisfied it follows that:

(*d*) The N's will again vary between two numbers, one above and one below the corresponding stable values q, and these oscillations will continue undamped indefinitely.

(*e*) The mean $N_r\to q_r$ as $t\to\infty$.

(*f*) If the system is started initially near the stationary state, a series of small oscillations about this will ensue, which are the sum of $n/2$ superposed periodic fluctuations in general not commensurable, so that the cycle is not truly periodic.

(*g*) For the initial assumption that (xi) gives a set of positive q's, some of the ϵ's must be positive and some negative. If all are negative, then all the species become extinct, and if all are positive, all the species increase indefinitely.

(*h*) If it is possible to divide the n species into p which feed on others but are not themselves eaten, and q which are eaten but do not feed on others, then if all the species are suddenly decreased in proportion to their numbers, the mean number in *some* of the eaten species will increase, and the mean in *some* of the eaters will decrease.

* One of these represents a case of unstable equilibrium.

† This is the special case, referred to above, considered by Lotka, who I think only reaches the first of the rules, (*a*), given by Volterra.

(j) If n is odd it is impossible theoretically for all the species to remain between finite limits. At least one, say N_k, must $\rightarrow 0$ or $\rightarrow \infty$. In the former event the number of species may become even again, while in the latter it becomes necessary to reconsider the initial hypothesis. Clearly in reality no species can increase indefinitely in size, and an approximation to the position may be reached by introducing into (viii) one or more of the limiting factors $-p_{rr}N_r^2$, which refers the solution back to that of the case when $F > 0$.

Consider now the bearing of these results on a population of fish. They provide of course one reason for some of the puzzling fluctuations observed in the relative numbers of fishes of different species, but they also suggest what may be the effect of intensive fishing. In trawling there is an approach to a uniform destruction of fish in proportion to their numbers; in the simple case of two species, the second of which lives by preying upon the first, it is suggested that the result of a moderate amount of trawling will be to increase the numbers of the first (or eaten) species and to decrease those of the second (or preying) species. In the case of many species so exact a result cannot be reached without inserting numerical values for the various coefficients, but it appears that there will again be a tendency for the numbers in the harmless species to increase and those in the predatory species to decrease. If the fishing is carried beyond a certain intensity, the system ceases to fluctuate in a stable cycle, and all the species tend to decrease indefinitely. There would be little difficulty in developing the theory so as to apply to the case of selective fishing if we had some measure of the intensity of this selection. In a great many areas the war was a period in which deep-sea fishing was completely or largely discontinued; this meant that the external destructive force was temporarily removed and consequently it might be expected that changes in the relative proportions of species would have occurred in the reverse manner to that just described. That is to say the stoppage of fishing during the war should, if Volterra's reasoning be correct, have favoured on the whole the predatory at the expense of the more harmless fish.

To test this point D'Ancona* has examined the statistics of the fish sold in the years between 1910 and 1924 in the three chief markets of the Upper Adriatic—Trieste, Fiume and Venice. He comes to the conclusion that an analysis of these figures definitely supports the mathematical results of Volterra. The problem is however one of great complexity and it is worth while examining some of the records in detail. These are given in terms of weight in kilograms of the fish of different species sold in each of the three markets during the year; the figures for Trieste are the most ample covering the 14 years 1910—23, with the exception of 1918, while those for Fiume cover the years 1914—24. D'Ancona has divided the species into three groups: (1) *forme pelagiche*, (2) *forme litoranee*, (3) *forme bentoniche*. The first of these are almost entirely migrant species about whose life-history and habitat comparatively little is known; the second group consists of the shallow-water fish caught in all manner of ways, which were not interfered with to so large an extent by the war; the third group contains the fish mainly caught in the trawl net and is therefore of special interest because the fluctuations in relative numbers may throw light on the effect of the uniform cessation of destruction by man. The difficulties which arise in interpreting the statistics can be illustrated from the figures given below for the species of group (3) brought in and sold in Trieste. I have classed together the eleven species of the family of *Selaci*, which D'Ancona considers to be the most predatory in the group. The first row of figures in the Table shows how the total catch fell during the war years to about one-tenth its normal value; in comparing the relative fluctuations in the numbers of different species we have to consider them as percentages of these totals. The figures in the second row are some of those which D'Ancona believes will support his contention that the predatory species were found on the whole in larger numbers at the end than before the war. But if we examine the percentages in his Table XXV we see some of the difficulties of the problem. As shown in the bottom row of my table, the species *Sepia officinalis* (the common cuttlefish), of the family of *Cefalopodia*, constituted in the years 1910, 11, 14, 15, 22, 23 about one-third of the total catch, but in

* *Dell' influenza della stasi peschereccia del periodo* 1914–18 *sul patrimonio ittico dell' Alto Adriatico*, Dott. Umberto D'Ancona, Memoria CXXVI, R. Comitato Talassografico Italiano, 1926.

Trieste. "Specie Bentoniche."

Year	*1910*	*1911*	*1912*	*1913*	*1914*	*1915*	*1916*	*1917*	*1919*	*1920*	*1921*	*1922*	*1923*
Total of all fish sold, in kilograms	666,280	722,365	792,511	635,706	516,193	91,425	84,635	74,372	249,851	574,655	644,940	821,614	759,301
Selaci; °/₀ of total	5·54	8·64	9·45	15·78	14·72	7·57	16·29	15·53	19·38	15·60	13·22	10·56	10·14
Selaci; °/₀ of total, less *Sepia officinalis*	8·39	11·55	11·15	18·74	21·74	11·29	16·80	16·13	20·94	18·25	16·67	14·13	15·84
Sepia officinalis; °/₀ of total	34·01	25·17	15·21	15·79	32·31	32·98	3·01	3·70	7·42	14·54	20·70	25·28	35·99

1916—1919 it was caught in comparatively insignificant numbers. The percentages for the *Selaci* given in the second row of my table are therefore very dependent on the catch of *Sepia officinalis*; how much so is seen by examining the percentages in the third row. These are now comparatively level, and the maximum of 1919—20 is no more emphasised than that of the pre-war years 1913—14*. Now the deduction of *Sepia* from the total may not be legitimate, but to clear the position let us consider the alternatives. We must either suppose that there is some connection between the increase in the *Selaci* and the decrease in *Sepia*, that the former either prey directly on the latter, or upon fish which in turn prey upon the latter. Or else if the two are unconnected we must admit that the increase in the *Selaci* was something which began in 1913 and appears to have had a secondary minimum in the middle of the war. D'Ancona states that *Sepia officinalis* is itself a relatively voracious species but does not I think say whether it has itself enemies.

Turning to the figures for Fiume we find similar difficulties of interpretation; the percentages of the catch of *Selaci* of the total are now

1914	1915	1916	1917	1918	1919	1920	1921	1922	1923	1924
11·78	21·45	23·09	21·02	36·50	25·87	16·00	15·77	14·54	10·53	13·25

These figures as they stand show a marked increase in the *Selaci* for the war years 1915—19. The position is quite different from that at Trieste, for at Fiume the catches of *Sepia officinalis* never amount to more than 5·6 °/ₒ of the total. It is found that 60—80 °/ₒ of the catch is made up by, (1) the *Selaci*, (2) *Nephrops norvegicus*, (3) the two species of *Gadidi*, *Gadus merlangus* and *Merluccius vulgaris*. Further the rise in the *Selaci* percentage from 11·78 in 1914 to 21·45 in 1915, is found to correspond to a great drop in the proportional catch of the *Gadidi* from 43·85 to 12·21 °/ₒ, while the rise in the *Selaci* from 21·02 in 1917 to 36·50 in 1918 corresponds to a fall in the *Nephrops* percentage from 41·44 in the first year to 16·61 in the second. In terms of Volterra's theory we must interpret these changes as mainly due to the fact that in these years the *Selaci* destroyed either directly or indirectly a very great number of the *Nephrops* and the two species of *Gadidi*. Is this interpretation biologically probable? D'Ancona does not appear to face the point, but merely states that these three species while all "relatively voracious" have "other enemies." It is difficult not to suspect that these relative changes may be due to other factors, such as changes in the method, area or season of fishing forced on the trawlers by the circumstances of the war, or even to migration of the fish. The problem is undoubtedly one of great interest, but the situation is of such complexity that one feels doubtful whether with even more detailed analysis than D'Ancona has given, it would be possible to obtain any definite confirmation of Volterra's laws from this material.

It has only been possible here to give an outline of Volterra's work which covers some 80 pages in which many other special cases and approximations are considered. Some of these have been referred to briefly by Volterra in a recent article in *Nature*†. Lotka has also suggested the application of this method of analysis to various fields, including that of parasitology. For the moment the theoretical results can hardly be advanced further without some check from the observational side which will test the reality of the simplifications and assumptions involved.

* The origin of the great drop in 1915 is easily traced; in that year the species *Pleuronectes flesus* contributed a percentage of 25·45 to the total catch while in the preceding and succeeding years it varies between 3 and 8. Removing from the total for 1915 the figures for *Sepia* and *Pleuronectes*, the percentage for the *Selaci* becomes 18·21.

† *Nature*, October 16, 1926, p. 558; see also January 1, 1927, p. 12, for correspondence between Lotka and Volterra as to priority.

Further Note on the "Linear Correlation Ratio."

By EGON S. PEARSON, D.Sc.

In a recent paper in this Journal* a new method was suggested of finding an approximation to the product-moment coefficient of correlation, when dealing with data for which one variable (y) is given quantitatively and the other (x) only qualitatively. In the proof there given, it was assumed that the correlation surface must be normal, and the formula was

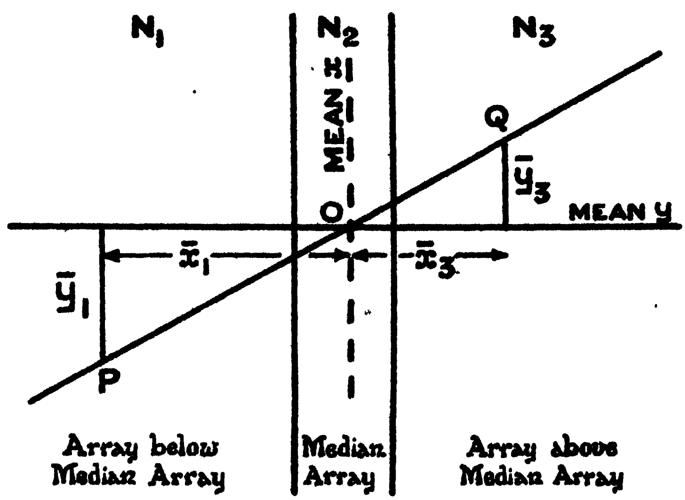
$$r = \frac{1}{z_1 + z_2} \frac{S(n_x(\bar{y}_x \sim \bar{y}))}{N\sigma_y} \quad \text{.........(i)},$$

for all arrays, excluding the median and taken always with the same sign, right and left of the median.

z_1 and z_2 are the ordinates on either side of the median array obtained by fitting a normal scale to the x-distribution.

It may however be shown that the result (i) follows from somewhat simpler assumptions, namely: (a) that the x-marginal distribution alone is normal, (b) that the regression of y on x is linear.

Suppose that the table is divided into three arrays parallel to the axis of y, containing frequencies N_1, N_2, and N_3, and that the mean, P, of the first array lies at $(\bar{x}_1, \bar{y}_1)$ and the mean, Q, of the third array at $(\bar{x}_3, \bar{y}_3)$, where these coordinates are referred to the mean $(\bar{x}, \bar{y})$ of the whole table as origin. $\bar{y}_1$ and $\bar{y}_3$ are known numerically, but not $\bar{x}_1$ and $\bar{x}_3$, since the x character is qualitative and the margin divided into three broad categories. The position is illustrated in the diagram; in the proof that follows it is not necessary that the central array should contain the median.



As the regression is linear, the means P and Q of the broad arrays must lie on the regression line POQ. This is the line

$$y = r\sigma_y/\sigma_x \cdot x.$$

Hence
$$r\frac{\sigma_y}{\sigma_x} = \frac{\bar{y}_1}{\bar{x}_1} = \frac{\bar{y}_3}{\bar{x}_3} = \frac{|N_1\bar{y}_1| + N_3\bar{y}_3}{|N_1\bar{x}_1| + N_3\bar{x}_3},$$

where $|N_1\bar{y}_1|$ and $|N_1\bar{x}_1|$ are quantities to be taken with positive signs. But if a normal scale be fitted to the three frequencies N_1, N_2, and N_3, we have with the usual notation

$$\bar{x}_1 = \frac{-z_1}{N_1/N}\sigma_x, \qquad \bar{x}_3 = \frac{z_2}{N_3/N}\sigma_x.$$

* K. Pearson, *Biometrika*, Vol. XVII. pp. 459—461.

It follows that $$r=\frac{|N_1\bar{y}_1|+N_3\bar{y}_3}{N\sigma_y(z_1+z_2)} \quad\text{.......................(ii),}$$

a form which is exactly equivalent to (i). If there are a number of x-arrays, it will be possible to group these together into three in a variety of ways. Having decided upon the method of division, the procedure is therefore as follows:

(1) Find $\bar{y}_1$ and $\bar{y}_3$, the deviations of the means of arrays N_1 and N_3 from the mean of the total y-distribution.

(2) Find σ_y, the standard deviation of this distribution.

(3) Find z_1 and z_2 by entering the appropriate table of the Normal Probability Function with the *per milles* N_1/N and N_3/N.

If the central array, N_2, vanishes (ii) reduces to

$$r=\frac{N_3\bar{y}_3}{N\sigma_y\,.\,z} \quad\text{.......................(iii),}$$

which is the *Biserial* coefficient of correlation*.

Without obtaining an expression for the probable error it is impossible to say whether a more reliable estimate of r is obtained by taking three groups rather than the two groups of the biserial method, and further in what manner it is best to distribute the observations in the three groups. If the x-distribution is not normal, then z_1+z_2 is an approximation for the true value

$$\zeta=\frac{1}{\sigma_x}\left(\frac{-N_1}{N}\bar{x}_1+\frac{N_3}{N}\bar{x}_3\right),$$

and it might be possible to find theoretically the division N_1, N_2, N_3 which would make this approximation the best for, let us say, curves of moderate skewness.

The "linear correlation ratio" is of course only likely to be of use in cases where the x-variable is qualitative, but it is of interest to examine the effect of different array groupings in a case where x is measured numerically so that a check on the approximations is possible. I have taken the correlation table for 2922 measures of Barometric Heights at Laudale and Southampton for which the regression is closely linear but the marginal distributions are skew†. The x-distribution (as defined above) is that for Southampton with frequency constants $\beta_1=\cdot 17$ and $\beta_2=3\cdot 61$, Product-moment $r=\cdot 7802\pm\cdot 0050$.

Table of Constants for Laudale-Southampton Barometric Heights Correlation Table.

N_1	N_2	N_3	Linear Correlation Ratio	z_1+z_2	ζ	$\frac{\zeta-(z_1+z_2)}{\zeta}\times 100$
1287·5	382·5	1252	·7927	·7870	·7751	−1·54
1287·5	722	912·5	·7962	·7484	·7451	−0·44
1287·5	1010	624·5	·8006	·6857	·6893	+0·52
1287·5	1211	423·5	·7966	·6225	·6292	+1·06
892	778	1252	·7841	·7429	·7267	−2·23
340	1330	1252	·7726	·5882	·5762	−2·08
340	2158·5	423·5	·7705	·4236	·4303	+1·56
1287·5	(biserial)	1634·5	·7854	·7890	·7686	−2·65
1670	"	1252	·8000	·7850	·7816	−0·44

Mean $r=\cdot 7887$

The linear correlation ratio gives a quite adequate approximation to the product-moment r in all cases, but it does not appear possible to draw straight off any conclusions as to the best method of division from a comparison of z_1+z_2 and ζ.

* *Biometrika*, Vol. VII. p. 97 and Vol. X. p. 384.

† *Phil. Trans.* A. Vol. 190, p. 423; *Biometrika*, Vol. XVII. p. 291, etc.

VOLUME XIX DECEMBER, 1927 PARTS III AND IV

BIOMETRIKA

ON THE FREQUENCY DISTRIBUTION OF THE MEANS OF SAMPLES FROM A POPULATION HAVING ANY LAW OF FREQUENCY WITH FINITE MOMENTS, WITH SPECIAL REFERENCE TO PEARSON'S TYPE II.

BY J. O. IRWIN, M.A., M.Sc.

SUPPOSE a population to have n_{x_1} individuals with a value x_1 of a character, n_{x_2} with value $x_2, \dots n_{x_s}$ with value $x_s, \dots n_{x_w}$ with value x_w, and that we take samples at random containing n individuals; the probability that the sum of the values of the character in a sample will be X is the coefficient of A^X in

$$\frac{(n_{x_1}A^{x_1} + n_{x_2}A^{x_2} + \dots + n_{x_s}A^{x_s} + \dots)^n}{N^n} \quad \dots\dots\dots\dots\dots\dots(1),$$

where N is the total population. This may be written

$$(p_{x_1}A^{x_1} + p_{x_2}A^{x_2} + \dots + p_{x_s}A^{x_s} + \dots)^n \quad \dots\dots\dots\dots\dots (1\ bis),$$

where $$p_{x_1} = \frac{n_{x_1}}{N}, \quad p_{x_2} = \frac{n_{x_2}}{N}, \quad \dots, \quad p_{x_s} = \frac{n_{x_s}}{N}, \quad \dots, \quad p_{x_w} = \frac{n_{x_w}}{N}.$$

This follows because there are N^n ways of forming the sample, while the number which satisfies the condition that the total should be X is clearly the coefficient of A^X in the numerator of (1). It has been pointed out by Pearson that this theorem was originally due to Euler, but it is well known and has been used by H. E. Soper in his *Frequency Arrays**.

The theorem provides a ready method of obtaining the distribution of totals or means of samples from discrete populations; it has been shown by Pearson, for instance, that we may readily deduce from the theorem that the distribution of the means of samples of r from the binomial $(p+q)^n$ plotted to unit h is the binomial $(p+q)^{rn}$ plotted to unit h/r, and that a similar result is true for a Poisson series.

The purpose of the present paper is to adapt the process to obtaining the distributions of totals or means in samples from populations in which the variate has a continuous distribution.

The rth moment coefficient about the origin of the original population will be given by

$$p_{x_1}x_1^r + p_{x_2}x_2^r + p_{x_3}x_3^r + \dots + p_{x_s}x_s^r + \dots + p_{x_w}x_w^r,$$

which is equal to the coefficient of

$$\frac{\alpha^r}{r!} \text{ in } \sum_{s=1}^{s=w} p_{x_s}e^{\alpha x_s} \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots(2).$$

* *Frequency Arrays*, p. 10, 1922. [Euler's merit lies in thinking of the problem long before the age was ripe for it. K. P.]

In H. E. Soper's elegant notation

$$\sum_{s=1}^{s=w} p_{x_s} A^{x_s}$$

would be termed the frequency array of the distribution in the original population, while

$$\sum_{s=1}^{s=w} p_{x_s} e^{\alpha x_s}$$

would be its moment array.

The expression (1 *bis*) may be written after expansion

$$\sum_{t=1}^{t=v} P_{X_t} A^{X_t},$$

where X_t is one of the possible values of the total (i.e. n times the mean), P_{X_t} is its corresponding probability and the summation is extended to all possible values $X_1, X_2, \ldots X_v$ of the totals. Thus

$$\left(\sum_{s=1}^{s=w} p_{x_s} A^{x_s}\right)^n = \left(\sum_{t=1}^{t=v} P_{X_t} A^{X_t}\right),$$

or on putting $A = e^{\alpha}$

$$\left(\sum_{s=1}^{s=w} p_{x_s} e^{\alpha x_s}\right)^n = \left(\sum_{t=1}^{t=v} P_{X_t} e^{\alpha X_t}\right) \quad \text{.........................(3)},$$

an expression in which we are really equating two different forms of the moment array of the distribution of totals.

Now if we suppose the distribution $\sum_{s=1}^{s=w} p_{x_s} A^{x_s}$ to pass into a continuous frequency distribution from $x = a$ to $x = b$ which may be denoted symbolically by

$$\int_a^b f(x)\, dx\, A^x,$$

the distribution of totals will also pass into a continuous distribution from $x = na$ to $x = nb$ which may be denoted by

$$\int_{na}^{nb} \psi(x)\, dX\, A^X,$$

and equation (3) will become

$$\left(\int_a^b f(x)\, e^{\alpha x} dx\right)^n = \int_{na}^{nb} \psi(x)\, e^{\alpha x} dx \quad \text{.......................(4)}.$$

Now if $f(x)$ is known (4) may be written

$$F(\alpha) = \int_{na}^{nb} \psi(x)\, e^{\alpha x} dx \quad \text{..........................(4 *bis*)},$$

and to obtain ψ we have to solve the integral equation (4 *bis*).

Let $\alpha = i\beta$ and let $\phi(x)$ be a function which is equal to $\psi(x)$ when $na \leqslant x \leqslant nb$ and zero when x is outside these limits, and let $F(i\beta) = \chi(\beta)$. Then

$$\chi(\beta) = \int_{-\infty}^{\infty} \phi(x)\, e^{i\beta x} dx,$$

and applying Fourier's Integral Theorem

$$\phi(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\chi(\beta)\,e^{-i\beta x}d\beta\,*,$$

or

$$\psi(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-i\beta x}F(i\beta)\,d\beta\,\dagger,$$

when

$$na\leqslant x\leqslant nb\qquad\qquad(5).$$

Thus we have the distribution of totals in samples of n, from which the distribution of means can be at once obtained. The value of this solution in practice depends of course on the possibility of evaluating the integral (5) and it will now be shown how (5) may be used to obtain the distribution when the original population is

(A) The Normal Curve,
(B) Pearson's Type III,
(C) Pearson's Type II.

The distributions of means of (A) and (B) have already been obtained otherwise.

(A) *The Normal Curve.*

In this case we put

$$f(x)=\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2},\quad a=-\infty,\quad b=\infty.$$

Then

$$F(\alpha)=\left(\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}e^{\alpha x}dx\right)^n=e^{\frac{1}{2}n\alpha^2},$$

and from (5)

$$\begin{aligned}\psi(x)&=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-i\beta x}\,e^{-\frac{1}{2}n\beta^2}\,d\beta\\&=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-\frac{1}{2}n\beta^2}(\cos\beta x+i\sin\beta x)\,d\beta\\&=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-\frac{1}{2}n\beta^2}\cos\beta x\,d\beta.\end{aligned}$$

* In order that this solution may be valid it is necessary to show independently that

$$\int_{-\infty}^{\infty}e^{-i\beta x}\chi(\beta)\,d\beta$$

is zero when x is $>nb$ or $<na$. In the case of the limited range curves discussed below this may be shown by contour integration.

† It is worth while noting that this result is real, for if $F(i\beta)$ can be expanded in the form

$$a_0+a_1(i\beta)+a_2(i\beta)^2+\ldots$$

$$\begin{aligned}\int_{-\infty}^{\infty}e^{-i\beta x}F(i\beta)\,d\beta&=\int_{-\infty}^{\infty}(\cos\beta x+i\sin\beta x)\{a_0-a_2\beta^2+a_4\beta^4+\ldots+i(a_1-a_3\beta^3+a_5\beta^5-\ldots)\}\,d\beta\\&=\int_{-\infty}^{\infty}[\cos\beta x\,(a_0-a_2\beta^2+a_4\beta^4-\ldots)-\sin\beta x\,(a_1-a_3\beta^3+a_5\beta^5-\ldots)]\,d\beta\\&\quad+i\int_{-\infty}^{\infty}[\sin\beta x\,(a_0-a_2\beta^2+a_4\beta^4-\ldots)+\cos\beta x\,(a_1-a_3\beta^3+a_5\beta^5-\ldots)]\,d\beta,\end{aligned}$$

and

$$\int_{-\infty}^{\infty}\sin\beta x\,(a_0-a_2\beta^2+a_4\beta^4-\ldots)\,d\beta,$$

$$\int_{-\infty}^{\infty}\cos\beta x\,(a_1-a_3\beta^3+a_5\beta^5-\ldots)\,d\beta$$

vanish because the integrands are both odd functions of β.

Now it may be shown* that

$$\int_{-\infty}^{\infty} e^{-\lambda x^2}\cos(2\lambda a x)\,dx = e^{-\lambda a^2}\int_{-\infty}^{\infty} e^{-\lambda x^2}\,dx,$$

or

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}n\beta^2}\cos(\beta x)\,d\beta = e^{-\frac{1}{2}\frac{x^2}{n}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}n\beta^2}\,d\beta$$

$$= \frac{\sqrt{2\pi}}{\sqrt{n}}\,e^{-\frac{1}{2}\frac{x^2}{n}},$$

or

$$\psi(x) = \frac{1}{\sqrt{2\pi n}}\,e^{-\frac{1}{2}\frac{x^2}{n}},$$

which gives the distribution of totals in samples of n.

Writing $n\bar{x} = x$,

$$\frac{1}{\sqrt{2\pi n}}\,e^{-\frac{1}{2}\frac{x^2}{n}}\,dx \text{ becomes } \frac{1}{\sqrt{2\pi n}}\,e^{-\frac{1}{2}n\bar{x}^2}\,n\,d\bar{x},$$

or

$$\frac{1}{\sqrt{2\pi}\Sigma}\,e^{-\frac{1}{2}\frac{\bar{x}^2}{\Sigma^2}} \text{ where } \Sigma = \frac{1}{\sqrt{n}} \quad \text{.........................(6).}$$

This gives the frequency distribution of the means of samples taken at random from a normal population with unit standard deviation and is a result which is known otherwise to be correct.

(B) *Pearson's Type III.*

Now suppose $\quad f(x) = \frac{e^{-x}x^{p-1}}{\Gamma(p)}, \quad a = 0, \quad b = \infty.$

Then

$$F(\alpha) = \left\{\int_0^{\infty} \frac{e^{-x}x^{p-1}}{\Gamma(p)}\,e^{\alpha x}\,dx\right\}^n$$

$$= \left\{\frac{1}{(1-\alpha)^p}\right\}^n = \frac{1}{(1-\alpha)^{np}}$$

$$= \frac{1}{(1-\alpha)^P} \text{ say, where } P = np.$$

Substituting this in (5)

$$\psi(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-i\beta x}\,d\beta}{(1-i\beta)^P}.$$

This integral is best evaluated by contour integration.

Let $Z = u + iv$ and let us integrate $\int \frac{e^{-ixz}\,dz}{(1-iz)^P}$ round a contour in the z plane bounded by the real axis and a semi-circle C of radius R below it.

Then by Cauchy's Theorem:

$\int_C \frac{e^{-ixz}\,dz}{(1-iz)^P} = 2\pi i$ (the residue at the pole $z = \frac{1}{i}$) $= 2\pi i$ (the coefficient of $\frac{1}{t}$ in the Laurent expansion of the integrand at $z = \frac{1}{i}$).

* Whittaker and Watson, *Modern Analysis*, p. 114.

We obtain this by writing

$$z = \frac{1}{i} + t,$$

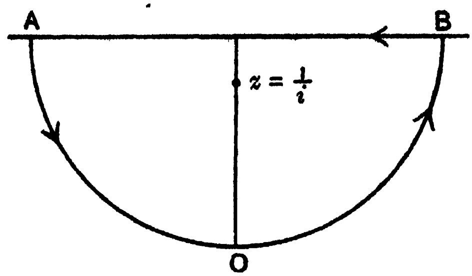
and expanding the integral we have

$$\frac{e^{-x(1+ti)}}{(-it)^P} = \frac{e^{-x}}{(-it)^P}\left\{1 - xti + \frac{(xti)^2}{2!} + \ldots\right\},$$

and the coefficient of

$$\frac{1}{t} = \frac{e^{-x}(xi)^{P-1}(-1)^{P-1}}{\Gamma(P)(-i)^P} = \frac{-e^{-x}x^{P-1}}{i\Gamma(P)},$$

therefore
$$\int_O \frac{e^{-ixz}\,dz}{(1-iz)^P} = \frac{-2\pi e^{-x}x^{P-1}}{\Gamma(P)}.$$



By making $R \to \infty$ it may be shown that provided $x > 0$*

$$\int_{AOB} \frac{e^{-ixz}\,dz}{(1-iz)^P} \to 0 \text{ as } R \to \infty,$$

and thus
$$+\int_{-\infty}^{\infty} \frac{e^{-ixu}\,du}{(1-iu)^P} = \frac{2\pi e^{-x}x^{P-1}}{\Gamma(P)}$$

and
$$\psi(x) = \frac{e^{-x}x^{P-1}}{\Gamma(P)}.$$

This gives the distribution of totals, for on putting $x = n\bar{x}$ we obtain

$$\frac{e^{-x}x^{P-1}}{\Gamma(P)}\,dx = \frac{ne^{-n\bar{x}}(n\bar{x})^{P-1}}{\Gamma(P)}\,d\bar{x}$$

$$= \frac{ne^{-n\bar{x}}(n\bar{x})^{np-1}}{\Gamma(np)}\,d\bar{x},$$

and the distribution of the means of samples from the curve $y = e^{-x}x^{p-1}/\Gamma(p)$ is given by†

$$y = \frac{ne^{-n\bar{x}}(n\bar{x})^{np-1}}{\Gamma(np)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7).$$

* When x is negative it may be shown, by taking a semi-circular contour *above* the real axis, that $\int_{-\infty}^{\infty} \frac{e^{-ixu}}{(1-iu)^P}\,du = 0$ as it should do.

† A more direct method of reaching this result has already been given in a paper by A. E. R. Church, *Biometrika*, Vol. XVIII. pp. 385—388.

(C) *Pearson's Type II.*

Let us put $f(x) = x^{p-1}(1-x)^{p-1}$, where $p > 0$.

Then $$F(\alpha) = \left\{\int_0^1 x^{p-1}(1-x)^{p-1} e^{\alpha x}\,dx\right\}^n.$$

Put $x = \sin^2\theta$, then

$$\int_0^1 x^{p-1}(1-x)^{p-1} e^{\alpha x}\,dx = \int_0^{\pi/2} \sin^{2p-2}\theta \cos^{2p-2}\theta e^{\alpha\sin^2\theta}(2\sin\theta\cos\theta\,d\theta)$$

$$= 2\int_0^{\pi/2} \sin^{2p-1}\theta\cos^{2p-1}\theta e^{\alpha\sin^2\theta}\,d\theta.$$

On putting $2\theta = \phi$ we have

$$\int_0^1 x^{p-1}(1-x)^{p-1} e^{\alpha x}\,dx = \frac{1}{2^{2p-1}}\int_0^{\pi} \sin^{2p-1}\phi e^{\frac{\alpha}{2}(1-\cos\phi)}\,d\phi$$

$$= \frac{e^{\frac{\alpha}{2}}}{2^{2p-1}}\int_0^{\pi} \sin^{2p-1}\phi e^{-\frac{\alpha}{2}\cos\phi}\,d\phi.$$

Now if $I_n(z)$ is Bessel's function of the nth order and imaginary argument it is known that*

$$I_n(z) = \frac{z^n}{2^n\Gamma(n+\frac{1}{2})\Gamma(\frac{1}{2})}\int_0^{\pi} e^{-z\cos\phi}\sin^{2n}\phi\,d\phi,$$

provided $n + \frac{1}{2} > 0$.

Thus $$\int_0^{\pi} e^{-\frac{\alpha}{2}\cos\phi}\sin^{2p-1}\phi\,d\phi = \frac{2^{p-\frac{1}{2}}\Gamma(p)\Gamma(\frac{1}{2})}{\left(\frac{\alpha}{2}\right)^{p-\frac{1}{2}}} I_{p-\frac{1}{2}}\left(\frac{\alpha}{2}\right),$$

provided $p > 0$; and

$$\int_0^1 f(x) e^{\alpha x}\,dx = e^{\frac{\alpha}{2}}\Gamma(p)\Gamma(\tfrac{1}{2})\alpha^{-p+\frac{1}{2}} I_{p-\frac{1}{2}}\left(\frac{\alpha}{2}\right) \quad\text{..................(8)},$$

and $$F(\alpha) = \frac{\{\Gamma(p)\}^n\{\Gamma(\frac{1}{2})\}^n}{\alpha^{n(p-\frac{1}{2})}} e^{\frac{n\alpha}{2}}\left\{I_{p-\frac{1}{2}}\left(\frac{\alpha}{2}\right)\right\}^n \quad\text{..................(9)}.$$

Substituting in (5)†

$$\psi(x) = \frac{\{\Gamma(p)\}^n\{\Gamma(\frac{1}{2})\}^n}{2\pi}\int_{-\infty}^{\infty} e^{-i\beta x} e^{\frac{ni\beta}{2}}(i\beta)^{-n(p-\frac{1}{2})}\left[I_{p-\frac{1}{2}}\left\{\frac{i\beta}{2}\right\}\right]^n d\beta,$$

when $0 \leqslant x \leqslant n$.

* Whittaker and Watson, *Modern Analysis*, p. 360.

† In order that this solution of the integral equation may be valid, it is necessary to show that

$$\int_{-\infty}^{\infty} \frac{e^{-i\beta\left(x-\frac{n}{2}\right)}}{\beta^{n(p-\frac{1}{2})}}\left[J_{p-\frac{1}{2}}\left(\frac{\beta}{2}\right)\right]^n d\beta = 0,$$

if $x > n$ or negative.

Writing $$x - \frac{n}{2} = y,$$

it must be shown that

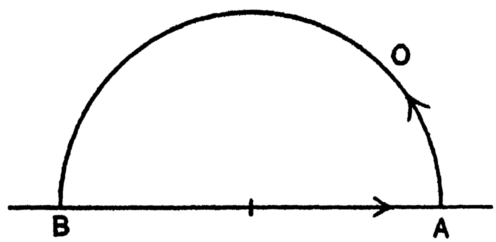
$$\int_{-\infty}^{\infty}\phi(\beta)\,d\beta = \int_{-\infty}^{\infty}\frac{e^{-i\beta y}}{\beta^{n(p-\frac{1}{2})}}\left[J_{p-\frac{1}{2}}\left(\frac{\beta}{2}\right)\right]^n d\beta = 0 \text{ if } y > \frac{n}{2} \text{ or } < -\frac{n}{2}.$$

Now
$$I_{p-\frac{1}{2}}\left(\frac{i\beta}{2}\right) = i^{p-\frac{1}{2}} J_{p-\frac{1}{2}}\left(\frac{\beta}{2}\right).$$

Thus
$$\psi(x) = \frac{\{\Gamma(p)\}^n \{\Gamma(\frac{1}{2})\}^n}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-i\beta\left(x-\frac{n}{2}\right)}}{\beta^{n(p-\frac{1}{2})}} \left\{J_{p-\frac{1}{2}}\left(\frac{\beta}{2}\right)\right\}^n d\beta \ldots\ldots\ldots (10).$$

I am indebted to Mr E. C. Titchmarsh for outlining the following proof that this is true.

Taking β as a complex variable let us integrate the above integral round a semi-circular contour with centre at the origin and lying above the real axis.



The function has no singularities within the contour, thus

$$\int_{-\infty}^{\infty} \phi(\beta)\, d\beta = -\lim_{|\beta|\to\infty} \int_{AOB} \phi(\beta)\, d\beta.$$

Let
$$\beta = \lambda + i\mu.$$

Then
$$|e^{-i\beta y}| = |e^{-iy(\lambda+i\mu)}| = e^{\mu y}.$$

Now
$$\left|\left\{J_{p-\frac{1}{2}}\left(\frac{\beta}{2}\right)\right\}\right| \quad \left| \frac{\left(\frac{\beta}{2}\right)}{\sqrt{\frac{\beta}{2}}} \right.$$

$$\sim \frac{e^{\frac{\mu n}{2}}}{|\beta|^{\frac{n}{2}}}.$$

Thus
$$|\phi(\beta)| \sim \frac{e^{\mu\left(y+\frac{n}{2}\right)}}{|\beta|^{np}}.$$

Thus if $y < -\frac{n}{2}$, remembering $p = 0$,

$$|\phi(\beta)| < \frac{Ae^{-kr\sin\theta}}{r}.$$

for sufficiently large r where $\lambda = r\cos\theta$, $\mu = r\sin\theta$ and A is a constant, and thus

$$\int_{AOB} \phi(\beta)\, d\beta \to 0 \text{ as } r \to \infty,$$

therefore
$$\int_{-\infty}^{\infty} \phi(\beta)\, d\beta = 0.$$

If we take a similar contour below the real axis

$$|e^{-i\beta y}| = e^{\mu y}$$

as before, but

$$J_{p-\frac{1}{2}}\left(\frac{\beta}{2}\right) \qquad \frac{e^{-\frac{\mu n}{2}}}{|\beta|^{n}},$$

Now $\quad J_n(z) = \dfrac{z^n}{2^n\,\Gamma(n+1)}\left\{1 - \dfrac{(z/2)^2}{1!\,(n+1)} + \dfrac{(z/2)^4}{2!\,(n+1)(n+2)} - \dots\right\}.$

Thus $\quad J_{p-\frac{1}{2}}(z) = \dfrac{z^{p-\frac{1}{2}}}{2^{p-\frac{1}{2}}\,\Gamma(p+\frac{1}{2})}\left\{1 - \dfrac{(z/2)^2}{1!\,(p+\frac{1}{2})} + \dfrac{(z/2)^4}{2!\,(p+\frac{1}{2})(p+\frac{3}{2})} - \dots\right\},$

therefore $\quad \dfrac{\left\{J_{p-\frac{1}{2}}\left(\frac{\beta}{2}\right)\right\}^n}{\beta^{n(p-\frac{1}{2})}} = \text{Const.} \times (\text{Series in even powers of } \beta).$

Accordingly $\quad \displaystyle\int_{-\infty}^{\infty} \sin\left(x - \frac{n}{2}\right)\beta\, \frac{\left\{J_{p-\frac{1}{2}}\left(\frac{\beta}{2}\right)\right\}^n}{\beta^{n(p-\frac{1}{2})}}\, d\beta = 0,$

since the integrand is an odd function of β.

We now have

$$\psi(x) = \tfrac{1}{2}(\sqrt{\pi})^{n-2}\{\Gamma(p)\}^n \int_{-\infty}^{\infty} \frac{\left\{J_{p-\frac{1}{2}}\left(\frac{\beta}{2}\right)\right\}^n}{\beta^{n(p-\frac{1}{2})}} \cos\left[\left(x - \frac{n}{2}\right)\beta\right] d\beta \;\dots\dots(11).$$

On transferring the origin to the mean of the distribution of totals, i.e. putting $x = \frac{n}{2} + x'$, $\psi(x) = w(x')$, we obtain the simpler form

$$w(x') = \tfrac{1}{2}(\sqrt{\pi})^{n-2}\{\Gamma(p)\}^n \int_{-\infty}^{\infty} \frac{\left\{J_{p-\frac{1}{2}}\left(\frac{\beta}{2}\right)\right\}^n}{\beta^{n(p-\frac{1}{2})}} \cos(x'\beta)\, d\beta \;\dots\dots(12).$$

In particular when $p = 1$ in virtue of the relation

$$J_{\frac{1}{2}}(z) = z^{\frac{1}{2}}\sqrt{\frac{2}{\pi}}\,\frac{\sin z}{z},$$

we have $$w(x') = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{\sin\frac{\beta}{2}}{\frac{\beta}{2}}\right)^n \cos(x'\beta)\, d\beta \;\dots\dots\dots\dots\dots\dots(13),$$

giving the distribution of totals when in the original population all values between $x = 0$ and $x = 1$ are equally likely.

and $$|\phi(\beta)| \sim \frac{e^{\mu\left(y - \frac{n}{2}\right)}}{|\beta|^{np}}.$$

Thus if $$y > \frac{n}{2}\; |\phi(\beta)| < \frac{A e^{kr\sin\theta}}{r},$$

for sufficiently large r, and remembering that $\sin\theta$ is negative

$$\int_{ABO'} \phi(\beta)\, d\beta \to 0 \text{ as } r \to$$

and $$\int_{-\infty}^{\infty} \phi(\beta)\, d\beta = 0$$

as before.

The relation (13) might have been proved *ab initio* as follows:

If $p=1$, $f(x)=1$ and the moment array of the original population about the mean is

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} e^{\alpha x}\, dx = \frac{\sinh \frac{\alpha}{2}}{\alpha}.$$

Thus
$$F(\alpha) = \left(\frac{\sinh \frac{\alpha}{2}}{\frac{\alpha}{2}}\right)^n$$

and using the corresponding relation to (5) with the mean as origin we obtain

$$w(x') = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\beta x}\left(\frac{i \sin \frac{\beta}{2}}{\frac{i\beta}{2}}\right)^n d\beta$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\beta x'}\left(\frac{\sin \frac{\beta}{2}}{\frac{\beta}{2}}\right)^n d\beta,$$

and remembering $\frac{\sin\theta}{\theta}$ is an even function of θ we have

$$\int_{-\infty}^{\infty} \sin \beta x' \left(\frac{\sin \frac{\beta}{2}}{\frac{\beta}{2}}\right)^n d\beta = 0,$$

therefore
$$w(x') = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left(\frac{\sin \frac{\beta}{2}}{\frac{\beta}{2}}\right)^n \cos(x'\beta)\, d\beta \quad \text{............(13 *bis*).}$$

The agreement of (13) and (13 *bis*) gives a check on the previous work.

The distribution of means is now easily obtained, writing $x' = n\bar{x}$ we have

$$w(x')\, dx' = \frac{n}{2}(\sqrt{\pi})^{n-2}\{\Gamma(p)\}^n\, d\bar{x}\int_{-\infty}^{\infty} \frac{\left\{J_{p-\frac{1}{2}}\left(\frac{\beta}{2}\right)\right\}^n}{\beta^{n(p-\frac{1}{2})}} \cos(n\bar{x}\beta)\, d\beta,$$

so that the distribution of means is given by

$$y = \frac{n}{2}(\sqrt{\pi})^{n-2}\{\Gamma(p)\}^n\int_{-\infty}^{\infty} \frac{\left\{J_{p-\frac{1}{2}}\left(\frac{\beta}{2}\right)\right\}^n}{\beta^{n(p-\frac{1}{2})}} \cos(n\bar{x}\beta)\, d\beta \quad \text{............(14),}$$

and in particular when $p=1$

$$y = \frac{n}{2\pi}\int_{-\infty}^{\infty} \left(\frac{\sin \frac{\beta}{2}}{\frac{\beta}{2}}\right)^n \cos(n\bar{x}\beta)\, d\beta \quad \text{....................(15).}$$

We will now consider this special case in some detail.

The integral (13) admits of exact determination. On writing $\frac{\beta}{2}=\theta$ (13) becomes

$$w(x')=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin^n\theta}{\theta^n}\cos(2x'\theta)\,d\theta \quad\text{.....................(16).}$$

There are two cases to be considered: (i) when n is even, (ii) when n is odd.

(i) Let n be even $=2r$ say.

Now by writing $\cos\theta+i\sin\theta=t$, we have the identity

$$(2i\sin\theta)^{2r}=\left(t-\frac{1}{t}\right)^{2r},$$

from which it is easily shown that

$$\sin^{2r}\theta=\frac{(-1)^r}{2^{2r-1}}\{\cos 2r\theta-{}^{2r}C_1\cos(2r-2)\theta+\ldots+(-1)^s\,{}^{2r}C_s\cos 2(r-s)\theta$$
$$+\ldots+(-1)^{r-1}\,{}^{2r}C_{r-1}\cos 2\theta+\tfrac{1}{2}(r)^r\,{}^{2r}C_r\}$$

Now $\qquad \cos 2s\theta\cos 2x\theta=\tfrac{1}{2}\{\cos 2(s+x)\theta+\cos 2(s-x)\theta\}.$

Thus

$$\sin^{2r}\theta\cos 2x\theta=\frac{(-1)^r}{2^{2r-1}}\sum_0^{r-1}\tfrac{1}{2}\,{}^{2r}C_s[\cos 2(r-s+x)\theta$$
$$+\cos 2(r-s-x)\theta]+\frac{1}{2^r}\,{}^{2r}C_r\cos 2x\theta,$$

and in order to integrate (16) we shall require

$$\int_{-\infty}^{\infty}\frac{\cos 2k\theta}{\theta^n}d\theta=\left[\frac{-\cos 2k\theta}{(n-1)\theta^{n-1}}\right]_{-\infty}^{\infty}-\int_{-\infty}^{\infty}\frac{2k\sin 2k\theta}{(n-1)\theta^{n-1}}d\theta$$
$$=\frac{-2k}{(n-1)}\int_{-\infty}^{\infty}\frac{\sin 2k\theta}{\theta^{n-1}}d\theta=\frac{-2k}{(n-1)}\left\{\left[\frac{-\sin 2k\theta}{(n-2)\theta^{n-2}}\right]_{-\infty}^{\infty}+\int_{-\infty}^{\infty}\frac{2k\cos 2k\theta}{(n-2)\theta^{n-2}}d\theta\right\}$$
$$=\frac{-2k}{(n-1)}\int_{-}^{\infty}\frac{2k\cos 2k\theta}{(n-2)\theta^{n-2}}d\theta$$
$$-\frac{-(2k)^2}{(n-1)(n-2)}\int_{-\infty}^{\alpha}\frac{\cos 2k\theta}{\theta^{n-2}}d\theta.$$

Thus $\qquad \displaystyle\int_{-\infty}^{\infty}\frac{\cos 2k\theta\,d\theta}{\theta^{2r}}=\frac{(-1)^{r-1}(2k)^{2r-2}}{(2r-1)(2r-2)\ldots 2}\int_{-\infty}^{\infty}\frac{\cos 2k\theta}{\theta^2}d\theta,$

and $\qquad \displaystyle\int_{-\infty}^{\infty}\frac{\cos 2k\theta\,d\theta}{\theta^2}=\left[-\frac{\cos 2k\theta}{\theta}-\int\frac{2k\sin 2k\theta}{\theta}d\theta\right]_{-\infty}^{\infty}$

$$=-\int_{-\infty}^{\infty}\frac{2k\sin 2k\theta}{\theta}d\theta.$$

Now $$\int_{-\infty}^{\infty} \frac{\sin 2k\theta}{\theta}\,d\theta = \pi \text{ if } 2k > 0, -\pi \text{ if } 2k < 0,$$

therefore $$\int_{-\infty}^{\infty} \frac{\cos 2k\theta\, d\theta}{\theta^{2r}} = \frac{(-1)^r (2k)^{2r-1}}{(2r-1)!}\pi \text{ if } 2k > 0,$$

and $$= \frac{(-1)^{r-1}(2k)^{2r-1}}{(2r-1)!}\pi \text{ if } 2k < 0.$$

Thus $$\int_{-\infty}^{\infty} \sin^{2r}\theta \cos 2x\theta \frac{d\theta}{\theta^{2r}}$$

$$= \frac{(-1)^r}{2^{2r}} \int_{-\infty}^{\infty} \left[\sum_0^{r-1} (-1)^s\, {}^{2r}C_s \{\cos 2(r-s+x)\theta + \cos 2(r-s-x)\theta\} + (-1)^r\, {}^{2r}C_r \cos 2x\theta\right] \frac{d\theta}{\theta^r}$$

$$= \frac{1}{2^{2r}} \left[\sum_0^{r-1} (-1)^s\, {}^{2r}C_s \left\{\pm \frac{[2(r-s+x)]^{2r-1}}{(2r-1)!} \pm \frac{[2(r-s-x)]^{2r-1}}{(2r-1)!}\right\} + (-1)^r\, {}^{2r}C_r \left[\pm \frac{(2x)^{2r-1}}{(2r-1)!}\right]\right]\pi$$

$$= \frac{\pi/2}{2r-1!} \left[\sum_0^{r-1} (-1)^s\, {}^{2r}C_s \{\pm (r-s+x)^{2r-1} \pm (r-s-x)^{2r-1}\} + (-1)^r\, {}^{2r}C_r [\pm x^{2r-1}]\right] \quad\text{..............(17)},$$

where the positive sign is to be taken in front of a bracket if the expression within the bracket is itself positive and the negative sign in the contrary case.

Thus if n is even

$$\int_{-\infty}^{\infty} \frac{\sin^n\theta}{\theta^n} \cos 2x'\theta\, d\theta = \frac{\pi/2}{(n-1)!} \left[\sum_0^{\frac{n}{2}-1} (-1)^s\, {}^nC_s \left\{\pm \left(\frac{n}{2}-s+x'\right)^{n-1} \pm \left(\frac{n}{2}-s-x'\right)^{n-1}\right\} + (-1)^{n/2}\, {}^nC_{n/2} [\pm x']^{n-1}\right],$$

with the above rule with regard to signs.

(ii) Let n be odd $= (2r-1)$ say.

Then it is easily shown that

$$\sin^{2r-1}\theta = \frac{(-1)^{r-1}}{2^{2r-2}} \{\sin(2r-1)\theta - {}^{2r-1}C_1 \sin(2r-3)\theta + \ldots + (-1)^s\, {}^{2r-1}C_s \sin(2r-2s-1)\theta + \ldots + (-1)^{r-1}\, {}^{2r-1}C_{r-1} \sin\theta\},$$

and

$$\sin(2r-2s-1)\theta \cos 2x\theta = \tfrac{1}{2}\{\sin[2(r-s+x)-1]\theta + \sin[2(r-s-x)-1]\theta\},$$

and in order to perform the integration we shall require

$$\int_{-\infty}^{\infty} \frac{\sin(2k-1)\theta\, d\theta}{\theta^{2r-1}}.$$

By the same method as employed above it is now easily shown that

$$\int_{-\infty}^{\infty} \frac{\sin (2k-1)\,\theta\, d\theta}{\theta^{2r-1}} = \frac{(-1)^{r-1}(2k-1)^{2r-2}}{(2r-2)!}\,\pi \text{ if } 2k-1>0,$$

and

$$= \frac{(-1)^{r}(2k-1)^{2r-2}}{(2r-2)!}\,\pi \text{ if } 2k-1<0;$$

therefore

$$\int_{-\infty}^{\infty} \sin^{2r-1}\theta \cos 2x\theta \frac{d\theta}{\theta^{2r-1}}$$

$$= \frac{(-1)^{r-1}}{2^{2r-1}} \int_{-\infty}^{\infty} \left[\sum_{s=0}^{r-1} (-1)^{s\,2r-1}C_s \{\sin [2(r-s+x-1)-1]\,\theta + \sin [2(r-s-x)-1]\,\theta\}\right] \frac{d\theta}{\theta^{2r-1}}$$

$$= \frac{(-1)^{r-1}\pi}{2^{2r-1}} \left[\sum_{s=0}^{r-1} (-1)^{s\,2r-1}C_s (-1)^{r-1} \left\{\pm \frac{[2(r-s+x)-1]^{2r-2}}{(2r-2)!} \pm \frac{[2(r-s-x)-1]^{2r-2}}{(2r-2)!}\right\}\right]$$

$$= \frac{\pi/2}{2r-2\,!} \left[\sum_{s=0}^{r-1} (-1)^{s\,2r-1}C_s \{\pm [r-s+x-\tfrac{1}{2}]^{2r-2} \pm [r-s-x-\tfrac{1}{2}]^{2r-2}\}\right] \quad \ldots(18),$$

where the positive sign is to be taken in front of a bracket if the expression within the bracket is itself positive and the negative sign in the contrary case.

Thus if n is odd

$$\int_{-\infty}^{\infty} \left(\frac{\sin\theta}{\theta}\right)^n \cos 2x'\theta\, d\theta = \frac{\pi/2}{(n-1)!} \left[\sum_{s=0}^{\frac{n-1}{2}} (-1)^{s\,n}C_s \left\{\pm \left(\frac{n}{2}-s+x'\right)^{n-1} \pm \left(\frac{n}{2}-s-x'\right)^{n-1}\right\}\right]$$

with the above rule with regard to signs.

If the sampled population be one in which all values of a variate between 0 and 1 are equally likely the equation (referred to the mean) of the distribution of the means of random samples of n will be given by

$$y = \frac{\frac{n}{2}}{n-1\,!} \left[\sum_{s=0}^{\frac{n-1}{2}} (-1)^{s\,n}C_s \left\{\pm \left(\frac{n}{2}-s+n\bar{x}\right)^{n-1} \pm \left(\frac{n}{2}-s-n\bar{x}\right)^{n-1}\right\}\right] + (-1)^{n/2\,n}C_{n/2} \{\pm (n\bar{x})\}^{n-1} \quad \ldots\ldots\ldots\ldots(19),$$

if n is even and

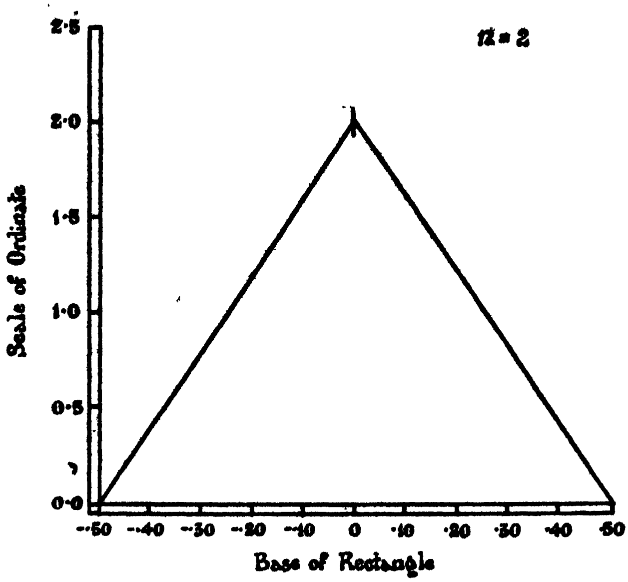
$$y = \frac{\frac{n}{2}}{n-1\,!} \left[\sum_{s=0}^{\frac{n-1}{2}} (-1)^{s\,n}C_s \left\{\pm \left(\frac{n}{2}-s+n\bar{x}\right)^{n-1} \pm \left(\frac{n}{2}-s-n\bar{x}\right)^{n-1}\right\}\right] \quad \ldots(19\ \textit{bis}),$$

if n is odd, and the ambiguous sign is to be determined as explained above. The range will be from $\bar{x} = -\tfrac{1}{2}$ to $\bar{x} = +\tfrac{1}{2}$.

On putting $n = 2, 3, 4$ in these formulae, paying attention to the above rule of signs, we obtain the following results for the frequency distribution of means in samples of 2, 3, 4 from a population in which all values of x between $-\tfrac{1}{2}$ and $+\tfrac{1}{2}$ are equally likely.

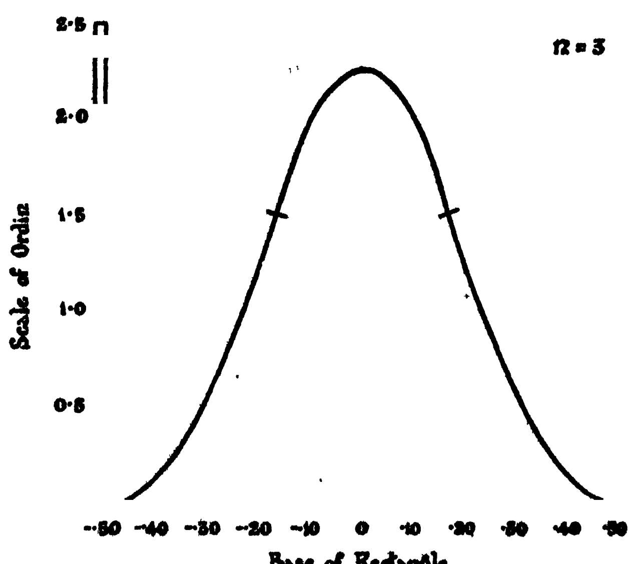
$n = 2$. $$y = 2\{1 - 2|x|\},$$
or the frequency distribution is an isosceles triangle.

FIG. I



$n = 3$. $$y = \tfrac{9}{4}(1 - 12x^2) \text{ if } -\tfrac{1}{6} < x < \tfrac{1}{6},$$
$$y = \tfrac{27}{8}(1 - 2|x|)^2 \text{ if } x > \tfrac{1}{6} \text{ or } < -\tfrac{1}{6},$$
or the frequency distribution is made up of three parabolas and has something of the appearance of a normal curve.

FIG. II

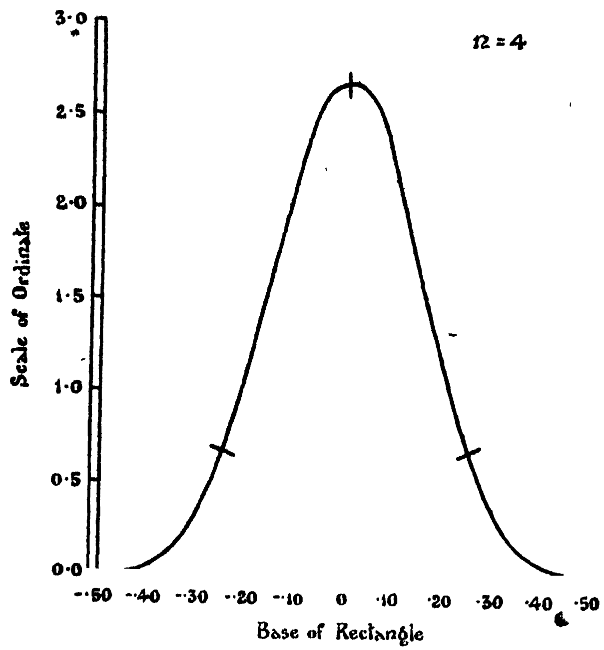


$n=4$. $\quad y=\frac{8}{3}\left[1-24x^2+48\,|x|^3\right]$ if $-\frac{1}{4}<x<\frac{1}{4}$,

$\qquad\qquad y=\frac{16}{3}\left[1-2\,|x|\right]^3$ if $x>\frac{1}{4}$ or $<-\frac{1}{4}$,

and the frequency distribution is made up of four cubic curves with a point of discontinuity at the vertex and two points where differential coefficients are discontinuous at $x=\frac{1}{4}$ and $x=-\frac{1}{4}$.

FIG. III



It is clear that when $n=5$, the distribution will be made up of five curves of the 4th degree joining at the points $x=\pm\frac{1}{10}$, $\pm\frac{3}{10}$ and in general that

(i) The distribution of means is symmetrical.

(ii) If n be even there is a point of abruptness at the vertex, but if n be odd there is not.

(iii) There are $n-1$ points of abruptness or points where differential coefficients become discontinuous.

(iv) These points of abruptness gradually become less marked as n increases and the appearance of the curves approaches rapidly to the normal. In fact when n is 10, $\beta_2=2{\cdot}88$*.

* This may be shown as follows:

The moment array of (19) about the mean is, as we have already seen, given by

$$F(\alpha)=\left(\frac{\sinh\frac{\alpha}{2}}{\frac{\alpha}{2}}\right)^n$$

$$\left\{\frac{\frac{\alpha}{2}+\frac{1}{3!}\left(\frac{\alpha}{2}\right)^3+\frac{1}{5!}\left(\frac{\alpha}{2}\right)^5}{\frac{\alpha}{2}}\right.$$

$$=1+\frac{n}{12}\frac{\alpha^2}{2!}+\frac{\alpha^4}{4!}\left(\frac{n}{80}-\frac{n}{48}+\frac{n^2}{48}\right),$$

We now return to the general case given by (14). It is known that

$$J_{-\frac{1}{2}}(z)=\left(\frac{2}{\pi z}\right)^{\frac{1}{2}}\cos z,$$

$$J_{\frac{1}{2}}(z)=\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\frac{\sin z}{z^{\frac{1}{2}}},$$

and when p is a positive integer

$$J_{p-\frac{1}{2}}(z)=\frac{(-1)^{p-1}(2z)^{p-\frac{1}{2}}}{\pi^{\frac{1}{2}}}\,\frac{d^{p-1}}{d\,(z^2)^{p-1}}\left(\frac{\sin z}{z}\right)^*.$$

Thus

$$\frac{J_{-\frac{1}{2}}\left(\frac{\beta}{2}\right)}{\beta^{-\frac{1}{2}}}=\frac{2}{\sqrt{\pi}}\cos\frac{\beta}{2},$$

$$\frac{J_{\frac{1}{2}}\left(\frac{\beta}{2}\right)}{\beta^{\frac{1}{2}}}=\frac{2}{\sqrt{\pi}}\,\frac{\sin\frac{\beta}{2}}{\beta},$$

$$\frac{J_{p-\frac{1}{2}}\left(\frac{\beta}{2}\right)}{\beta^{p-\frac{1}{2}}}=\frac{(-1)^{p-1}}{\sqrt{\pi}}\,\frac{2^{2p-2}\,d^{p-1}}{d\,(\beta^2)^{p-1}}\left(\frac{\sin\frac{\beta}{2}}{\frac{\beta}{2}}\right).$$

It is thus possible by substitution of this expression in (14) and simplifying to reduce the integrand to the sum of terms of the type $\frac{\sin k\beta}{\beta^s}$ or $\frac{\cos k\beta}{\beta^s}$ which are each integrable. Thus (14) is integrable when p is a positive integer and for low values of n and p (say up to 4 or 5) it should be quite possible to evaluate the integrals†. For higher values of n the number of terms in the simplified expression would render the process more difficult, but since when p is a positive integer the distribution of means will approach more rapidly to the normal than in the case of the rectangle treated above, we should not in practice require to take n greater than at most 10 and p than 3 or 4. This should be sufficient to give a good idea in practice of the nature of the distribution of means of small samples taken at random from the Type II curve.

therefore $\mu_2=\frac{n}{12},\ \mu_4=\frac{1}{240}(5n^2-2n).$

Whence $\beta_1=0,\ \beta_2=3-\frac{6}{5n},$

and $\beta_2\to 3$ as $n\to\infty$.

and in particular when $n=10$ $\beta_2=2{\cdot}88,$

when $n=50$ $\beta_2=2{\cdot}976.$

* Whittaker and Watson, *Modern Analysis*, 2nd Edition, p. 358.

† It is hoped that at a later date this may be done.

THE DISTRIBUTION OF MEANS FOR SAMPLES OF SIZE N DRAWN FROM A POPULATION IN WHICH THE VARIATE TAKES VALUES BETWEEN 0 AND 1, ALL SUCH VALUES BEING EQUALLY PROBABLE.

By PHILIP HALL, B.A.

A PARTICULAR sample in which the variate x takes the values $x_1, x_2, \ldots, x_N$ may be represented by the point $P=(x_1, x_2, \ldots, x_N)$ of N-dimensional space. The representative points of a large number of samples will tend to be evenly distributed throughout the interior of the unit hypercube

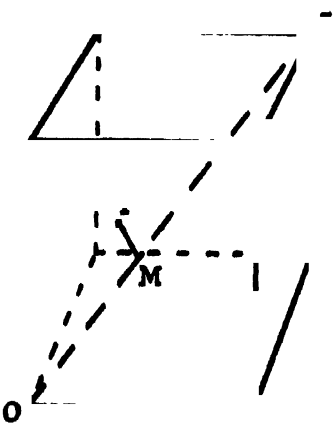
$$0 \leqslant x_i \leqslant 1, \quad (i=1, 2, \ldots, N) \quad \ldots\ldots(1).$$

Let O be the origin and I the point $(1, 1, \ldots, 1)$. Then if M is the foot of the perpendicular from P on to OI, we have

$$\text{Mean of sample} = m = \frac{OM}{OI} \quad \ldots\ldots(2),$$

and

$$\text{Standard deviation of sample} = \sigma = \frac{MP}{OI}.$$



We shall write $\quad p=Nm \quad \ldots\ldots(3).$

Then the points whose mean is equal to m all lie in the region cut off by (1) on the hyperplane

$$\sum_{i=1}^{N} x_i = p \quad \ldots\ldots(4),$$

and conversely. Hence in order to find the frequency distribution of m it is only necessary to calculate the $N-1$-dimensional content of this region.

With this object, we consider the "quadrants"

$$\left.\begin{array}{l} x_i \geqslant r_i, \\ r_i = 0 \text{ or } 1, \end{array}\right. \quad (i=1, 2, \ldots, N) \quad \ldots\ldots(5),$$

whose corners are the corners of (1). These we divide into $N+1$ sets according to the value of

$$r = \sum_{i=1}^{N} r_i \quad \ldots\ldots(6),$$

so that $r=0, 1, 2, \ldots, N$. A quadrant of the ith set will be called a Q_i. Thus the number of Q_i's is $\binom{N}{i}$.

Let S be any point of Q_0, i.e. any point whose coordinates are all $\geqslant 0$, and let just s of its coordinates be $\geqslant 1$. Then S will belong to just $\binom{s}{1}$ Q_1's, $\binom{s}{2}$ Q_2's, ... and $\binom{s}{s} = 1 Q_s$. Now, if $s > 0$, we have

$$\sum_{r=0}^{s} (-1)^r \binom{s}{r} = 0 \qquad \text{.............................(7).}$$

Hence if whenever a point of (4) belongs to a Q_r we give it a "density" $(-1)^r$ and then sum over all Q, the resultant density will be equal to 1 or 0 according as the point in question belongs to (1) or not. Let the segment of (4) lying in Q_0 have the content $V_N(p)$. Then the segment of (4) lying in (5) will have the content $V_N(p-r)$, (which is 0 when $r \geqslant p$). And the segment of (4) lying in (1) will have the content

$$\sum_{r=0}^{\kappa} (-1)^r \binom{N}{r} V_N(p-r) \qquad \text{..........................(8),}$$

where $\kappa = [p]$ is the greatest integer less than p.

It remains to find $V_N(p)$. Let $V_{N-1}(p)$ be the content of the projection of $V_N(p)$ perpendicular to one of the axes, so that

$$V_N(p) = \sqrt{N} \times \overline{V}_{N-1}(p).$$

Now $\overline{V}_N(p)$ is the content of the N-dimensional region bounded by (4) and the coordinate hyperplanes, a region whose base is therefore of content $V_N(p)$. The perpendicular from O on to this base is equal to $p/\sqrt{N}$. Hence

$$\overline{V}_N(p) = \frac{1}{N} \times \frac{p}{\sqrt{N}} \times V_N(p) \qquad \text{..........................(9).}$$

Since $V_2(p) = \sqrt{2}p$, we get

$$V_N(p) = \frac{\sqrt{N}}{(N-1)!} p^{N-1} \qquad \text{..........................(10).}$$

Substituting in (8), we find for the content of the region common to (1) and (4) the value

$$f(p) = \frac{\sqrt{N}}{(N-1)!} \sum_{r=0}^{\kappa} (-1)^r \binom{N}{r} (p-r)^{N-1} \qquad \text{..............(11),}$$

for values of p between κ and $\kappa+1$. To find the actual frequency distribution of m, we note that $\int_0^N f(p)\, d\left(\frac{p}{\sqrt{N}}\right) =$ the volume of the hypercube $= 1$ and hence $\int_0^1 f(Nm)\, dm = \frac{1}{\sqrt{N}}$. The distribution $F(m)$ of m is, therefore, given by

$$\left.\begin{aligned} F(m) &= \frac{N^N}{(N-1)!} \sum_{r=0}^{\kappa} (-1)^r \binom{N}{r} \left(m - \frac{r}{N}\right)^{N-1} \\ &\frac{\kappa}{N} \leqslant m \leqslant \frac{\kappa+1}{N} \end{aligned}\right\} \qquad \text{(12).}$$

Naturally $y = F(m)$ is symmetrical about $m = \frac{1}{2}$. It consists of N arcs of degree $N-1$ having $N-1$-point contact at their joins, viz. at the points

$$\frac{\kappa}{N} \, (\kappa = 1, 2, \ldots, N-1).$$

The Moments.

These are defined by

$$\bar{M}_a = \int_0^1 (m - \tfrac{1}{2})^a F(m)\, dm \quad \text{.........................(13)}$$

Owing to the symmetry of F, we have

$$\bar{M}_{2a+1} = 0, \quad (a = 0, 1, \ldots) \quad \text{.........................(14)},$$

while
$$\bar{M}_{2a} = \frac{2N^N}{(N-1)!} \sum_{r=0}^{\left[\frac{N}{2}\right]} (-1)^r \binom{N}{r} \int_{\frac{r}{N}}^{\frac{1}{2}} (m - \tfrac{1}{2})^{2a} \left(m - \frac{r}{N}\right)^{N-1} dm$$

$$= \frac{2N^N}{(N-1)!} \sum_{r=0}^{\left[\frac{N}{2}\right]} (-1)^r \binom{N}{r} \left(\frac{1}{2} - \frac{r}{N}\right)^{N+2a} B(2a+1, N),$$

or using Δ^N to denote the nth difference,

$$\bar{M}_{2a} = \frac{B(2a+1, N)}{(N-1)!\, N^{2a}} \sum_{r=0}^{N} (-1)^r \binom{N}{r} \left(\frac{N}{2} - r\right)^{N+2a} \quad \text{...........(15)},$$

$$\bar{M}_{2a} = \frac{(2a)!}{(N+2a)!\, N^{2a}} \Delta^N_{x=-\frac{N}{2}} (x^{N+2a}) \quad \text{..........................(15)}',$$

or, in terms of the generalised polynomials of Bernoulli* defined by

$$B_\nu^{(-N)}(x) = \frac{\nu!}{(N+\nu)!} \Delta^N x^{N+\nu} \quad \text{......................(16)},$$

$$\bar{M}_{2a} = \frac{1}{N^{2a}} B_{2a}^{(-N)}\left(-\frac{N}{2}\right) \quad \text{......................(17)}.$$

These moments can, however, be obtained in a different form and by an entirely different method.

We consider an *arbitrary* frequency distribution $f(x)$ of a variate x and write

$$\mu_a = \int_{-\infty}^{+\infty} x^a f(x)\, dx \quad \text{............................(18)}.$$

We may suppose $\mu_0 = 1$ and $\mu_1 = 0$. Then the distribution of means $m = \frac{1}{N}\sum_{i=1}^{N} x_i$ for samples of size N will be given by

$$F(m) = A \int f_1 f_2 \ldots f_N\, d\tau \quad \text{.........................(19)},$$

where $f_i = f(x_i)$ and $d\tau$ is the $N-1$-dimensional volume-element in the hyperplane (4), the integral being taken completely over (4). Let us denote by M_a the moments of this distribution. Then

$$M_0 = \int_{-\infty}^{+\infty} F(m)\, dm = 1,$$

if $A = \sqrt{N}$, which we accordingly suppose, and

$$M_a = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} m^a f_1 f_2 \ldots f_N\, dx_1 \ldots dx_N,$$

i.e.
$$M_a = \frac{1}{N^a} \sum \frac{a!}{s_1!\, s_2! \ldots s_N!} \mu_{s_1} \mu_{s_2} \ldots \mu_{s_N} \quad \text{......................(20)},$$

* Not to be confused with the B of the complete Beta-function in formula (15).

the summation being taken over all integer solutions of

$$\left.\begin{aligned}\sum_{i=1}^{N}{}^{*}\, s_i = \alpha\\ s_i \geqslant 0, \quad (i=1,2,\ldots,N)\end{aligned}\right\} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(21).$$

Since $\mu_1 = 0$, we may also add the condition $s_i \neq 0$. Now in our case

$$\left.\begin{aligned}\mu_{2s-1} &= 0 \quad (s=1,2,\ldots)\\ \mu_{2s} &= \int_0^1 (m-\tfrac{1}{2})^{2s}\,dm = \frac{1}{2s+1}(\tfrac{1}{2})^{2s}\end{aligned}\right\} \quad\ldots\ldots\ldots\ldots\ldots\ldots(22).$$

Hence, we get

$$\bar{M}_{2\alpha} = \frac{1}{(2N)^{2\alpha}} \Sigma \frac{(2\alpha)!}{(2s_1+1)!\,(2s_2+1)!\ldots(2s_N+1)!} \quad\ldots\ldots\ldots\ldots(23),$$

taken as before over all integer solutions of (21)*.

We return now to the general formula (20) true for any frequency distribution. To express M_α as a polynomial in the μ, let us take any partition of α into N non-negative summands, and let there be a_0 0's, a_1 1's, a_2 2's and so on, so that

$$\Sigma a_i = N, \quad \Sigma i a_i = \alpha, \quad a_i \geqslant 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(24).$$

The number of terms of (20) corresponding to this partition will be

$$\frac{N!}{a_0!\,a_1!\ldots}.$$

Hence we have

$$M_\alpha = \frac{1}{N^\alpha} \Sigma \frac{N!}{a_0!\,a_1!\ldots} \frac{\alpha!}{(2!)^{a_2}(3!)^{a_3}\ldots} \mu_0^{a_0}\mu_1^{a_1}\mu_2^{a_2}\ldots \quad\ldots\ldots\ldots\ldots(25),$$

the summation being now extended to all possible partitions of (24). For example, the first few cases may be written down (remembering $\mu_1 = 0$) as follows:

$$\left.\begin{aligned}M_1 &= 0\\ M_2 &= \frac{\mu_2}{N}\\ M_3 &= \frac{\mu_3}{N^2}\\ M_4 &= \frac{1}{N^3}[\mu_4 + 3(N-1)\mu_2^2]\\ M_5 &= \frac{1}{N^4}[\mu_5 + 10(N-1)\mu_3\mu_2]\\ M_6 &= \frac{1}{N^5}[\mu_6 + 15(N-1)\mu_4\mu_2 + 10(N-1)\mu_3^2 + 15(N-1)(N-2)\mu_2^3]\end{aligned}\right\}\ldots(26).$$

* That (17) and (23) are consistent may be seen, e.g. by consulting Nörlund's *Differenzenrechnung* and comparing the sixth equation of p. 139 with the last equation of the same section, on p. 140. The actual values of the $\bar{M}_{2\alpha}$ for the first few values of α may be read off from Nörlund's table 6, p. 460, remembering that in his notation, $\bar{M}_{2\alpha} = \frac{D_{2\alpha}^{(-N)}}{(2N)^{2\alpha}}$. Thus e.g. $\bar{M}_2 = \frac{1}{12N}$, as follows also from (26) and (22).

Introducing the β's by the formulæ

$$\beta_{2s-2} = \frac{\mu_{2s}}{\mu_2^s}, \quad \beta_{2s-1} = \frac{\mu_{2s+1}\mu_3}{\mu_2^{s+2}} \quad \text{......................(27)},$$

and similarly for the B's in terms of the M's, we have

$$\left.\begin{aligned} B_1 &= \frac{\beta_1}{N} \\ B_2 - 3 &= \frac{\beta_2 - 3}{N} \\ B_3 - 10B_1 &= \frac{\beta_3 - 10\beta_1}{N^2} \\ B_4 - 15B_2 - 10B_1 + 30 &= \frac{\beta_4 - 15\beta_2 - 10\beta_1 + 30}{N^2} \\ &\cdots\cdots\cdots\cdots \end{aligned}\right\} \quad \text{...........(28)}.$$

As $N \to \infty$, the dominant term in (25) is that which corresponds to a partition having the smallest possible value for a_0. This is the partition (2 2 ...) if α is even and (3 2 2 ...) if α is odd. Hence, we have as $N \to \infty$

$$M_{2\alpha} \sim \frac{1}{N^{2\alpha}} \frac{(2\alpha)!}{(2!)^\alpha} \frac{N!}{\alpha!(N-\alpha)!} \mu_2^\alpha,$$

or $$M_{2\alpha} \sim (2\alpha - 1)(2\alpha - 3) \dots 5 . 3 \left(\frac{\mu_2}{N}\right)^\alpha = \prod_{r=1}^{\alpha} \{(2r-1) M_2\} \quad \text{......(29)},$$

while $$M_{2\alpha+1} \sim \frac{1}{N^{2\alpha+1}} \frac{(2\alpha+1)!}{(2!)^{\alpha-1} 3!} \frac{N!}{(N-\alpha)!(\alpha-1)!} \mu_2^{\alpha-1}\mu_3,$$

or $$M_{2\alpha+1} \sim \tfrac{1}{3}\alpha(2\alpha+1)(2\alpha-1)(2\alpha-3)\dots 5 . 3 \frac{\mu_2^{\alpha-1}\mu_3}{N^{\alpha+1}} \quad \text{..............(30)}.$$

Whence $$B_{2\alpha-2} \sim (2\alpha-1)(2\alpha-3)\dots 7 . 5 . 3 \quad \text{..................................(31)},$$

$$B_{2\alpha-3} \sim \tfrac{1}{3}(\alpha-1)(2\alpha-1)(2\alpha-3)\dots 7 . 5 . 3 \frac{\beta_1}{N} = \tfrac{1}{3}(\alpha-1)B_1 B_{2\alpha-2} \quad \text{...(32)}.$$

If we change our unit of length in the ratio $1 : \sqrt{N}$*, we see at once from these equations that the new moments tend to normal-curve values as $N \to \infty$.

EDITORIAL NOTE.

Proceeding by completely different methods, Irwin and Hall have obtained equations for the frequency distribution of means in samples from a population following a "rectangular" law of distribution (p. 236, (19) and p. 241, (12)). Their results are of considerable interest, not only in themselves, but for the light they throw upon the distribution of moments in samples from populations in which the variable lies within a finite or limited range.

Hall makes use of a conception which has been employed in connection with the sampling distribution of means and standard deviations from a Normal population †,

* So as to maintain a constant standard deviation as N increases.

† R. A. Fisher, *Biometrika*, Vol. x. p. 507.

representing the sample of N by a point $(x_1, x_2, \ldots, x_N)$ in an N-dimensional density space. For a population of limited range, obeying a law $y = f(x)$, we have, choosing suitable units, $0 \leqslant x \leqslant 1$, so that the appropriate density field will be the unit hypercube, with point density at $(x_1, x_2, \ldots, x_N)$ of $f(x_1) f(x_2) \ldots f(x_N)$. In the case of the "rectangular" population for which $f(x)$ is constant, the density is uniform throughout the hypercube, but for other forms of distribution, as for example that of the general Type I curve, this will not be so. The frequency distribution of means, m, in samples of N, will still however be obtained by integrating the density throughout the $N-1$-dimensioned region within the hypercube lying at right angles to the diagonal axis $x_1 = x_2 = \ldots = x_N$. Just as for Hall's simple case, it would appear that the law representing the content of this region will change its form at the $N-1$ points $m = \kappa/N$ $(\kappa = 1, 2, \ldots, N-1)$ so that the resulting distribution of means will consist of N connected arcs.

In the simple case of $N = 3$, the position can be grasped visually from Hall's diagram (p. 240). The frequency of m is here proportional to the integral of a density $f(x_1) f(x_2) f(x_3)$ over the plane perpendicular to the diagonal OI. This plane is represented as cutting OI in M where $OM/OI = m$. As M moves away from O the law of distribution remains of the same form until $OM = \frac{1}{3} OI$ when the section of the plane by the cube changes from a triangle to a hexagon. The new law holds until $OM = \frac{2}{3} OI$, when the section changes back from a hexagon to a triangle, and the third law holds until M reaches I and $m = 1$. For the special case of the "rectangular" population, Irwin deduces these three laws from the general equations for $N = 2$, 3 and 4 on pp. 237—238.

It seems probable that the solution outlined by Irwin at the end of his paper, for Type II curves, will lead also to a system of separate arcs, as the consideration of the hypercube suggests.

The case of the Type III curve is of interest as forming a transition type; one limit of range is removed to infinity, so that only one corner of the hypercube lies at a finite point. As M moves away from O along the diagonal (Hall's figure) the shape of the region lying at right angles to this line, throughout which the density is to be integrated, does not change as long as m remains finite. Consequently there is a *single* law of distribution of means, as given first by Church and later by Irwin (p. 229, (7)). This should also be true for Type VI curves (limited in one direction), as well as for Type IV where the density field will fill the whole of the N-dimensional space, as for the Normal curve.

A consideration of the hypercube also suggests that the distribution of standard deviations in samples from a limited range population will also consist of connected arcs. With the lettering of Hall's figure, the standard deviation of the sample is $\sigma = MP/OI$, and the frequency of a given value of σ will be obtained by integrating the density throughout the region for which MP is constant, which may be described as a "hypercylinder," with axis OI, and cross-section an $(N-1)$-dimensioned hypersphere. As MP increases the law representing the form of this region will change according to the number of faces of the hypercube which it cuts. E. S. P.

THE MATHEMATICS OF INTELLIGENCE.

I. THE SAMPLING ERRORS IN THE THEORY OF A GENERALISED FACTOR.

BY KARL PEARSON AND MARGARET MOUL.

CONTENTS.

(1) *Introductory.*

In a paper under the above title by Professor Spearman and Dr Holzinger in *The British Journal of Psychology** various approximations are given for the probable error of a "tetrad"† of correlation coefficients which we think require a little ampler consideration before they are adopted into psychological practice. The need for a reconsideration of the matter is the more essential because Professor Spearman writes in his recent work, *The Abilities of Man*, as if his mathematical theory was beyond impeachment and his experimental data justified his theory:

"Even with the degree of exactness attained already, however, the agreement of the observed values with those required by theory must be admitted by any unbiased person to have been surprisingly close. In general, it seems quite as good as, if not better than, that usually reached in determining the mechanical equivalent of heat and thus establishing the law of conservation of energy."

The matter is of special interest to one of the present writers, because formulae are cited for which he is largely responsible, and the exact limitation of which appears to be overlooked.

* Vol. xv. pp. 17—19, 1925; *The Abilities of Man*, p. 160.

† There is no reason why the term "tetrad" should not be used for brevity's sake instead of "tetrad difference" for the cross product of the four correlations.

(2) *Nature of Problem.*

We will suppose that ρ_{st} represents the correlation between the sth and tth variates in an indefinitely large population which is going to be sampled; that r_{st} is the corresponding correlation in any particular sample, and that $\bar{r}_{st}$ is the mean value of r_{st} for many samples. Then $\bar{r}_{st}$ is *not* equal to ρ_{st} and, unless this point is borne in mind, it is not possible to deduce the accurate value to higher approximations of the probable error of functions of the correlation coefficients. Actually we do not know the relation between $\bar{r}_{st}$ and ρ_{st} unless we make some hypothesis as to the frequency surface of the variates under discussion. The only frequency surface for which the relation between $\bar{r}_{st}$ and ρ_{st} has been fully determined is the normal surface. In this case the full expression for $\bar{r}_{st}$ in terms of ρ_{st} will be found in *Biometrika*, Vol. XI. p. 336. An approximate expression, namely,

$$\bar{r}_{st} = \rho_{st}\left[1 - \frac{1-\rho^2_{st}}{2n} + (\ldots)\frac{1}{n^2} + \ldots\right] \quad \text{.....................(i)},$$

was given earlier by Mr Soper in the same Journal, Vol. IX. p. 105, where n is the size of the sample. Now Spearman and Holzinger apparently measure their δr_{st} from ρ_{st}, the value in the entire or sampled population*, but they put the mean values of quantities like δr_{st}, which we will represent by $\{\delta r_{st}\}$, zero, or they tacitly assume that δr_{st} is measured from its mean. In doing this they are neglecting the difference between $\bar{r}_{st}$ and ρ_{st}, or terms of the order $1/n$ as compared with 1. Further, the Pearson-Filon formulae† which they cite and use are only correct for $\{\delta r_{st}\delta r_{uv}\}$ and $\{\delta r^2_{st}\}$, (i) when these differentials are measured from their means in samples, (ii) when we suppose normal distribution of the variates, and (iii) when we neglect terms in $1/n^2$ as compared with $1/n$. If we are going to proceed to terms in $1/n^2$ as well as those in $1/n$, then we cannot, as the above authors have done, put $\{\delta r_{st}\}$ zero, nor apply the Pearson-Filon formulae to determine the value of $\{\delta r_{st}\delta r_{uv}\}$. Still less can we assume, knowing that the distribution of r_{st} and r_{uv} is *not* normal, that their combined distribution will be normal, even when we retain terms of the order $1/n^2$. Accordingly we may safely say that their determination of the terms in $1/n^2$ is in error, and we need not enter into a discussion of their attempt to improve their formula involving only terms in $1/n$ by introducing others in $1/n^2$. To obtain such terms correctly we should want to know at least to a second approximation many things of which we are at present ignorant. We can therefore rest content with the statement that we must dismiss as erroneous the formula professing to give terms in $1/n^2$ which Holzinger and Spearman provide‡. But if we could find the correct term in $1/n^2$—which at present we cannot—would it be worth finding? We answer, certainly not, for our authors replace the true values of the correlations, ρ, by the observed values, r. Now the variation of r round ρ due to random sampling is of the order of its standard deviation, namely $1/\sqrt{n}$, and therefore while terms of the order $1/(n\sqrt{n})$ are neglected, it is idle to attempt the valuation of terms of the order $1/n^2$.

* *loc. cit.* pp. 17—18.

† *Phil. Trans.* Vol. 191 A, p. 262.

‡ *British Journal of Psychology*, Vol. XV. pp. 18—19.

(3) *Variability of a Tetrad.*

Bearing these points in mind let us now turn to the first approximation to the variability of a tetrad. It is needful to find the relations between the various forms of tetrad that may appear and to adopt a notation to distinguish them. In an individual sample we have the tetrad

$$_rT_{st|uv} = r_{su}r_{tv} - r_{tu}r_{sv} \quad \text{.............................(ii)},$$

the correlations being those of the sample. When we may without confusion in the text drop the subscript $_{st|uv}$, we can write this $_rT$ simply. Again

$$_\rho T_{st|uv} = \rho_{su}\rho_{tv} - \rho_{tu}\rho_{sv} \quad \text{.............................(iii)}$$

will represent the corresponding tetrad in the sampled population.

If $\bar{r}_{su}$, $\bar{r}_{tv}$, $\bar{r}_{tu}$, $\bar{r}_{sv}$ be the mean correlation coefficients in samples we shall require

$$_{\bar{r}}T_{st|uv} = \bar{r}_{su}\bar{r}_{tv} - \bar{r}_{tu}\bar{r}_{sv} \quad \text{.............................(iv)},$$

or the tetrad with the mean values of the correlation coefficients in samples.

Finally we may need the mean value of the tetrad in samples $\bar{T}_{st|uv}$, which is not the same thing as $_{\bar{r}}T_{st|uv}$. All these various values must be carefully distinguished and interrelated, if we require in the case of small samples to proceed to an approximation involving $1/n^2$ as well as $1/n$.

In order to find the standard deviation of a tetrad, it is needful to measure its variation from $\bar{T}$, its mean value in samples, and not from its value in the sampled population. In order to apply the known results for $\{\delta r^2_{st}\}$ and $\{\delta r_{st}\delta r_{uv}\}$ it is necessary to measure the variations of r_{st}, r_{uv}, etc. from their mean values in samples. Accordingly we have:

$$\begin{aligned} _rT_{st|uv} &= \bar{T}_{st|uv} + \delta T_{st|uv} = (\bar{r}_{su} + \delta r_{su})(\bar{r}_{tv} + \delta r_{tv}) - (\bar{r}_{tu} + \delta r_{tu})(\bar{r}_{sv} + \delta r_{sv}) \\ &= {_{\bar{r}}T_{st|uv}} + \bar{r}_{su}\delta r_{tv} + \bar{r}_{tv}\delta r_{su} - \bar{r}_{sv}\delta r_{tu} - \bar{r}_{tu}\delta r_{sv} + \delta r_{su}\delta r_{tv} - \delta r_{tu}\delta r_{sv} \quad \text{......(v)}. \end{aligned}$$

Now if we take the mean value of both sides,

$$\{\delta T_{st|uv}\} = 0,\ \{\delta r_{su}\} = 0,\ \{\delta r_{tv}\} = 0,\ \{\delta r_{tu}\} = 0,\ \{\delta r_{sv}\} = 0,$$

because the variations are all measured from their mean values. Accordingly

$$\bar{T}_{st|uv} = {_{\bar{r}}T_{st|uv}} + \{\delta r_{su}\delta r_{tv}\} - \{\delta r_{tu}\delta r_{sv}\} \quad \text{..................(vi)}.$$

To the latter two mean products the Pearson-Filon formulae will apply as the correlation coefficient variations are measured from their means. With regard to the first term

$$\begin{aligned} _{\bar{r}}T_{st|uv} &= \bar{r}_{su}\bar{r}_{tv} - \bar{r}_{tu}\bar{r}_{sv} \\ &= \rho_{su}\rho_{tv}\left(1 - \frac{1-\rho^2_{su}}{2n}\right)\left(1 - \frac{1-\rho^2_{tv}}{2n}\right) - \rho_{tu}\rho_{sv}\left(1 - \frac{1-\rho^2_{tu}}{2n}\right)\left(1 - \frac{1-\rho^2_{sv}}{2n}\right) \\ &= {_\rho T_{st|uv}}\left(1 - \frac{1}{n}\right) + \frac{\rho_{su}\rho_{tv}(\rho^2_{su} + \rho^2_{tv}) - \rho_{tu}\rho_{sv}(\rho^2_{tu} + \rho^2_{sv})}{2n} \quad \text{..............(vii)}, \end{aligned}$$

using Soper's formula to a first approximation and retaining only terms of order $1/n$. It will be clear accordingly that $_{\bar{r}}T_{st|uv}$ is not equal to $_\rho T_{st|uv}$, and this distinction must be preserved when we proceed to second order approximations.

We next require $\{\delta r_{su}\delta r_{tv}\} - \{\delta r_{tu}\delta r_{sv}\}$. The complete value of $\{\delta r_{su}\delta r_{tv}\}$ is unknown to us. If we neglect terms of the order $1/n^2$ as compared to $1/n$, which in practice amounts to using only large samples, then we may apply the Pearson-Filon formula, using either $\bar{r}$'s or ρ's, i.e. we can put

$$\{\delta r_{su}\delta r_{tv}\} = \frac{\begin{cases} (\rho_{st}-\rho_{su}\rho_{tu})(\rho_{uv}-\rho_{ut}\rho_{vt})+(\rho_{sv}-\rho_{st}\rho_{vt})(\rho_{ut}-\rho_{su}\rho_{st}) \\ +(\rho_{st}-\rho_{sv}\rho_{tv})(\rho_{uv}-\rho_{su}\rho_{sv})+(\rho_{sv}-\rho_{su}\rho_{vu})(\rho_{ut}-\rho_{uv}\rho_{tv}) \end{cases}}{2n}$$

$$\{\delta r_{tu}\delta r_{sv}\} = \frac{\begin{cases} (\rho_{st}-\rho_{tu}\rho_{su})(\rho_{uv}-\rho_{us}\rho_{vs})+(\rho_{tv}-\rho_{st}\rho_{vs})(\rho_{us}-\rho_{ut}\rho_{st}) \\ +(\rho_{st}-\rho_{tv}\rho_{sv})(\rho_{uv}-\rho_{tu}\rho_{tv})+(\rho_{tv}-\rho_{tu}\rho_{vu})(\rho_{us}-\rho_{uv}\rho_{sv}) \end{cases}}{2n}$$

.................(viii).

Expanding the right-hand sides, subtracting and cancelling, we find:

$$\{\delta r_{su}\delta r_{tv}\} - \{\delta r_{tu}\delta r_{sv}\} = {}_\rho T_{st|uv}(S(\rho^2)-2)/(2n) - [\rho_{su}\rho_{tv}(\rho^2_{su}+\rho^2_{tv}) - \rho_{tu}\rho_{sv}(\rho^2_{tu}+\rho^2_{sv})]/(2n),$$

where $$S(\rho^2) = \rho^2_{st}+\rho^2_{uv}+\rho^2_{su}+\rho^2_{tv}+\rho^2_{sv}+\rho^2_{tu}.$$

Hence by (vi) and (vii)

$$\overline{T}_{st|uv} = {}_\rho T_{st|uv}\left(1+\frac{S(\rho^2)-4}{2n}\right) \qquad \text{......................(ix).}$$

Accordingly we conclude that in general $\overline{T}$, ${}_\rho T$ and ${}_{\bar{r}}T$ are not equal, and that in the special case when the tetrad vanishes for the sampled population, although $\overline{T}$ will then be zero, ${}_{\bar{r}}T$ will not be zero unless we can neglect terms of the order $1/n$ as compared with unity. Thus the problem of determining the probable error of T does not reduce to finding merely the third and fourth order mean products, but to something even more fundamental, namely, to the consideration of what δT and the δr's are measured from, as well as the ascertaining of $\{\delta r^2\}$ and $\{\delta r_{st}\delta r_{uv}\}$, etc. to the $1/n^2$ order. It is not possible to put $\bar{r}_{su}\bar{r}_{tv} - \bar{r}_{tu}\bar{r}_{sv}$ zero, because $\rho_{su}\rho_{tv} - \rho_{tu}\rho_{sv}$ is zero, if one desires to retain higher terms. We need not stay to criticise the Spearman-Holzinger treatment* of the higher terms; they really assume $\rho^2_{su}+\rho^2_{tv} = \rho^2_{tu}+\rho^2_{sv}$ for every four variates, which is exceedingly unlikely to be true, nor do they realise that their standard deviation, if true, would be about the mean value in samples and not about the population value. Thus the Spearman-Holzinger equation for σ_T^2, i.e. σ_F^2 in their notation, is in error, and their attempt to allow for terms in $1/n^2$ (their $1/N^2$) fails. Let us now return to their fundamental formula which gives the standard deviation of the tetrad to a first approximation.

(4) *Standard Deviation of a Tetrad.*

Writing ω for "of order" we have from (vii)

$${}_{\bar{r}}T_{st|uv} = {}_\rho T_{st|uv}\left(1-\frac{1}{n}\right)+\omega\left(\frac{1}{n}\right),$$

and then from (ix)

$${}_{\bar{r}}T_{st|uv} = \overline{T}_{st|uv}\left(1-\frac{1}{n}-\frac{S(\rho^2)-4}{2n}\right)+\omega\left(\frac{1}{n}\right)$$

$$= \overline{T}_{st|uv}\left(1-\frac{S(\rho^2)-2}{2n}\right)+\omega\left(\frac{1}{n}\right) \qquad \text{....................(x).}$$

* *British Journal of Psychology*, Vol. XVI. pp. 87—88.

Accordingly from (v) we have, $\delta T_{st|uv}$ being measured from its true mean value for samples,

$$\delta T_{st|uv} = - T_{st|uv} \frac{S(\rho^2) - 2}{2n} + \omega\left(\frac{1}{n}\right) + \text{etc.}$$

$$= - {}_{\rho}T_{st|uv} \frac{S(\rho^2) - 2}{2n} + \omega\left(\frac{1}{n}\right) + \text{etc.},$$

or, if ${}_{\rho}T_{st|uv} = 0$ *, then we have, replacing ω,

$$\delta T_{st|uv} = \frac{\rho_{su}\rho_{tv}(\rho^2_{su} + \rho^2_{tv}) - \rho_{tu}\rho_{sv}(\rho^2_{tu} + \rho^2_{sv})}{2n}$$

$$+ \bar{r}_{su}\delta r_{tv} + \bar{r}_{tv}\delta r_{su} - \bar{r}_{sv}\delta r_{tu} - \bar{r}_{tu}\delta r_{sv}$$

$$+ \delta r_{su}\delta r_{tv} - \delta r_{tu}\delta r_{sv} \quad \text{.......(xi).}$$

Now when we square δT and take its mean value, the linear terms, whatever the order of approximation we go to, will always vanish, being measured from their means, but no other terms disappear absolutely. Writing

$$\delta T = A + B + C,$$

where A, B, C correspond to the three lines of (xi), we have:

$$\sigma_T{}^2 = \{A^2\} + 2\{AB\} + 2\{AC\} + \{B^2\} + 2\{BC\} + \{C^2\} \quad \text{.........(xii),}$$

and we note $\{A^2\}$ is of order $1/n^2$, $\{AB\}$ vanishes, $\{AC\}$ is of order $1/n^2$. Thus it follows that A must be retained if we wish to go to order $1/n^2$, but may be dropped if we only wish to go to order $1/n$. Now consider $\{B^2\}$; this involves terms of the forms $\bar{r}^2_{su}\{\delta r^2_{tv}\}$ and $\bar{r}_{su}\bar{r}_{tv}\{\delta r_{tv}\delta r_{su}\}$. If we wish to go only to terms in $1/n$, it will be sufficient to write $\bar{r}_{su} = \rho_{su}$, etc., $\{\delta r^2_{tv}\} = \dfrac{(1 - \rho^2_{tv})^2}{n}$ and for products like $\{\delta r_{tv}\delta r_{su}\}$, the values given by (viii). But should we wish to go to terms in $1/n^2$ we must write

$$= \rho_{su}\left(1 - \frac{1 - \rho^2_{st}}{2n}\right)$$

and for $\{\delta r^2_{tv}\}$ we must use the expansions provided in *Biometrika*, Vol. XI. pp. 335—6 (xx*bis*) and (xxv). But a new difficulty arises with regard to $\{\delta r_{tv}\delta r_{su}\}$; we know this to the order $1/n$, but as far as we are aware nobody has yet deduced it to the order $1/n^2$. Until this is done we cannot hope to reach Spearman and Holzinger's desired goal of including terms in $1/n^2$. Next we have threefold products such as $\{\delta r_{su}\delta r^2_{sv}\}$ and $\{\delta r_{su}\delta r_{tv}\delta r_{tu}\}$; these we should anticipate would be of the order $1/n^2$. If we assume the regression linear, the first mean will depend upon $\{\delta r^3_{sv}\}$, or on the β_1 of the distribution of r_{sv}. The wide divergence of the distribution of r_{sv} from a normal curve even for a sample of 25, if r_{sv} is ·5 or over, is well known†. Again some terms in the fourfold products will be of order $1/n^2$. But what their form is we do not yet know.

* If ${}_{\rho}T_{st|uv}$ be not zero, there will be an additional term in $1/n$, but this only modifies A (see below) and $\sigma_T{}^2$ will still depend on $\{B^2\}$ to a first approximation.

† *Biometrika*, Vol. XI. pp. 396—408, for values of β_1 and β_2.

Finally we would suggest that it is impossible to solve Spearman and Holzinger's problem of the expansion of σ_T^2 to terms in $1/n^2$ as well as $1/n$ in order to meet the case of small samples, until the correlation surface of correlation coefficients is known, which it is not at present. Even then we would point out two grave difficulties which arise, namely (*a*) all the results used apply only to sampling from normal populations, and the errors introduced by the grade of anormality of the distribution of many mental characters will certainly exceed anything capable of being corrected by using additional terms in $1/n^2$. But the second difficulty is still graver: (*b*) What values do Spearman and Holzinger propose to use for the population values ρ? Presumably they use those of the individual sample, which is all they have knowledge of. But, as we have already pointed out, these values differ from the true or population values by terms of the order σ_r, i.e. by terms of the order $1/\sqrt{n}$. It is therefore idle to retain terms of the order $1/n^2$, when we neglect terms of the order $1/n \times 1/\sqrt{n}$, which occur in connection with replacing ρ_{st} by r_{st} in the terms of order $1/n$. The terms in $1/n^2$ would only be of value provided we were dealing with small sampling from a population with *a priori* known values of the ρ's.

Clearly if we wish to obtain σ_T to the order $1/n$ only, it will be adequate to retain only the B^2 terms, and this is precisely what Spearman and Holzinger have done without adequately describing why they neglect the A terms, and that inadequacy causes them to overlook points necessary to establish their second result.

Remembering that if we neglect terms in $1/n$ as compared with unity, $\bar{r}_{st}$ may be replaced by ρ_{st}, we easily find for $\sigma_{T_{st|uv}}$:

$$\begin{aligned}\sigma^2_{T_{st|uv}} = 1/n\,[&\rho^2_{tv}(1-\rho^2_{su}) + \rho^2_{su}(1-\rho^2_{tv}) + \rho^2_{sv}(1-\rho^2_{tu}) + \rho^2_{tu}(1-\rho^2_{sv}) \\ &+ \rho_{tv}\rho_{su}Q_{su,tv} + \rho_{sv}\rho_{tu}Q_{tu,sv} - \rho_{tv}\rho_{sv}Q_{su,tu} \\ &- \rho_{tv}\rho_{tu}Q_{su,sv} - \rho_{su}\rho_{sv}Q_{tv,tu} - \rho_{su}\rho_{tu}Q_{tv,sv}] \quad \text{..................(xiii)},\end{aligned}$$

where $Q_{su,tv}$, etc. are the numerators in the Pearson-Filon formulae for the mean values $\{\delta r_{su}\delta r_{tv}\}$, etc. given by Equation (viii) above.

If now we substitute the ρ values for the Q's we obtain after very lengthy but straightforward algebra—which it seems unnecessary to reproduce here—and on rearrangement:

$$\begin{aligned}\sigma^2_{T_{st|uv}} = 1/n\,[&\rho^2_{su} + \rho^2_{sv} + \rho^2_{tu} + \rho^2_{tv} \\ &- 2(\rho_{st}\rho_{su}\rho_{tu} + \rho_{st}\rho_{sv}\rho_{tv} + \rho_{su}\rho_{sv}\rho_{uv} + \rho_{tv}\rho_{tu}\rho_{uv}) \\ &+ 2\rho_{st}\rho_{uv}(\rho_{su}\rho_{tv} + \rho_{sv}\rho_{tu}) \\ &+ (\rho_{su}\rho_{tv} - \rho_{tu}\rho_{sv})^2(S(\rho^2) - 4)] \quad \text{..............................(xiv)}.\end{aligned}$$

If the tetrad $T_{st|uv}$ be really zero in the sampled population the last term on the fourth line vanishes, and our result is then in entire agreement with the Spearman-Holzinger formula for the terms in $1/n$. Further, Equation (ix) shows us that while $\bar{T}_{st|uv}$—the mean about which $\sigma_{T_{st|uv}}$ is taken—is not equal to ${}_\rho T_{st|uv}$ in general, yet, in the particular case of the tetrad vanishing, $\bar{T}_{st|uv}$ does equal ${}_\rho T_{st|uv}$, because both vanish together. It thus seems to us that it is necessary to establish (ix) before we can show that $\sigma_{T_{st|uv}}$ is really taken about the zero value of the sampled

population. Actually Equations (ix) and (xiv) give the mean and variance of a tetrad whether it is or it is not zero in the sampled population*.

Now consider the four variates s, t, u, v leading to the six correlation coefficients

$$\rho_{st},\ \rho_{uv},\ \rho_{su},\ \rho_{sv},\ \rho_{tu},\ \rho_{tv}$$

connected with the tetrad $T_{st|uv}$; they are equally connected with the two further tetrads $T_{tu|vs}$ and $T_{us|vt}$. We have in fact:

$$\left.\begin{aligned} T_{st|uv} &= \rho_{su}\rho_{tv} - \rho_{tu}\rho_{sv} \\ T_{tu|vs} &= \rho_{ts}\rho_{uv} - \rho_{tv}\rho_{us} \\ T_{us|vt} &= \rho_{ut}\rho_{vs} - \rho_{uv}\rho_{st} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(xv)}.$$

We shall speak of these tetrads which arise from the four variates as a triplet of tetrads. We see accordingly that whether the tetrads do or do not vanish, if the correlation products on the right are taken in cyclical order,

$$T_{st|uv} + T_{tu|vs} + T_{us|vt} = 0,$$

or in a long series of tetrads each triplet sum if taken in proper order will vanish, or the mean tetrad of a system of tetrads will be actually zero, if the triplets are properly taken. If not taken in cyclical order, but at random without regard to sign, then the mean tetrad of a system will be very nearly zero. In other words: *The vanishing of the mean tetrad is no proof whatever of the individual tetrads being zero or small; it is merely a result of their method of construction and the inter-relation of the tetrads in triplets.*

(5) *Approximation to Mean Square Tetrad.*

Now let us consider the mean squared standard deviation of a triplet of tetrads. Clearly from (xiv) we have:

$$\begin{aligned} &\tfrac{1}{3}\left[\sigma^2_{T_{st|uv}} + \sigma^2_{T_{tu|vs}} + \sigma^2_{T_{us|vt}}\right] \\ &\quad = 1/n\,\big[\tfrac{2}{3}S(\rho^2) - 2\left(\rho_{st}\rho_{su}\rho_{tu} + \rho_{st}\rho_{tv}\rho_{vs} + \rho_{su}\rho_{uv}\rho_{vs} + \rho_{tu}\rho_{uv}\rho_{vt}\right) \\ &\qquad + \tfrac{4}{3}\left(\rho_{st}\rho_{uv}\rho_{us}\rho_{tv} + \rho_{us}\rho_{tv}\rho_{tu}\rho_{sv} + \rho_{tu}\rho_{sv}\rho_{st}\rho_{uv}\right) \\ &\qquad + \tfrac{1}{3}\left((\rho_{su}\rho_{tv} - \rho_{tu}\rho_{sv})^2 + (\rho_{ts}\rho_{uv} - \rho_{tv}\rho_{us})^2 + (\rho_{tu}\rho_{sv} - \rho_{uv}\rho_{st})^2\right) \times \left(S(\rho^2) - 4\right)\big] \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \ldots\ldots\ldots\ldots\ldots\ldots\text{(xvi)}. \end{aligned}$$

Here the last line will be zero if the tetrads vanish in the sampled population.

To obtain the squared standard deviation of all the tetrads we have to sum (xvi) for all triplets and divide by their number, $\frac{1}{8}m(m-1)(m-2)(m-3)$.

Now although there is nothing more to hinder the calculation of the expression in (xvi) for a hundred or a thousand triplets of tetrads† than the immense labour of the arithmetic, it would undoubtedly be a gain could we throw back the determination of the mean square tetrad on constants of the original distribution of

* This follows because the term ${}_\rho T_{st|uv}(S(\rho^2) - 2)$ only adds to the A or $1/n$ terms in (xi), which disappear on taking the variance to a first approximation. Of course, when ${}_\rho T_{st|uv}$ is not zero, the variance is measured round the value of ${}_\rho T_{st|uv}$ to a first approximation and not round zero.

† We have actually carried this out in the case of the 378 tetrads of Holzinger's data and the 3003 tetrads of Simpson's data.

correlation coefficients. Holzinger and Spearman have attempted this by a method which we shall generalise in this section of our discussion. We beg the reader's careful attention for this analysis, as, though it appears plausible enough, we fear that it is really erroneous. First a preliminary proposition: *To determine the sums of products of variates $X_1, X_2, \ldots X_m$ for which $X_1 + X_2 + \ldots + X_m = 0$.*

Since $X_1 + X_2 + \ldots + X_m = 0$,

$$S(X_1^2) + 2S(X_1 X_2) = 0.$$

Now let curled brackets denote, as before, the mean of the sum of a quantity contained in them. Then the above equation becomes

$$m\{X_1^2\} = -2\frac{m(m-1)}{2}\{X_1 X_2\}.$$

We shall write $\{X_1^s\}^2 = \mu_s$, and accordingly

$$\{X_1 X_2\} = -\mu_2/(m-1) \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(xvii)}.$$

Again:

$$(X_1 + X_2 + \ldots + X_m)(X_1^2 + X_2^2 + \ldots + X_m^2) = 0,$$

$$S(X_1^3) + S(X_1^2 X_2) = 0,$$

$$m\mu_3 + m(m-1)\{X_1^2 X_2\} = 0,$$

$$\{X_1^2 X_2\} = -\mu_3/(m-1) \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(xviii)}.$$

Further:

$$(X_1 + X_2 + \ldots + X_m)^3 = 0,$$

therefore:

$$S(X_1^3) + 3S(X_1^2 X_2) + 6S(X_1 X_2 X_3) = 0,$$

or

$$2S(X_1^3) = 6S(X_1 X_2 X_3),$$

$$m\mu_3 = \frac{3m(m-1)(m-2)}{6}\{X_1 X_2 X_3\},$$

$$\{X_1 X_2 X_3\} = \frac{2}{(m-1)(m-2)}\mu_3 \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(xix)},$$

which completes the third order products. Again:

$$(X_1 + X_2 + \ldots + X_m)^4 = 0,$$

$$S(X_1^4) + 4S(X_1^3 X_2) + 6S(X_1^2 X_2^2) + 12S(X_1^2 X_2 X_3) + 24S(X_1 X_2 X_3 X_4) = 0.$$

But

$$(S(X_1^2))^2 = (X_1^2 + X_2^2 + \ldots + X_m^2)^2 = S(X_1^4) + 2S(X_1^2 X_2^2),$$

or

$$m^2\mu_2^2 = m\mu_4 + \frac{2m(m-1)}{2}\{X_1^2 X_2^2\};$$

thus:

$$\{X_1^2 X_2^2\} = \frac{m\mu_2^2 - \mu_4}{m-1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(xx)}.$$

Further:

$$(X_1^2 + X_2^2 + \ldots + X_m^2)(X_1 + X_2 + \ldots + X_m)^2 = 0,$$

$$S(X_1^4) + 2S(X_1^2 X_2^2) + 2S(X_1^3 X_2) + 2S(X_1^2 X_2 X_3) = 0,$$

and

$$S(X_1^3)(X_1 + X_2 + \ldots + X_m) = 0,$$

$$S(X_1^4) + S(X_1^3 X_2) = 0.$$

Accordingly:
$$S(X_1^3 X_2) = -S(X_1^4),$$
$$S(X_1^2 X_2 X_3) = S(X_1^4) - \tfrac{1}{2}(S(X_1^2))^2,$$
and
$$24S(X_1 X_2 X_3 X_4) = 3(S(X_1^2))^2 - 6S(X_1^4).$$
Thus:
$$\{X_1^3 X_2\} = -\mu_4/(m-1) \quad\text{(xxi)},$$
$$\{X_1^2 X_2 X_3\} = -\frac{(m\mu_2^2 - 2\mu_4)}{(m-1)(m-2)} \quad\text{(xxii)},$$
$$\{X_1 X_2 X_3 X_4\} = \frac{3(m\mu_2^2 - 2\mu_4)}{(m-1)(m-2)(m-3)} \quad\text{(xxiii)}.$$

So far so good; we do not think that there is any exception to be taken to these results. Now let us endeavour to apply these results to the mean squared standard deviation of a triplet in (xvi), omitting the term involving the squared tetrads. For brevity let us write:

$$\rho_{su} = \rho_1 = \bar{\rho} + X_1,\quad \rho_{sv} = \rho_2 = \bar{\rho} + X_2,\quad \rho_{st} = \rho_3 = \bar{\rho} + X_3,$$
$$\rho_{tu} = \rho_4 = \bar{\rho} + X_4,\quad \rho_{tv} = \rho_5 = \bar{\rho} + X_5,\quad \rho_{uv} = \rho_6 = \bar{\rho} + X_6,$$

and Σ^2_{stuv} for the left-hand side. Then:

$$\begin{aligned}
\Sigma^2_{stuv} = 1/n\,[&\tfrac{2}{3}(6\bar{\rho}^2 + 2\bar{\rho}(X_1 + X_2 + X_3 + X_4 + X_5 + X_6) \\
&\qquad + X_1^2 + X_2^2 + X_3^2 + X_4^2 + X_5^2 + X_6^2) \\
&- 8\bar{\rho}^3 - 4\bar{\rho}^2(X_1 + X_2 + X_3 + X_4 + X_5 + X_6) \\
&- 2\bar{\rho}(X_1X_3 + X_3X_4 + X_4X_1 + X_3X_5 + X_3X_2 + X_2X_5 + X_1X_6 + X_6X_2 \\
&\qquad + X_2X_1 + X_4X_6 + X_6X_5 + X_5X_4) \\
&- 2(X_1X_3X_4 + X_3X_5X_2 + X_1X_6X_2 + X_4X_6X_5) \\
&+ 4\bar{\rho}^4 + \tfrac{8}{3}\bar{\rho}^3(X_1 + X_2 + X_3 + X_4 + X_5 + X_6) \\
&+ \tfrac{4}{3}\bar{\rho}^2(X_1X_3 + X_1X_6 + X_3X_5 + X_5X_6 + X_1X_4 + X_1X_2 + X_4X_5 + X_2X \\
&\qquad + X_4X_3 + X_4X_6 + X_2X_3 + X_2X_6 \\
&\qquad + 2(X_3X_6 + X_1X_5 + X_2X_4)). \\
&+ \tfrac{4}{3}\bar{\rho}(X_3X_6X_1 + X_3X_6X_5 + X_3X_1X_5 + X_6X_1X_5 + X_1X_5X_4 + X_5X_4X_2 \\
&\qquad + X_1X_5X_2 + X_1X_4X_2 + X_4X_3X_6 + X_2X_3X_6 + X_4X_2X_6 + X_4X_2X_3) \\
&+ \tfrac{4}{3}(X_3X_6X_1X_5 + X_1X_5X_4X_2 + X_4X_2X_3X_6)] \quad\text{(xxiv)}.
\end{aligned}$$

Now let us suppose, first, that we have only this triplet of tetrads and $\bar{\rho}$ is the mean of the six correlations, then $X_1 + X_2 + X_3 + X_4 + X_5 + X_6 = 0$. Also $X_1^2 + X_2^2 + X_3^2 + X_4^2 + X_5^2 + X_6^2 = \sigma_\rho^2$. But the product of the X's two at a time does not contain all the couplets: it has 12 instead of 15, the omitted couplets being X_1X_5, X_2X_4 and X_3X_6; it cannot therefore be represented by $S(X_1X_2)$. Next, the product of the X's three at a time contains only four out of 20 such products. It cannot therefore possibly be represented by $S(X_1X_2X_3)$. Turning to the fourth order terms, the coefficient of $\frac{8}{3}\bar{\rho}^3$ is $S(X_1)$ and vanishes; the coefficient of $\frac{4}{3}\bar{\rho}^2$ contains all the 15 couplets, but three of them occur twice; it cannot therefore be represented by $S(X_1X_2)$. The coefficient of $\frac{4}{3}\rho$ contains only 12 out of the 20 X triplets and again cannot be represented by $S(X_1X_2X_3)$. Lastly, there

are only three quadruplets out of the 15 requisite to form $S(X_1X_2X_3X_4)$. Further, these differences will still persist and to the same degree of inexactitude when we sum all possible triplets of tetrads. It is clear therefore that the process by which Spearman and Holzinger suppose these X products to correspond to $S(X_1X_2)$, $S(X_1X_2X_3)$ and $S(X_1X_2X_3X_4)$ is in error; and it is not easy to say what is the extent of the error. Their method is difficult to follow, but it appears to be of this nature: They take a product in a single triplet, say $X_3X_6X_1X_5$ or $(\rho_{su}-\bar{\rho})(\rho_{tv}-\bar{\rho})(\rho_{st}-\bar{\rho})(\rho_{uv}-\bar{\rho})$, and taking the mean for all tetrads suppose it to be $\{X_1X_2X_3X_4\}$, but this it clearly is not, because 12 products of type, say $(\rho_{su}-\bar{\rho})(\rho_{tu}-\bar{\rho})(\rho_{tv}-\bar{\rho})(\rho_{uv}-\bar{\rho})$, which contribute to form $\{X_1X_2X_3X_4\}$ can never appear in the mean square tetrad expansion, and we have no right to assume the mean without these terms will be the same as the mean including them. It must, we think, be clear that when 12 terms are used instead of 15, four instead of 20 and three instead of 15 to find means, it cannot be done without risk of very considerable error.

If we may assume that the sum of X_1X_3 or $X_1X_2X_4$ or $X_3X_6X_1X_5$ throughout the whole series of triplets of tetrads divided by the number of triplets is $\{X_1X_2\}$ or $\{X_1X_2X_3\}$ or $\{X_1X_2X_3X_4\}$—which it clearly is not, for it excludes values essential to the latter—we can easily arrive at an expression for the mean square tetrad in terms of the constants of the distribution of correlation coefficients. We have

$$\begin{aligned}\{\psi_T^2\} = \sigma_T^2 = 1/n\,[&4\bar{\rho}^2(1-\bar{\rho})^2 + 4\{X^2\} - 24\bar{\rho}\{X_1X_2\}\\ &- 8\{X_1X_2X_3\} + \tfrac{4}{3}\bar{\rho}^2 . 18\{X_1X_2\} + \tfrac{4}{3}\bar{\rho} . 12\{X_1X_2X_3\} + \tfrac{4}{3} . 3\{X_1X_2X_3X_4\}],\end{aligned}$$

or, using our values above for the mean products in (xvii), (xix) and (xxiii),

$$\sigma_T^2 = \frac{4}{n}\left[\bar{\rho}^2(1-\bar{\rho})^2 + \mu_2\left(1+\frac{6\bar{\rho}(1-\bar{\rho})}{m-1}\right) - \frac{4(1-2\bar{\rho})\mu_3}{(m-1)(m-2)} + \frac{3(m\mu_2^2-2\mu_4)}{(m-1)(m-2)(m-3)}\right] \quad \text{..................(xxv).}$$

There is no difficulty or excessive labour in finding $\bar{r}$, μ_2, μ_3 and μ_4 for the actual distribution of correlation coefficients, but the question arises as to how far we are justified in retaining the terms with denominators in $m-1$, $m-2$ and $m-3$, when (a) we are replacing the true $\bar{\rho}$, μ_2, μ_3, μ_4 by the observed values. We have indicated that this is very doubtful, and that the errors so introduced may be of the same order as the retained terms*. (b) We have deduced these terms by a fallacious reasoning. It is true that this reasoning is wider of the mark for the terms in which the third and fourth order products occur than for those in which the second order products occur, but it is just possible that with certain distributions they might rise into importance. However, we have not found cases in which they do†. But the case of the second order products is far more serious, arising partly from the fact that certain terms of $S(X_1X_2)$ do not occur, and partly from the fact that those that do occur are repeated a number of times, which the above form of

* See p. 247.

† We have tested on a variety of data, Holzinger's, Simpson's, Macdonell's, etc. They may affect σ_T in the sixth decimal place.

reasoning, following Holzinger and Spearman, appears to neglect. Let us consider the matter a little more closely. If p variates be measured, then the number of correlation coefficients is $m = \frac{1}{2}p(p-1)$ and the number of pairs of correlation coefficients is $\frac{1}{2}(\frac{1}{2}p(p-1))(\frac{1}{2}p(p-1)-1) = \frac{1}{8}p(p-1)(p-2)(p+1)$. This represents the number of terms in $S(X_1X_2)$. Now in any triplet of tetrads there are only 12 instead of 15 terms in the coefficient of $2\bar{\rho}$, the three omitted products being those in which the whole four variates occur as subscripts, i.e. such as $X_{st}X_{uv}$. Now there are $\frac{1}{24}p(p-1)(p-2)(p-3)$ triplets, and three such omitted products occur in each triplet; thus we have altogether $\frac{1}{8}p(p-1)(p-2)(p-3)$ omitted products. Now these products never repeat themselves and we will call their sum $\Sigma(X_1X_2)$. Of the non-omitted products, we have

$$\tfrac{1}{8}p(p-1)(p-2)(p+1) - \tfrac{1}{8}p(p-1)(p-2)(p-3) = \tfrac{1}{2}p(p-1)(p-2).$$

But the total number of such products on the right of our tetrad equation is $12 \times$ number of triplets or is $\frac{1}{2}p(p-1)(p-2)(p-3)$. Hence every one of the non-omitted products occurs $(p-3)$ times. This point appears to be overlooked in the Holzinger-Spearman argument. Now let us consider the terms involving the X_1X_2 products. They are:

$$\begin{aligned}
&-2\bar{\rho}\ ((p-3)S(X_1X_2) - (p-3)\Sigma(X_1X_2))\\
&+\tfrac{4}{3}\bar{\rho}^2((p-3)S(X_1X_2) - (p-3)\Sigma(X_1X_2) + 2\Sigma(X_1X_2))\\
&= -(2\bar{\rho} - \tfrac{4}{3}\bar{\rho}^2)(p-3)S(X_1X_2) + (2\bar{\rho}(p-3) - \tfrac{4}{3}\bar{\rho}^2(p-5))\Sigma(X_1X_2)\\
&= -(2\bar{\rho} - \tfrac{4}{3}\bar{\rho}^2)(p-3)\tfrac{1}{8}p(p-1)(p-2)(p+1)\{X_1X_2\}\\
&\qquad\qquad + (2\bar{\rho}(p-3) - \tfrac{4}{3}\bar{\rho}^2(p-5))\Sigma(X_1X_2)\\
&= (2\bar{\rho} - \tfrac{4}{3}\bar{\rho}^2)\tfrac{1}{8}(p-3)p(p-2)(p+1)\mu_2\\
&\qquad\qquad + (2\bar{\rho}(p-3) - \tfrac{4}{3}\bar{\rho}^2(p-5))\tfrac{1}{8}p(p-1)(p-2)(p-3)[X_1X_2],
\end{aligned}$$

where $[X_1X_2]$ stands for the mean value of the omitted products. Now to obtain the mean square tetrad we must divide by the number of triplets, i.e. $\frac{1}{24}p(p-1)(p-2)(p-3)$. We note in doing this that the X^2 terms are

$$\tfrac{1}{2}p(p-1) \times \tfrac{1}{2}(p-2)(p-3)$$

in number, for there are $\frac{1}{2}p(p-1)$ correlation coefficients, and any one X_{st} of them occurs in the tetrad $T_{st|uv}$ and uv may take all possible values, i.e.

$$\tfrac{1}{2}(p-2)(p-3).$$

Hence the term is

$$\frac{2}{3}\frac{p(p-1)(p-2)(p-3)}{4}\{X^2\} = \tfrac{1}{6}p(p-1)(p-2)(p-3)\mu_2.$$

Dividing by the number of triplets we have accordingly

$$\begin{aligned}
\sigma_T{}^2 = \frac{1}{n}\Big(4\bar{\rho}^2(1-\bar{\rho})^2 + 4\mu_2 + (2\bar{\rho} - \tfrac{4}{3}\bar{\rho}^2)\frac{3p+1}{m-1}\mu_2\\
+ 3(2\bar{\rho}(p-3) - \tfrac{4}{3}\bar{\rho}^2(p-5))[X_1X_2]\Big).
\end{aligned}$$

This is, we believe, correct and if the mean of the "omitted" X_1X_2 were computed, this should give a close value for the $\sigma_T{}^2$. But the labour of finding the mean of

two or three thousand products is almost as great as finding the actual value of the mean square tetrad. The only solution is to assume $[X_1 X_2]$ will not differ very widely from $\{X_1 X_2\}$, i.e. may be put $-\mu_2/(m-1)$. In this case

$$\sigma_T^2 = \frac{1}{n}\left(4\bar{p}^2(1-\bar{p})^2 + 4\mu_2 + 6\bar{p}\mu_2\left(\frac{p+1}{m-1} - \frac{p-3}{m-1}\right) - 4\bar{p}^2\mu_2\left(\frac{p+1}{m-1} - \frac{p-5}{m-1}\right)\right)$$

$$= \frac{4}{n}\left(\bar{p}^2(1-\bar{p})^2 + \mu_2\left(1 + \frac{6\bar{p}(1-\bar{p})}{m-1}\right)\right) \quad \text{..............(xxvi).}$$

This is in agreement with the value given by Holzinger and Spearman*. The practical limitations of this formula must, however, be recognised. First, we are going to replace the ρ's by the r's; secondly, the term $6\bar{p}(1-\bar{p})/(m-1)$ is only a fairly rough approximation as we have assumed that $[X_1 X_2]$ may be taken equal to $\{X_1 X_2\}$, so that the "omitted" products have the mean value $-\mu_2/(m-1)$. Now the X's of the omitted products do have the same mean and the same variance as the total X's, but they do not satisfy the relation $\Sigma(X_1 X_2) = -\frac{1}{2}\Sigma(X^2)$ because $\Sigma(X^2)$ is really $S(X^2)$, but $\Sigma(X_1 X_2)$ is very far from being $S(X_1 X_2)$†. We in fact can only tell the degree of approximation of (xxvi) to the mean square tetrad by actual calculation of the latter. This we have done in several cases which will be referred to later. The mischief is that the deviation of (xxvi) from the true mean square tetrad may be of the order of the probable error of the latter, and this adds to the difficulty of determining whether the observed mean square tetrad is within reasonable distance of the theoretical value.

(6) *Probable Error of the Mean Square Tetrad.*

One use of (xxvi) is, however, possible. It is the mean square tetrad with *some* approximation and may accordingly be used to find the probable error of the mean square tetrad. We cannot be content to write down the observed mean square tetrad and the above theoretical value, without any measure of the probable error of their difference, and say "How near they are!" if we have no measure of the variation of their difference. Thus: ·010 and ·012 may or may not be "near" each other according as the probable error of their difference is of the order ·002 or ·0002.

Taking as an adequate approximation for probable error purposes

$$\sigma_T^2 = \frac{4}{n}\left(\bar{r}^2(1-\bar{r})^2 + \mu_2\left(1 + \frac{6\bar{r}(1-\bar{r})}{m-1}\right)\right),$$

we have

$$\delta\sigma_T^2 = \frac{4}{n}\left(2\bar{p}(1-\bar{p})(1-2\bar{p}) + \frac{6(1-2\bar{p})}{m-1}\mu_2\right)\delta\bar{r} + \frac{4}{n}\left(1 + \frac{6\bar{p}(1-\bar{p})}{m-1}\right)\delta\mu_2,$$

* *British Journal of Psychology*, Vol. XVI. p. 88. Still another value is suggested by Spearman, *The Abilities of Man* (Appendix, p. xi), but we have not seen his proof of it.

† For example, in Simpson's data $S(X_1 X_2)$ contains 4095 products, while $\Sigma(X_1 X_2)$ contains only 3003. How far may the mean of the 3003 products be taken equal to that of the 4095 products? It seems impossible to say without experiment.

and accordingly

$$\sigma^2_{\sigma_{T^2}} = \frac{64}{n^2}\left(\bar{\rho}(1-\bar{\rho})(1-2\bar{\rho}) + \frac{3(1-2\bar{\rho})}{m-1}\mu_2\right)^2 \sigma_{\bar{r}}^2 + \frac{16}{n^2}\left(1 + \frac{6\bar{\rho}(1-\bar{\rho})}{m-1}\right)^2 \sigma^2_{\mu_2}$$
$$+ \frac{64}{n^2}\left(\bar{\rho}(1-\bar{\rho})(1-2\bar{\rho}) + \frac{3(1-2\bar{\rho})}{m-1}\mu_2\right)\left(1 + \frac{6\bar{\rho}(1-\bar{\rho})}{m-1}\right)\{\delta\bar{r}\,\delta\mu_2\} \quad \text{...............(xxvii).}$$

If the distribution of r's were normal and without intercorrelation, then there would be no correlation between the mean and variance of r, or $\{\delta\bar{r}\,\delta\mu_2\}$ would be zero. Hence approximately

$$\sigma_{\sigma_{T^2}} = \frac{8}{n}\left[\left(\bar{\rho}(1-\bar{\rho})(1-2\bar{\rho}) + \frac{3(1-2\bar{\rho})}{m-1}\mu_2\right)^2 \sigma_{\bar{r}}^2 + \frac{1}{4}\left(1 + \frac{6\bar{\rho}(1-\bar{\rho})}{m-1}\right)^2 \sigma^2_{\mu_2}\right]^{\frac{1}{2}} \quad \text{...............(xxviii).}$$

Under ordinary circumstances the standard deviation squared of a mean based upon m variants is μ_2/m and of its variance $(\mu_4 - \mu_2^2)/m = 2\mu_2^2/m$ approximately, and our result would be

$$\sigma_{\sigma_{T^2}} = \frac{8}{n\sqrt{m}}\left[\left(\bar{\rho} + \frac{3(1-2\bar{\rho})}{m-1}\mu_2\right)^2 \mu_2 + \frac{1}{2}\left(1 + \frac{6\bar{\rho}(1-\bar{\rho})}{m-1}\right)^2 \mu_2^2\right]^{\frac{1}{2}},$$

but this gives a value of quite the wrong order, because we have omitted to consider that the m variants on which $\bar{r}$ and μ_2 are based are correlated together. We will accordingly now endeavour to find at least approximate values for $\sigma_{\bar{r}}^2$, $\sigma^2_{\mu_2}$ and $\{\delta\bar{r}\,\delta\mu_2\}$. Let us take the last first in order to reach some measure of its magnitude and how far it may reasonably be neglected. We have

$$\delta\bar{r} = S(\delta r_s)/m, \text{ where } r_s = \rho_s + \delta r_s,$$
$$\delta\mu_2 = \frac{2}{m}S(\rho_s - \bar{\rho})\,\delta r_s - \frac{2}{m}\delta\bar{r}\,S(\rho_s - \bar{\rho}),$$

and the last summation vanishes. Accordingly

$$\delta\bar{r}\,\delta\mu_2 = \frac{2}{m^2}S(\delta r_s) \times S(\rho_s - \bar{\rho})\,\delta r_s$$
$$= \frac{2}{m^2}S(\rho_s - \bar{\rho})\,\delta r_s^2 + \frac{2}{m^2}S((\rho_s - \bar{\rho}) + (\rho_t - \bar{\rho}))(\delta r_s \delta r_t),$$

or, taking mean values,

$$\{\delta\bar{r}\,\delta\mu_2\} = \frac{2}{m^2}\left[S(\rho_s - \bar{\rho})\frac{(1-\rho_s^2)^2}{n} + S((\rho_s - \bar{\rho}) + (\rho_t - \bar{\rho}))\{\delta r_s \delta r_t\}\right]$$
$$= \frac{2}{mn}\left[-4\bar{\rho}(1-\bar{\rho}^2)\mu_2 - 2(1-3\bar{\rho}^2)\mu_3 + 4\bar{\rho}\mu_4 + \mu_5 + \frac{S((\rho_s - \bar{\rho}) + (\rho_t - \bar{\rho}))\{n\delta r_s \delta r_t\}}{m}\right] \quad \text{..................(xxix).}$$

Now it seems probable that the summation term is very small. If we give $\{n\delta r_s \delta r_t\}$ its mean value for all correlation pairs and then sum the summation vanishes. If r_s and r_t do not contain any common variate, then $\{\delta r_s \delta r_t\}$ is of the

fourth order in quantities less than unity and these are multiplied by differences of correlation coefficients which are as often positive as negative. When r_s and r_t do contain a common variate, then a portion of $\{\delta r_s \delta r_t\}$ rises to the cubic order in correlation coefficients, or, if we take $\rho_s - \bar{\rho}$ to be of the second order, to the fifth order. Again these terms of the fifth order will be in part negative and in part positive, and we consider that the summation term as well as μ_5 may with high probability be negligible. We have not thus far succeeded in reducing the summation term to a simple expression, nor do we think that its value could be computed without enormous labour from the observations*.

With the above assumptions we may write

$$\{\delta \bar{r}\, \delta \mu_2\} = \frac{2}{mn}[-4\bar{\rho}(1-\bar{\rho}^2)\mu_2 - 2(1-3\bar{\rho}^2)\mu_3 + 4\bar{\rho}\mu_4] \quad \text{.........(xxx).}$$

We will now consider $\sigma_{\bar{r}}^2$. We have, from $\bar{r} = S(r_s)/m$,

$$\delta \bar{r} = \frac{1}{m} S(\delta r_s),$$

and therefore

$$\frac{1}{m^2}\{S(\delta r_s)^2\} + \frac{2}{m^2}\{S(\delta r_s \delta r_t)\}$$
$$= \frac{1}{m^2}\left(\frac{S(1-\rho_s^2)^2}{n}\right) + \frac{2}{m^2} S\{\delta r_s \delta r_t\}$$
$$\frac{1}{mn}\left(\frac{S(1-(\overline{\rho_s - \bar{\rho}} + \bar{\rho})^2)^2}{m} + \frac{2S\{n\,\delta r_s \delta r_t\}}{m}\right.$$
$$= \frac{1}{mn}\left((1-\bar{\rho}^2)^2 - 2\mu_2(1-3\bar{\rho}^2) + 4\bar{\rho}\mu_3 + \mu_4 + \frac{2S(n\{\delta r_s \delta r_t\})}{m}\right) \quad \text{...(xxxi).}$$

Again we see the same difficulty arising about the evaluation of the great number of terms in $\{\delta r_s \delta r_t\}$. To appreciate the order of magnitude of this term in relation to the others, we will determine its value if all the correlation coefficients were given the mean value $\bar{\rho}$. Now there are two types of products $\{\delta r_{ab} \delta r_{cd}\}$ and $\{\delta r_{ab} \delta r_{ad}\}$; the former gives rise to the mean value $2\bar{\rho}^2(1-\bar{\rho})^2/n$ and the latter to the mean value $\bar{\rho}(1-\bar{\rho})^2(2+3\bar{\rho})/(2n)$. Of the former there are

$$\tfrac{1}{8}p(p-1)(p-2)(p-3)$$

and of the latter $\frac{1}{2}p(p-1)(p-2)$, the total number being

$$\tfrac{1}{8}p(p-1)(p-2)(p+1) = \tfrac{1}{2}m(m-1),$$

since m, the total number of correlation coefficients, $= \frac{1}{2}p(p-1)$, where p is the number of variates. Hence we may take as approximating to the mean value of $\{\delta r_s \delta r_t\}$:

$$\frac{\frac{1}{8}p(p-1)(p-2)(p-3)\,2\bar{\rho}^2(1-\bar{\rho})^2/n + \frac{1}{2}p(p-1)(p-2)\,\bar{\rho}(1-\bar{\rho})^2(2+3\bar{\rho})/(2n)}{\frac{1}{8}p(p-1)(p-2)(p+1)}$$
$$= \frac{2}{n(p+1)}\bar{\rho}(1-\bar{\rho})^2(2+p\bar{\rho}) \quad \text{....................(xxxii).}$$

* Holzinger's data would involve the finding of $((\rho_s - \bar{\rho}) + (\rho_t - \bar{\rho}))\{n\delta r_s \delta r_t\}$ 680 times and Simpson's no less than 4095 times!

In $S(\delta r_s \delta r_t)$ there are $\frac{1}{2}m(m-1)$ product terms. Hence

$$\frac{2S(n\,\delta r_s\delta r_t)}{m} = \frac{2(m-1)}{p+1}\bar{\rho}(1-\bar{\rho})^2(2+p\bar{\rho}) = (p-2)\bar{\rho}(1-\bar{\rho})^2(2+p\bar{\rho}).$$

Thus approximately*

$$\sigma_{\bar{r}}^2 = \frac{1}{mn}[(1-\bar{\rho}^2)^2 - 2\mu_2(1-3\bar{\rho}^2) + 4\bar{\rho}\mu_3 + \mu_4 + (p-2)\bar{\rho}(1-\bar{\rho})^2(2+p\bar{\rho})] \quad \text{..............(xxxiii).}$$

We turn lastly to $\sigma^2_{\mu_2}$. We have $\mu_2 = S(r_s - \bar{r})^2/m$,

$$\delta\mu_2 = \frac{2}{m}S(\rho_s - \bar{\rho})\,\delta r_s - \frac{2}{m}\delta\bar{r}\,S(\rho_s - \bar{\rho}),$$

but $S(\rho_s - \bar{\rho}) = 0$, accordingly

$$\sigma^2_{\mu_2} = \frac{4}{m^2}S(\rho_s - \bar{\rho})^2\,\sigma^2_{r_s} + \frac{8}{m^2}S(\rho_s - \bar{\rho})(\rho_t - \bar{\rho})\{\delta r_s \delta r_t\};$$

thus
$$\sigma^2_{\mu_2} = \frac{4}{mn}\left[\frac{S(\rho_s - \bar{\rho})^2(1-\rho_s^2)^2}{m} + \frac{2S(\rho_s - \bar{\rho})(\rho_t - \bar{\rho})\{n\,\delta r_s\delta r_t\}}{m}\right]$$

$$= \frac{4}{mn}\Big[(1-\bar{\rho}^2)^2\mu_2 - 4\bar{\rho}(1-\bar{\rho}^2)\mu_3 + 2(1+\bar{\rho}^2)\mu_4 - 4\bar{\rho}\mu_6 + \frac{2}{m}S(\rho_s - \bar{\rho})(\rho_t - \bar{\rho})\{n\,\delta r_s\delta r_t\}\Big] \quad \text{..................(xxxiv).}$$

The summation term is one of a type which we have seen reasons for supposing very small when it occurred in the expression $\{\delta\bar{r}\,\delta\mu_2\}$, and we think it may again be neglected here. Also the term in μ_6 will be small, since it depends on the sixth powers of the differences of correlation coefficients, and, perhaps, will be adequately accounted for by putting $\beta_4 = 15$, the normal value, i.e. $\mu_6 = 15\mu_2^3$†, or more accurately by taking β_4 from Table XLII (*b*) on p 78 of the *Tables for Statisticians and Biometricians*. We have then

$$\sigma^2_{\mu_2} = \frac{4}{mn}[(1-\bar{\rho}^2)^2\mu_2 - 4\bar{\rho}(1-\bar{\rho}^2)\mu_3 + 2(1+\bar{\rho}^2)\mu_4 - 4\bar{\rho}\beta_4\mu_2^3] \quad \text{..............(xxxiv}\,bis\text{),}$$

* Not a bad approximation to $\sigma_{\bar{r}}^2$ may be made as follows:

$$\sigma_{\bar{r}}^2 = \frac{1}{m}\left(\frac{S(\sigma^2_{r_s})}{m} + (m-1)\{\sigma_{r_s}\sigma_{r_t}R_{r_s r_t}\}\right).$$

Now put all the correlations in σ_{r_s} and σ_{r_t}, etc. equal to $\bar{\rho}$ and give $R_{r_s r_t}$ its mean value, i.e.

$$\text{Mean } R_{r_s r_t} = \frac{2}{p+1}\,\frac{\bar{\rho}(2+p\bar{\rho})}{(1+\bar{\rho})^2}.$$

Then
$$\sigma_{\bar{r}}^2 = \frac{(1-\bar{\rho}^2)^2}{mn}(1 + (m-1)\text{ Mean } R_{r_s r_t}).$$

For example, in the Holzinger data $\quad$ Mean $R_{r_s r_t} = \cdot 196{,}482$,

and we find $\sigma_{\bar{r}}^2 = \cdot 0083{,}8074$, not so far removed from the fuller value of (xxxiii), i.e. $\cdot 0088{,}7072$, showing that the terms $-2\mu_2(1-3\bar{\rho}^2) + 4\bar{\rho}\mu_3 + \mu_4$ contribute little to the result.

† $\beta_2 = 2{\cdot}0561$ for Holzinger's data, and $= 2{\cdot}3104$ for Simpson's data, β_1 being $\cdot 0008$ and $\cdot 0715$ respectively. Hence β_4 is nearer 5 in the first case and 7·8 in the second than 15.

a formula open to correction should anyone evaluate successfully the summation term. We conclude therefore that if $\{\delta\bar{r}\,\delta\mu_2\}$, $\sigma_{\bar{r}}^2$ and $\sigma^2_{\mu_2}$ be computed from (xxx), (xxxi) and (xxxiv) respectively and substituted in (xxvii) we shall have the best value at present available of $\sigma_{\sigma_T^2}$, or in more popular and incorrect parlance we shall be able to find the probable error of σ_T^2, i.e. of the mean square tetrad, and so judge to some extent how far the deviation between its observed and theoretical values is or is not significant.

We have seen that in obtaining the conditions for testing σ_T as well as determining the value of σ_T itself we have neglected terms of the order $1/(n\sqrt{n})$ as compared with $1/n$. It is accordingly essential that in a really valuable test of the two-factor hypothesis n should be large. This is hardly the case in the two chief series on which the hypothesis has so far been examined, i.e. the data of Holzinger, where $n=50$, and those of Simpson where $n=37$*. Further, our formulae indicate that the size of m, the number of correlations, ought also to be considerable†. The number of correlations seems adequate in Simpson's data (91), less so in Holzinger's (36). This number depends on the number of mental abilities measured and very rapidly increases with that. At the same time, of course, the number of tetrads runs up to very unmanageable numbers, but that evil cannot be avoided.

We suggest that some 12 to 15 abilities (66 to 105 correlations, 1485 to 4095 tetrads), the abilities being settled by psychologists *a priori* to avoid "overlaps," are essential to a satisfactory test, the observations to be made on a homogeneous population of several hundreds. Short series involve such large probable errors that a mere statement that theory and observation are in accordance within the limits of the probable errors can carry no conviction with it.

(7) *Restatement of Problem.*

We are not underrating the importance of the problem‡, but really appreciating it, when we state that it is worthy a thorough and direct test of this nature, in which mathematicians and psychologists must work to a common end. The labour involved is great, but not so great that it could not be surmounted by co-operative action.

We should like to word the problem somewhat differently to the Holzinger-Spearman approach to it. Consider the quantity

$$\psi_r^2 = S\,(r_{su}r_{tv} - r_{tu}r_{sv})^2\,\frac{8\,(m-4)\,!}{m\,!}\,.$$

ψ_r^2 is the mean square tetrad and is supposed to be zero for the sampled population; it will only be zero in the sample if every tetrad vanishes. But it will not be zero in

* Simpson's material appears to have been also somewhat heterogeneous, his 37 tested individuals consisting partly of academic and partly of commercial adults of various ages.

† m must be considerable to keep low the two last terms in (xxv) which cannot otherwise be neglected.

‡ The factor problem has bearings not only on mental ability, but on craniometry, where in the cranial correlations there is some reason to believe that a limited number of factors may be concerned.

general for the sample owing to the fact that it consists of a number of squares and therefore is essentially positive, even when the r's only differ from the ρ's by the variations of random sampling. Our first criterion therefore is that ψ_r^2 should be zero within the limits of random sampling. If it is not, then the sample is not compatible with the sampled population, being of a two-factor constitution. If it is within the limits of a random sample, then the sampled population *may be* of a two-factor constitution; it does not *prove* that it is, it only proves compatibility. There can exist no possible proof that because $(r_{su}r_{tv} - r_{tu}r_{sv})^2$ does not exceed zero by more than, say, twice its probable error, that $\rho_{su}\rho_{tv} - \rho_{tu}\rho_{sv} = 0$, and a two-factor constitution holds for the four variables s, u, t, v in the population sampled. All we can venture to affirm is that it is compatible with it. In the same way ψ_r^2 lying within reasonable limits of zero judged by its probable error can never prove a general and specific factor constitution, it can only indicate compatibility with such an hypothesis. It is thus easier to show the improbability of a two-factor hypothesis than to demonstrate its existence*.

Let us take our material in triplets of tetrads, and consider first the value for a single tetrad given in (v), i.e.

$$ {}_rT_{st|uv} = ({}_{\bar{r}}T_{st|uv}) + (\bar{r}_{su}\delta r_{tv} + \bar{r}_{tv}\delta r_{su} - \bar{r}_{sv}\delta r_{tu} - \bar{r}_{tu}\delta r_{sv}) + (\delta r_{su}\delta r_{tv} - \delta r_{tu}\delta r_{sv}). $$

We will term the three bracketed expressions A', B' and C' respectively. Now squaring:

$$ ({}_rT_{st|uv})^2 = A'^2 + 2A'B' + 2A'C' + B'^2 + 2B'C' + C'^2, $$

or taking mean values:

$$ \{({}_rT_{st|uv})^2\} = A'^2 + 2A'\{B'\} + 2A'\{C'\} + \{B'^2\} + 2\{B'C'\} + \{C'^2\}. $$

Now $$ A' = {}_{\bar{r}}T_{st|uv} = \frac{\rho_{su}\rho_{tv}(\rho^2_{su} + \rho^2_{tv}) - \rho_{tu}\rho_{sv}(\rho^2_{tu} + \rho^2_{sv})}{2n}. $$

Therefore A'^2 is of the order $1/n^2$. $\{B'\} = 0$. C' is of the order $1/n$ and accordingly $A'\{C'\}$ is of the order $1/n^2$. $\{B'^2\}$ is of the order $1/n$, $\{B'C'\}$ and $\{C'^2\}$ of the order $1/n^2$. Hence to a first approximation:

$$ \text{Mean (tetrad)}^2 = \{B'^2\}. $$

But $\{B'^2\}$ = mean value of $(\bar{r}_{su}\delta r_{tv} + \bar{r}_{tv}\delta r_{su} - \bar{r}_{sv}\delta r_{tu} - \bar{r}_{tu}\delta r_{sv})^2$, i.e. precisely (as we might anticipate) what we have evaluated in (xiii) and (xiv). We have for mean variation of the triplet of tetrads squared:

$$ \{S({}_rT)^2\} = \{(r_{su}r_{tv} - r_{tu}r_{sv})^2 + (r_{su}r_{tv} - r_{uv}r_{st})^2 + (r_{st}r_{uv} - r_{tu}r_{sv})^2\} $$
$$ = (1/n)(2S(\rho^2) - 6P_3 + 12\lambda^2) \quad\text{.....................(xxxv)}, $$

* Because a tetrad vanishes within the limits of its probable error we cannot assert, as some authorities have done, that it is clearly demonstrated that what is common to s, t and u is also shared by v. Such writers forget that we can increase the probable error to any extent by reducing the size, n, of the sample and so can demonstrate for $n=30$ or 50 what would be most improbable for $n=500$ or 1000.

where P_3 = third order products and $\lambda = \rho_{su}\rho_{tv} = \rho_{tu}\rho_{sv} = \rho_{st}\rho_{uv}$. Now, if

$$\Delta_{stuv} = \begin{vmatrix} 1, & \rho_{st}, & \rho_{su}, & \rho_{sv} \\ \rho_{ts}, & 1, & \rho_{tu}, & \rho_{tv} \\ \rho_{us}, & \rho_{ut}, & 1, & \rho_{uv} \\ \rho_{vs}, & \rho_{vt}, & \rho_{vu}, & 1 \end{vmatrix}$$

we have on expansion $\Delta_{stuv} = 1 - S(\rho^2) + 2P_3 - 3\lambda^2$, and can get rid of P_3. Accordingly

$$\{S({}_rT)^2\} = (1/n)(s - 3\Delta_{stuv} - S(\rho^2) + 3\lambda^2)$$

$$= (1/n)((1-\rho^2_{su})(1-\rho^2_{tv}) + (1-\rho^2_{sv})(1-\rho^2_{tu}) + (1-\rho^2_{st})(1-\rho^2_{uv})) - 3\Delta_{stuv}.$$

Hence if D be the general determinant of the correlation system, i.e. $|\rho|$, and we denote by $D_{(abcd..)}$ the minor obtained by striking out all constituents not in both the *abcd*... rows and columns, we shall have

$$\{\psi_r^2\} = \frac{8\times(m-4)!}{m!} \frac{\Sigma(D_{(su)}D_{(tv)} + D_{(sv)}D_{(tu)} + D_{(st)}D_{(uv)} - 3D_{(stuv)})}{n} \quad \text{..............(xxxvi).}$$

The summation Σ refers to every quadruplet of four variables out of the total m*.

We now need to find the standard deviation of ψ_r^2.

Let us write $\xi = ({}_rT_{st|uv})^2 = (A' + B' + C')^2$, then what we need is $\{S(\xi^2)\} - \{\psi_r^2\}^2$. But $\xi^2 = (A' + B' + C')^4$ and $\{\xi^2\} = \{B'^4\}$ + terms of orders $1/n^3$ and $1/n^4$, while $\{B'^4\}$ is of order $1/n^2$. But to discover the value of $\{B'^4\}$ we want to know all the fourth order products of the variations in the correlation coefficients. These are at present unknown, and having regard to the known values of β_1 and β_2 for distributions of r (see Tables, *Biometrika*, Vol. XI. pp. 379 *et seq.*) it seems rash to assume normality. Actually each tetrad is supposed to be varying round zero with its own standard deviation due to random sampling. We have as yet no proof that this variation takes the form of a normal curve. And if any single correlation coefficient does not, what chance is there that the tetrad $r_{su}r_{tv} - r_{sv}r_{tu}$, based upon four inter-correlated non-normally distributed variates, is going to do so? If we assume that it does so roughly and approximately, what evidence have we that a distribution based upon the variations of a large number of tetrads, *each with its own standard deviation*†, will follow a *normal curve* the standard deviation of which is based upon an approximation to the variance of the mean sum of the squared tetrads? If the deviation of each tetrad from zero had been divided by its own standard deviation we should have more hope of the resulting ratios obeying a normal law. It does not seem accordingly to be any demonstration of the validity of the theory of a general and specific factor to place a normal curve on the top of a frequency distribution of tetrads and argue from its appearance to the truth of the hypothesis. No proof

* We have given this form in the hope that some abler mathematician might possibly throw back the summation Σ of minors into a simple determinantal form.

† A very little consideration shows how diverse are the variances of tetrads. If the six correlations concerned in the tetrad are all equal the standard deviation is $0/\sqrt{n}$, $\cdot 375/\sqrt{n}$, $\cdot 5/\sqrt{n}$, $\cdot 375/\sqrt{n}$ and $0/\sqrt{n}$ as the correlations take the value 0, $\cdot 25$, $\cdot 5$, $\cdot 75$ and 1 in succession.

has yet been provided that the distribution ought to follow a normal curve at all, and if such a proof can be provided, then the degree of accuracy of fit must be properly tested. It is quite conceivable also that 378 or 1000 tetrads arising from a number of correlation coefficients with random sampling values might when "tetraded" give a distribution very akin to a normal curve. We do not say that they would or would not, but it is quite clear that confident proofs of an hypothesis cannot be based on a curve which it has not been demonstrated that the distribution ought really to satisfy. Writers on this subject do not tell us whether they suppose single tetrads to follow the normal law, nor how if they do a number of tetrads not measured in terms of their own standard deviations will also follow a normal law, and this with a standard deviation which is not even a true mean of the standard deviations of the contributions. Whether a theory be true or false can only be ascertained when the theory has been adequately developed so that it can be tested against observed facts. Here the facts are placed against a result which has not been shown to flow from the theory—and the accordance then is said to be good!

(8) *Fourth Moment of Tetrad Distribution.*

If notwithstanding the warning given as to the nature of the distributions of r_{st} and r_{uv}, we assume that as a first approximation we may treat them as normally distributed, then if r_1, r_2, r_3, r_4 be any four correlation coefficients and R_{12}, R_{13}, R_{14}, R_{23}, R_{24}, R_{34} their correlations as determined by the Pearson-Filon formulae, we require the following mean products:

$$\{\delta r_1^4\} = \frac{3}{n^2}(1-r_1^2)^4, \quad \{\delta r_1^3 \delta r_2\} = \frac{3}{n^2} R_{12}(1-r_1^2)^3(1-r_2^2),$$

$$\{\delta r_1^2 \delta r_2 \delta r_3\} = \frac{1}{n^2}(2R_{12}R_{13} + R_{23})(1-r_1^2)^2(1-r_2^2)(1-r_3^2),$$

$$\{\delta r_1^2 \delta r_2^2\} = \frac{1}{n^2}(1+2R_{12}^2)(1-r_1^2)^2(1-r_2^2)^2,$$

$$\{\delta r_1 \delta r_2 \delta r_3 \delta r_4\} = \frac{1}{n^2}(1-r_1^2)(1-r_2^2)(1-r_3^2)(1-r_4^2)(R_{12}R_{34} + R_{13}R_{24} + R_{14}R_{23}) \qquad \text{...........}\ \text{...(xxxvii)},$$

where it is clear that the last contains implicitly the first four*. It is accordingly a problem whether the last is reducible to any more manageable form. Up to the present we have not succeeded in finding it.

If we write

$$a = r_{13}\delta r_{24}, \quad b = r_{24}\delta r_{13}, \quad c = r_{14}\delta r_{23}, \quad d = r_{23}\delta r_{14}, \quad e = r_{12}\delta r_{34}, \quad f = r_{34}\delta r_{12},$$

then the fourth moment coefficient of the triplet of tetrad differences for the variables 1, 2, 3, 4 is given by

$$3M_4' = \{(B')^4\} = \{(a+b-c-d)^4 + (e+f-a-b)^4 + (c+d-e-f)^4\},$$

* For proof of the last see Soper, *Biometrika*, Vol. IX. 1913, p. 100, and Isserlis, Vol. XI. 1916, p. 189.

or, expanding and rearranging,

$$3M_4' = \{2(a^4+b^4+c^4+d^4+e^4+f^4) - 4a^3(-2b+c+d+e+f) - 4b^3(-2a+c+d+e+f)$$
$$-4c^3(a+b-2d+e+f) - 4d^3(a+b-2c+e+f) - 4e^3(+a+b+c+d-2f)$$
$$-4f^3(a+b+c+d-2e) + 12(a^2b^2+c^2d^2+e^2f^2)$$
$$+6(a^2c^2+a^2d^2+a^2e^2+a^2f^2+b^2c^2+b^2d^2+b^2e^2+b^2f^2+c^2e^2+c^2f^2+d^2e^2+d^2f^2)$$
$$+12a^2(cd+ef-bc-bd-be-bf) + 12b^2(ef+cd-ac-ad-ae-af)$$
$$+12c^2(ab+ef-ad-bd-ed-fd) + 12d^2(ab+ef-ac-bc-ec-fc)$$
$$+12e^2(ab+cd-fa-fb-fc-fd) + 12f^2(ab+cd-ae-be-ce-de)$$
$$+24(abcd+abef+cdef)\} \quad \text{...(xxxviii)}.$$

If now we use (xxxviii) above to find the mean values, there results:

$$3n^2M_4' = 6[r_{13}^4(1-r_{24}^2)^4 + r_{24}^4(1-r_{13}^2)^4 + r_{14}^4(1-r_{23}^2)^4 + r_{23}^4(1-r_{14}^2)^4 + r_{12}^4(1-r_{34}^2)^4 + r_{34}^4(1-r_{12}^2)^4]$$
$$+[24r_{13}^3r_{24}(1-r_{24}^2)^3(1-r_{13}^2)R_{24|13} - 12r_{13}^3r_{14}(1-r_{24}^2)^3(1-r_{23}^2)R_{24|23}$$
$$-12r_{13}^3r_{23}(1-r_{24}^2)^3(1-r_{14}^2)R_{24|14} - 12r_{13}^3r_{12}(1-r_{24}^2)^3(1-r_{34}^2)R_{24|34}$$
$$-12r_{13}^3r_{34}(1-r_{24}^2)^3(1-r_{12}^2)R_{24|12}]$$
$$+\ 5 \text{ similar expressions from the } b^3, c^3, d^3, e^3 \text{ and } f^3 \text{ terms}$$
$$+12r_{13}^2r_{24}^2(1-r_{24}^2)^2(1-r_{13}^2)^2(1+2R^2_{24|13})$$
$$+12r_{14}^2r_{23}^2(1-r_{23}^2)^2(1-r_{14}^2)^2(1+2R^2_{23|14}) + 12r_{12}^2r_{34}^2(1-r_{12}^2)^2(1-r_{34}^2)^2(1+2R^2_{34|12})$$
$$+6r_{13}^2r_{14}^2(1-r_{24}^2)^2(1-r_{23}^2)^2(1+2R^2_{24|23})$$
$$+\ 11 \text{ similar expressions from products of squared terms, i.e. } a^2d^2, \text{ etc.,}$$
$$+[12r_{13}^2r_{14}r_{23}(1-r_{24}^2)^2(1-r_{14}^2)(1-r_{23}^2)(R_{23|14}+2R_{24|23}R_{24|14})$$
$$+12r_{13}^2r_{12}r_{34}(1-r_{24}^2)^2(1-r_{12}^2)(1-r_{34}^2)(R_{34|12}+2R_{24|34}R_{24|12})$$
$$-12r_{13}^2r_{24}r_{14}(1-r_{24}^2)^2(1-r_{13}^2)(1-r_{23}^2)(R_{13|23}+2R_{24|13}R_{24|23})$$
$$-12r_{13}^2r_{24}r_{23}(1-r_{24}^2)^2(1-r_{13}^2)(1-r_{14}^2)(R_{13|14}+2R_{24|13}R_{24|14})$$
$$-12r_{13}^2r_{24}r_{12}(1-r_{24}^2)^2(1-r_{34}^2)(1-r_{13}^2)(R_{13|34}+2R_{24|13}R_{24|34})$$
$$-12r_{13}^2r_{24}r_{34}(1-r_{24}^2)^2(1-r_{12}^2)(1-r_{13}^2)(R_{13|12}+2R_{24|13}R_{24|12})]$$
$$+\ 5 \text{ similar expressions arising from the } b^2, c^2, d^2, e^2 \text{ and } f^2 \text{ terms}$$
$$+24r_{13}r_{24}r_{14}r_{23}(1-r_{24}^2)(1-r_{13}^2)(1-r_{23}^2)(1-r_{14}^2)(R_{24|13}R_{23|14}+R_{24|23}R_{13|14}+R_{24|14}R_{13|23})$$
$$+\ 2 \text{ similar expressions arising from the } abef \text{ and } cdef \text{ terms} \quad \text{...............(xxxix)}.$$

Granted sufficient space this expression might be written down at length, and the values of the R's substituted, when no doubt certain of the terms would cancel; it might even take a simpler, but hardly a simple form*. The prospect of cal-

* We suggest the problem to some algebraic stalwart with leisure!

culating the value of M_4' even for a single triplet of tetrads is alarming enough, for a series of 1000 triplets of tetrads actually appalling! Nor are we certain that the great labour would be worth while, for the result is based on hypotheses which we know are inaccurate, and which do not lead to even reasonable approximations when the correlations are fairly large and the value of n small. To obtain, however, a rough estimate of the value of M_4', let us assume $r_{12} = r_{13} = r_{14} = r_{23} = r_{24} = r_{34} = \bar{\rho}$ (as we have done as a first approximation for the case of M_2). We find

$$\left.\begin{aligned} R_{24|13} = R_{21|43} = R_{23|14} &= \frac{2\bar{\rho}^2}{(1+\bar{\rho})^2} \\ R_{12|13} = R_{14|13} = R_{12|14} = \text{etc.} &= \frac{\bar{\rho}\,(2+3\bar{\rho})}{2\,(1+\bar{\rho})^2} \end{aligned}\right\} \quad \text{(xl)},$$

whence on substituting we have

$$3n^2 M_4' = 36\bar{\rho}^4 (1-\bar{\rho}^2)^4 \left(4 + \frac{16\bar{\rho}^2}{(1+\bar{\rho})^2} + \frac{16\bar{\rho}^4}{(1+\bar{\rho})^4} - \frac{8\bar{\rho}\,(2+3\bar{\rho})}{(1+\bar{\rho})^2} + \frac{4\bar{\rho}^2(2+3\bar{\rho})^2}{(1+\bar{\rho})^4} - \frac{16\bar{\rho}^3(2+3\bar{\rho})}{(1+\bar{\rho})^4}\right),$$

and finally

$$M_4' = \frac{48\bar{\rho}^4(1-\bar{\rho})^4}{n^2} \quad \text{(xli)},$$

a very simple form. Now on the same hypothesis (see the first term of (xxv)) the value of M_2' is given by

$$M_2' = \frac{4\bar{\rho}^2(1-\bar{\rho})^2}{n} \quad \text{(xlii)}.$$

Here for the simple tetrads* treated as a symmetrical distribution M_2' and M_4' would be the second and fourth moment coefficients. For the squared tetrads M_2' would be the mean and

$$\Sigma^2 = M_4' - M_2'^2 = \frac{32\bar{\rho}^4(1-\bar{\rho})^4}{n^2}$$

the squared standard deviation, i.e.

$$\Sigma = \frac{4\sqrt{2}}{n}\,\bar{\rho}^2(1-\bar{\rho})^2 \quad \text{(xliii)}.$$

Turning to the distribution of tetrads rendered symmetrical, we have of course $\beta_1 = 0$. Further $\beta_2 = M_4'/M_2'^2 = 3$, or we conclude that these conditions for normality as far as β_1 and β_2 are concerned would be satisfied if all the correlations had the same equal value, but in this case the standard deviations of all the individual tetrads due to random sampling become the same and there is some theoretical reason for pooling the tetrads which are based on correlations all selected from the same curve. This curve is known to be

$$y_n = \frac{(1-\rho^2)^{\frac{1}{2}(n-1)}}{\pi\,(n-3)!}(1-r^2)^{\frac{1}{2}(n-4)}\frac{d^{n-2}}{d\,(r\rho)^{n-2}}\left(\frac{\cos^{-1}(-\rho r)}{\sqrt{1-\rho^2 r^2}}\right),$$

of which the β_2 is *not* however 3.

* Of course if we treat all correlations as the same these values are true for the whole series of tetrads as well as for a triplet of three tetrads.

Why then does the distribution of the tetrads give $\beta_2 = 3$ in the present case? For the simple reason that the Pearson-Filon formulae (viii) and the formulae (xxxvii) are all based on the assumption that the distribution of r may to a first approximation—and *Biometrika*, Vol. XI. pp. 379 *et seq.* show it to be a *very* rough approximation—be treated as normal! We have dealt with this point at length because we wished to indicate that it is a fallacious theory and not the essential nature of the two-factor hypothesis which leads to the above suggestion of approximate normality. We do not think it at all possible, in the present state of our theoretical knowledge, to deduce any evidence in favour of the two-factor hypothesis from coincidence or non-coincidence of the distribution of tetrads with a normal curve. We shall see as we advance that such a coincidence is as observationally improbable as it is theoretically unreasonable.

We might learn something of the real nature of the distribution of tetrads were M_4' in (xxxix) evaluated as M_2' in (xxvi) at least as far as the terms in σ_r^2 and its powers. Until this not very promising task is accomplished, we must content ourselves with comparing (xli) and (xlii) with the results obtained from observations*. We shall not, however, be able to ascertain whether the divergences found are due to (i) neglect of terms or (ii) to failure of the two-factor hypothesis in our material or to a combination of both!

(9) *Results which flow if we assume the Tetrad Distribution to be Normal.*

In the present state of our knowledge it seems that we can only apply a *single* test to determine whether $\{\psi_r^2\}$ is zero within the limits of random sampling; we can compare its theoretical value computed by (xvi), or if they are found to be close enough by the short formulae (xxv) or (xxvi) with its observed value, and determine whether the difference is significant having regard to the approximate value of $\sigma_{\psi_r}{}^2$ or $\sigma_{\sigma_T}{}^2$ provided by (xxvii). But here we must remember that (a) not knowing the theoretical distribution of the tetrads we shall be judging of the identity of a theoretical and an observed distribution on the basis of a single constant, which is extremely inadequate, and (b) we shall be judging by a "probable error" which fails more and more in any meaning as the distribution of tetrads diverges more and more from normality. Professor Spearman appears to get over these difficulties (a) and (b) by assuming without any discussion or show of proof that the distribution of tetrads must follow a normal curve. We know that correlation coefficients only roughly approximate to a normal distribution; it is inconceivable therefore, even if the original variates follow normal distributions, that the tetrads should do so. Thus it seems worth while, since half the strength at least of the Spearman demonstration appears to turn on the distribution of tetrads being normal, to investigate a little more closely what must hold if the tetrad distribution were really a normal one. We can do this best by inquiring into what would be

* For Holzinger's data the observed standard deviation of the squared tetrads is $\Sigma = \cdot 007,465$, while the value obtained from (xliii) is $\Sigma = \cdot 005,821$. For Simpson's data the corresponding values are $\cdot 018,002$ and $\cdot 009,542$! It will therefore be clear that but little is to be learnt of the success or failure of the two-factor hypothesis from M_4' until we know its value far more closely.

the distribution of squared tetrads if we suppose the distribution of tetrads themselves to follow the normal curve

$$y = y_0 e^{-\frac{1}{2}\frac{T^2}{\sigma_T^2}},$$

where T is a tetrad measured from zero and σ_T is the standard deviation of tetrads.

Let u be a squared tetrad $= T^2$, then $2TdT = du$ or $dT = \frac{1}{2}u^{-\frac{1}{2}}du$, and consequently the frequency distribution of squared tetrads is given by

$$z = z_0 e^{-\frac{1}{2}\frac{u}{\sigma_T^2}} u^{-\frac{1}{2}}.$$

This is a curve of Pearson's Type III or

$$z = z_0 e^{-\gamma u} u^p \quad \text{.................................(xliv),}$$

where $p = -\frac{1}{2}$ and $\gamma = 1/(2\sigma_T^2)$.

Thus we have:

$$\text{Mean squared tetrad, } \mu_1' = \frac{p+1}{\gamma} = \sigma_T^2 \quad \text{...............(xlv),}$$

as might of course be foreseen.

If $\Sigma =$ the standard deviation of squared tetrads,

$$\Sigma^2 = \frac{p+1}{\gamma^2} = 2\sigma_T^4,$$

or

$$\Sigma = \sqrt{2}\,\sigma_T^2 \quad \text{.................................(xlvi).}$$

Further we have for the fundamental constants β_1 and β_2 of the distribution

$$\beta_1 = \frac{4}{p+1} = 8, \quad \beta_2 = \frac{3(p+3)}{p+1} = 15 \text{.................(xlvii).}$$

Apart from the general test of "goodness of fit" of the tetrad distribution to a normal curve, which we shall deal with later, a very simple test arises from the quartile of the normal curve. The quartile will be the median tetrad if we disregard sign. But for the normal curve the quartile is $\cdot 674{,}490\sigma_T$. Thus we should have

$$\text{Median tetrad} = \cdot 674{,}490\sigma_T \quad \text{.....................(xlviii),}$$

and the probable error of this median tetrad will be $(\cdot 674{,}490)^2\,\sigma_{\sigma_T}$, which can be determined from our knowledge of $\sigma^2_{\sigma_T^2}$ in (xxvii) with reasonable approximation.

Again if the distribution be normal, the mean tetrad $\bar{T}$, all tetrads being treated as positive, is given by

$$\bar{T} = \sqrt{\frac{2}{\pi}}\,\sigma_T = \cdot 797{,}884\sigma_T \quad \text{.......................(xlix),}$$

and the probable error of $\bar{T}$ is given by

$$\cdot 674{,}490\sigma_{\bar{T}} = \cdot 538{,}165\sigma_{\sigma_T}.$$

The tests involved in (xlviii) and (xlix) are easy to apply when $\sigma_{\sigma_T^2}$ has been computed, for very approximately

$$\sigma_{\sigma_T} = \frac{1}{2\sigma_T} \cdot \sigma_{\sigma_T^2}.$$

With only the single test such as the Spearman process provides one is apt to be suspicious of that test being easily satisfied by some very simple hypothesis other than an elaborate structural one. We might even devise one which gave practically the desired normal distribution of the tetrads. Let us suppose the correlations on which the tetrads are based were drawn *at random* from a symmetrical frequency distribution of mean ρ and standard deviation σ_ρ. In this case there would be no correlation between the correlations, or if $\rho+\delta r_s$, $\rho+\delta r_t$ be two correlations drawn at random, then $\{\delta r_s \delta r_t\}=0$.

Taking any four correlations $\rho+\delta r_1$, $\rho+\delta r_2$, $\rho+\delta r_3$ and $\rho+\delta r_4$ we have:

$$T=(\rho+\delta r_1)(\rho+\delta r_2)-(\rho+\delta r_3)(\rho+\delta r_4)$$
$$=\rho(\delta r_1+\delta r_2-\delta r_3-\delta r_4)+\delta r_1\delta r_2-\delta r_3\delta r_4;$$

taking the mean of both sides for all r's:

$$\{T\}=\rho(\{\delta r_1\}+\{\delta r_2\}-\{\delta r_3\}-\{\delta r_4\})+\{\delta r_1\delta r_2\}-\{\delta r_3\delta r_4\},$$

but all these curled bracket terms vanish. Hence the mean tetrad is zero. Again

$$\begin{aligned}T^2=\rho^2[&(\delta r_1)^2+(\delta r_2)^2+(\delta r_3)^2+(\delta r_4)^2+2(\delta r_1\delta r_2)\\&-2(\delta r_1\delta r_3)-2(\delta r_1\delta r_4)-2(\delta r_2\delta r_3)-2(\delta r_2\delta r_4)+2(\delta r_3\delta r_4)]\\&+2\rho(\delta r_1+\delta r_2-\delta r_3-\delta r_4)(\delta r_1\delta r_2-\delta r_3\delta r_4)\\&+(\delta r_1\delta r_2-\delta r_3\delta r_4)^2.\end{aligned}$$

Taking mean values:

$$\text{Mean } T^2=\sigma_T^2=\rho^2 4\sigma_\rho^2+2\rho\times 0+2\sigma_\rho^2\sigma_\rho^2,$$

or

$$\sigma_T^2=2\sigma_\rho^2(2\rho^2+\sigma_\rho^2) \quad\text{.......................................(l).}$$

Again mean T^3 will be zero, for it involves terms of types $\{r_s\}$ or $\{r_s^3\}$ in every constituent.

Next turning to the fourth power of the tetrad we have:

$$\begin{aligned}T^4=&\rho^4(\delta r_1+\delta r_2-\delta r_3-\delta r_4)^4\\&+4\rho^3(\delta r_1+\delta r_2-\delta r_3-\delta r_4)^3(\delta r_1\delta r_2-\delta r_3\delta r_4)\\&+6\rho^2(\delta r_1+\delta r_2-\delta r_3-\delta r_4)^2(\delta r_1\delta r_2-\delta r_3\delta r_4)^2\\&+4\rho(\delta r_1+\delta r_2-\delta r_3-\delta r_4)(\delta r_1\delta r_2-\delta r_3\delta r_4)^3\\&+(\delta r_1\delta r_2-\delta r_3\delta r_4)^4.\end{aligned}$$

The expressions in the second and fourth lines vanish, as they involve only odd powers of the δr's when we sum and take the mean. Suppose $\nu_4=\{\delta r_s^4\}$ and write ν_2 for $\{\delta r_s^2\}$, i.e. σ_ρ^2, then it follows easily that

$$\{\rho^4(\delta r_1+\delta r_2-\delta r_3-\delta r_4)^4\}=4\rho^4(\nu_4+9\nu_2^2).$$

Similarly:

$$6\rho^2(\delta r_1+\delta r_2-\delta r_3-\delta r_4)^2(\delta r_1\delta r_2-\delta r_3\delta r_4)^2=48\rho^2\nu_2^3,$$

and

$$(\delta r_1\delta r_2-\delta r_3\delta r_4)^4=2(\nu_4^2+3\nu_2^4).$$

Combining the above results we find:

$$\text{Mean } T^4=\{T^4\}=4\rho^4(\nu_4+9\nu_2^2)+48\rho^2\nu_2^3+2(\nu_4^2+3\nu_2^4) \quad\text{........(li).}$$

Thus far we have not assumed that the symmetrical distribution from which the r's are drawn is normal. If we now suppose it to be normal, then $\nu_4 = 3\nu_2^2 = 3\sigma_\rho^4$ and

$$\{T^4\} = 3\,(2\sigma_\rho^2\,(2\rho^2 + \sigma_\rho^2))^2 + 12\sigma_\rho^8 \quad \text{.....................(lii).}$$

Hence we have for the fundamental constant

$$B_2 = \{T^4\}/(\{T^2\})^2 = 3\left(1 + \left(\frac{\sigma_\rho^2}{2\rho^2 + \sigma_\rho^2}\right)^2\right) \quad \text{..................(liii).}$$

The system of tetrads deduced from a normal distribution of correlation coefficients is theoretically a leptokurtic curve, but it would be in appearance extremely like a normal curve with its tails in reality slightly emphasised. Indeed the resemblance would be very great if ρ and σ_ρ were to be so chosen that the chance of a randomly selected correlation coefficient being negative was very improbable. Thus*:

If $\rho = 2\sigma_\rho$ we should have $B_2 = 3{\cdot}037$,
$\rho = 2{\cdot}5\sigma_\rho$ „ „ „ $B_2 = 3{\cdot}016$,
$\rho = 3\sigma_\rho$ „ „ „ $B_2 = 3{\cdot}008$.

It may be safely asserted that even the most highly trained statistical judgment would not detect by an ocular examination of graphs such deviations from normality as are represented by these values of B_2. It may be reasonably doubted if it could discriminate in the same manner between $B_2 = 3{\cdot}0$ and $B_2 = 3{\cdot}1$.

We can easily measure the amount of variation there will be in random samples of M correlation coefficients from the distribution ρ, σ_ρ, for since

$$\sigma^2_{T_r} = 2\sigma_r^2\,(2\bar{r}^2 + \sigma_r^2),$$

where $\bar{r}$ is the mean r of samples as distinguished from the true ρ, it follows that

$$\sigma_T\,\delta\sigma_T = 4\,(\sigma_\rho\,(\rho^2 + \sigma_\rho^2)\,\delta\sigma_r + \sigma_r^2\rho\,\delta\bar{r}).$$

Hence squaring, taking the means and substituting the values

$$\{(\delta\sigma_r)^2\} = \sigma_\rho^2/(2M), \quad \{(\delta\bar{r})^2\} = \sigma_\rho^2/M$$

and $\{\delta\bar{r}\,\delta\sigma_r\} = 0$—for in the case of material following the normal curve there is no correlation between the mean and the standard deviation of samples—we have:

$$\sigma_{\sigma_T} = \frac{4\sigma_\rho^3}{\sqrt{M}}\sqrt{\frac{\frac{1}{2}(\rho^2 + \sigma_\rho^2)^2 + \rho^2\sigma_\rho^2}{2\sigma_\rho^2\,(2\bar{\rho}^2 + \sigma_\rho^2)}} \quad \text{.....................(liv).}$$

Now let us consider the following system and compare it with Holzinger's data. Suppose we had a normal curve of mean $\rho = {\cdot}340{,}000$ and of standard deviation $\sigma_\rho = {\cdot}111{,}000$, and let us suppose 36 correlations selected from this distribution at random, then the mean r of the samples would vary round ρ in the manner prescribed by

$${\cdot}340{,}000 \pm {\cdot}012{,}478,$$

and the standard deviations of these samples in the manner prescribed approximately by

$${\cdot}111{,}000 \pm {\cdot}008{,}823.$$

* For the ρ and σ_ρ of Holzinger's data $\rho = 2{\cdot}884\sigma_\rho$, while for Simpson's data $\rho = 2{\cdot}833\sigma_\rho$.

The tetrads would follow a very nearly normal curve round a zero tetrad having its

$$B_2 = 3{\cdot}0077,$$

and so be quite indistinguishable from a normal curve of errors.

Further the standard deviation (see (1)) of this distribution and its probable error would be

$$\cdot077{,}465 \pm \cdot010{,}386.$$

Now in Holzinger's data the mean of the correlations is ·347,778 and their standard deviation ·120,603. The first is only 0·62 times the probable error in excess of ρ, and the second is only 1·1 times the probable error in excess. Holzinger's data have a moderate resemblance to a normal curve distribution as a graph, and finally their σ_T is equal to ·077,253 by observation, or the deviation from ·077,465 is only about ·02 times the probable error. Reduced to probable errors the expected and observed values are both ·052, when stated only to two significant figures!

Now if to show a general resemblance by a graph to a normal curve and to state that the observed and expected values are ·051 and ·052* suffices to demonstrate the hypothesis of two factors on Holzinger's data, will not a similar reasoning enable us to assert that Holzinger's tetrad curve simply results from forming the tetrads of a random selection of 36 correlations drawn from a normal distribution of correlation coefficients with a mean and standard deviation well within the probable error limits of the like quantities for Holzinger's data?

Now we by no means suggest that drawings from a normal curve were the origin of Holzinger's correlations of mental abilities; we know they were not obtained in this way. But we think it reasonable to suggest that the one-constant proof adopted to demonstrate a delicate structural constitution for these correlations would equally well demonstrate that they arose merely from random samplings of a normal curve. The fact is that 36 correlation coefficients are not in such a case adequate to discriminate between random drawings of the correlations and an elaborate structural hypothesis as to the nature of the variates. The distribution of correlation coefficients is not generally even approximately normal. It may be seen at once that we could not apply the same reasoning to Simpson's data. Here $\bar{r} = \cdot486{,}703$ and $\sigma_r = \cdot208{,}617$, and σ_r is relatively too great for us to assume that the correlations are selected from a normal curve. Indeed an inspection of the following coarsely grouped frequency distribution of the 91 correlation coefficients indicates that it can hardly be looked upon as a sample from a normal

Central Values of Coefficients.

	·945	·845	·745	·645	·545	·445	·345	·245	·145	·045
Frequency ...	3	5	10	6	21	11	12	18	3	2

* Values given by Spearman, *The Abilities of Man*, p. 149, with the remark "Evidently the agreement could hardly be bettered."

distribution; we appear to be dealing with a bimodal if not a trimodal system, and the sample would be almost as probable from a rectangle ($\rho = \cdot 500{,}000$, $\sigma_\rho = \cdot 288{,}675$) as from a normal curve. But this very irregularity of the correlation coefficients, which may be quite well inherent in the mental abilities selected for testing, makes it *a priori* exceedingly improbable that any frequency distribution based on these correlations by forming their tetrads will give any approach to a normal curve. In the case of Holzinger's data there is likely to be more resemblance to a normal curve because his correlations are more nearly a sample from a normal distribution.

(10) *Tests to be applied to Data.*

We will now turn to a consideration of the data which are more or less sufficiently ample to allow of legitimate testing of the hypothesis of general and specific factors. These data practically reduce to two series, that of Holzinger and that of Simpson, although even here some criticism may be made of their complete suitability. In both series the number of persons tested is too small, and in Simpson's series it is possibly too heterogeneous. However it is on these series that the greatest stress has been laid and they have been spoken of as demonstrating with the accuracy of physical investigation the truth of the two-factor hypothesis. The demonstration consists in asserting (i) that the tetrad distributions are closely normal and (ii) that these normal distributions have a probable error value closely identical with what is theoretically to be expected, i.e. a value deducible from Equation (xxvi).

To test these demonstrations we have first formed the distribution of the tetrads for both series, we have then found the distribution of the squared tetrads*. We have finally found by brute force the value of $\sigma_T{}^2$ from Equation (xvi) to test the efficiency of Formula (xxvi).

(11) *Illustration I. Holzinger's Data.*

The following table is given by Spearman and Holzinger†. It is said to "describe an experiment based on 50 cases and comprising an unusually homo-

TABLE I.

Correlations between Tests.

Tests	*a*	*b*	*c*	*d*	*e*	*f*	*g*	*h*	*i*
a	—	·50	·54	·34	·47	·40	·50	·33	·24
b	·50	—	·39	·56	·51	·43	·36	·32	·15
c	·54	·39	—	·49	·52	·27	·27	·26	·27
d	·34	·56	·49	—	·30	·27	·52	·13	·35
e	·47	·51	·52	·30	—	·35	·14	·33	·18
f	·40	·43	·27	·27	·35	—	·38	·40	·19
g	·50	·36	·27	·52	·14	·38	—	·19	·38
h	·33	·32	·26	·13	·33	·40	·19	—	·29
i	·24	·15	·27	·35	·18	·19	·38	·29	—

* Not from the distribution of the tetrads, but from squaring each individual tetrad and then grouping.

† *British Journal of Psychology*, Vol. XVI. Part 2, p. 87.

geneous group of pupils aged 12 and 13 years. The nine tests used were the components of the Otis Intelligence Scale, the parts being very similar to the American Army Tests." The data are due to Holzinger.

There are 378 independent tetrads here leading to 126 triplets of tetrads, each triplet depending on a group of four characters, s, t, u, v, for which in the sampled population the relations

$$\rho_{su}\rho_{tv} = \rho_{tu}\rho_{sv} = \rho_{st}\rho_{uv}$$

are assumed to hold.

Thus by (xiv) each triplet of tetrads will contribute

$$\begin{aligned} 2/n\,[(\rho^2_{su} + \rho^2_{st} + \rho^2_{sv} + \rho^2_{tu} + \rho^2_{tv} + \rho^2_{uv}) \\ - 3\,(\rho_{tu}\rho_{tv}\rho_{uv} + \rho_{su}\rho_{sv}\rho_{uv} + \rho_{st}\rho_{sv}\rho_{vt} + \rho_{st}\rho_{tu}\rho_{su}) \\ + 2\,(\rho_{st}\rho_{uv}\rho_{sv}\rho_{tu} + \rho_{sv}\rho_{tu}\rho_{su}\rho_{tv} + \rho_{su}\rho_{tv}\rho_{st}\rho_{uv})] \\ = 2/n\,[S(\rho^2) - 3P_3 - 2P_4], \text{ say } \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\text{lv}). \end{aligned}$$

Since this paper has been in type Professor Spearman has stated that his book (*The Abilities of Man*) never made any assumption as to the theoretical normality of the tetrad distribution. He seems to overlook what he has himself written. On p. 140 he says: "The most perfect comparison between the two, the observed tetrad differences and those to be expected from sampling errors alone, is obtained by making a complete frequency distribution of each of these two sets of values. A more summary comparison is got by seeing whether or not about half of the observed tetrad differences are greater and half less than their probable error. Much the same thing is to see whether the median observed tetrad difference and the probable error are about equal." In the "most perfect comparison" Professor Spearman invariably puts a normal curve on his diagram as representing the frequency distribution of theory. His "more summary comparisons" are again entirely based on the theory being a normal curve. There is no meaning in the statement that the median tetrad difference should equal the probable error, unless the normal curve has been taken as the theoretical distribution.

Further illustrations are:

p. 142: "The theoretical distribution is given ... in the curve" [normal curve].

p. 143: "The two distributions are shown in the same way as in the preceding case" [normal curve].

p. 146: "The two distributions, curve [normal curve] and rectangles ... display one of the most striking agreements between theory and practice ever recorded in psychology."

p. 149: "The two distributions theoretical [normal curve] and observed.... Evidently the agreement could hardly be bettered."

p. 154: "The two distributions theoretical and observed are shown in the same way as on p. 142."

Again on this p. 154 in two diagrams curves which are presumably intended for normal curves are used, and approximation to the normal curve is treated as a test of the accordance of observation and theory. Further illustration is unnecessary.

With the six correlations of a quadruplet of variates arranged thus:

$$\begin{matrix} \rho_{su} & \rho_{sv} & \rho_{st} \\ \rho_{tu} & \rho_{tv} & \rho_{uv} \end{matrix}$$

it was found fairly easy, if laborious, to compute:

(*a*) the value of each tetrad,

(*b*) the square of each tetrad,

(*c*) 21 times the sum of the squares of each correlation coefficient,

(*d*) the sum of the triple products leaving out one variate each time, and

(*e*) the sum of the quadruple products.

Each of the operations (*c*), (*d*), (*e*) was done separately and continuously, so that the attention of the computer might be fixed on a single process at one time.

If Σ denotes a summation for the 126 triplets we found:

$$\Sigma(S(\rho^2)) = 102{\cdot}4338 = 21 \times 4{\cdot}8778,$$

$$\Sigma(P_3) = 21{\cdot}751{,}944, \quad \Sigma(P_4) = 5{\cdot}579{,}378.$$

Hence:
$$\{\psi_r^2\} = \frac{96{\cdot}673{,}448}{378 \times 50} = {\cdot}005{,}1150 = \sigma_T^2.$$

Thus:
$$\sigma_T = {\cdot}071{,}519.$$

The weak point of this result lies of course in the fact that the formula above holds for the ρ's and there is no alternative but replacing them by the observed r's. This involves neglecting terms of the order $1/\sqrt{n}$, i.e. in this case $1/\sqrt{50}$ as compared with unity, that is the terms neglected may be about $\frac{1}{7}$ of those retained. No stronger argument could be used for largely increasing the number of persons tested.

We now proceeded to determine the constants of the distribution of correlation coefficients*. We have:

$$\bar{r} = {\cdot}347{,}778, \qquad \sigma_r = {\cdot}120{,}603,$$

$$\mu_2 = {\cdot}0145{,}4500, \qquad \mu_3 = {\cdot}0000{,}3216, \qquad \mu_4 = {\cdot}0004{,}3499,$$

$$\beta_1 = {\cdot}000{,}336, \qquad \beta_2 = 2{\cdot}056{,}107.$$

Accordingly from (xxv):

$$\sigma_T^2 = \tfrac{4}{50}\,[{\cdot}0514{,}5115 + {\cdot}0151{,}1058 - {\cdot}0000{,}0003 + {\cdot}0000{,}0017],$$

where we indicate the magnitude of each contributory term. Hence:

$$\sigma_T^2 = {\cdot}0053{,}2495,$$

* Without grouping by squaring, cubing and fourth powering the correlation coefficients.

or if we omit the μ_3 and μ_4 terms, $\sigma_T^2 = \cdot 0053,2492_5$, thus they only make a difference in the eighth decimal place, and as far as (xxv) goes we should be justified in using (xxvi). We have then $\sigma_T = \cdot 072,972$. But we have seen that σ_T as ascertained from the full formula (lv) is $\cdot 071,519$. Thus we see that σ_T as found from the approximate formula differs from the exact value *in the second significant figure.* This can only be due to our replacing $[X_1 X_2]$ by $\{X_1 X_2\}$: see p. 256 But as we shall see the probable error of σ_T is itself only significant in the third place of decimals, the very place in which the theoretical value of σ_T as given by (xxv) is inexact. It is therefore very doubtful whether we can replace the more exact but very laborious (lv) by the Spearman-Holzinger formula (xxvi), which is a good enough approach to (xxv), for the error involved is of the order of the probable error of σ_T. The source of the trouble seems to be that Spearman and Holzinger have not recognised that their $\Sigma(X_1 X_2)$ is not the same thing as the $S(X_1 X_2)$ by which they replace it. We have seen so far no short method of evaluating $[X_1 X_2]$ except by forming the 378 products and taking their mean value. We next have to calculate $\sigma^2_{\sigma_T^2}$ from (xxvii), but this involves the determination of $\sigma_{\bar{r}}^2$, $\sigma^2_{\mu_2}$ and $\{\delta\bar{r}\,\delta\mu_2\}$. We note that $p = 9$, $m = 36$ and $n = 50$ for Holzinger's data.

We have from (xxxiii): $\quad \sigma_{\bar{r}}^2 = \cdot 0033,70716$,

and from (xxxiv *bis*): $\quad \sigma^2_{\mu_2} = \cdot 0000,27008$*,

while from (xxx): $\quad \{\delta\bar{r}\,\delta\mu_2\} = -\cdot 0000,19136$.

The necessary coefficients in (xxvii) are

$$\bar{p}(1-\bar{p})(1-2\bar{p}) + \frac{3(1-2\bar{p})}{m-1}\mu_2 = \cdot 071\cdot 589$$

and

$$1 + \frac{6\bar{p}(1-\bar{p})}{m-1} = 1\cdot 0388,8486.$$

Finally we obtain $\quad \sigma_{\sigma_T^2} = \cdot 0005,7994$,

or, in the language of "probable error,"

Probable Error of $\sigma_T^2 = \cdot 0003,9117$.

We have now to compare these theoretical results, i.e. $\sigma_T^2 = \cdot 005,115 \pm \cdot 000,391$, with the observed value of σ_T^2.

At first sight it might appear relatively easy to obtain the observed value of σ_T^2, but this is far from the case. It may be approached by three different paths. It should be:

(*a*) the squared standard deviation of the symmetrical distribution of tetrads,

(*b*) the mean of the distribution of squared tetrads,

or, (*c*) it may be obtained by adding up the squares of the individual tetrads and taking their average.

* Using $\beta_4 = 5$, as corresponding to $\beta_1 = 0$, $\beta_2 = 2$ approximately.

Now (c) should give the exact value to any required number of places. (a) and (b) will differ from each other and (c) owing to the subrange elements adopted for grouping, and even if corrected by Sheppard may fail to give adequate accordance, i.e. their divergence may be in the same decimal place as the first significant figure of the probable error of σ_T^2. We will now investigate this difficulty.

(a) We formed the 378 tetrads and grouped them in subranges of ·01; the following distribution was obtained:

Magnitude of Tetrads, taken as positive.

	·00—·01	·01—·02	·02—·03	·03—·04	·04—·05	·05—·06	·06—·07	·07—·08	·08—·09	·09—·10	·10—·11	·11—·12	·12—·13	·13—·14	·14—·15	·15—·16	·16—·17	·17—·18	·18—·19	·19—·20	·20—·21	Total
Frequency	37	38	33	20	37	41·5	29·5	30	16	24	17	17	6	8	5	8	3	1	5	1	1	378

The constants of this distribution are as follows:

Mean (i.e. mean deviation from zero) ·062,341,

Median (Quartile of Symmetrical Distribution*) ·0570,

σ_T = Standard Deviation (for Symmetrical Distribution) ·076,117,

$\mu_2 = \sigma_T^2 = \cdot 0057,9392$, $\mu_3 = 0$, $\mu_4 = \cdot 0000,8717,6119$,

and hence $\beta_2 = 2\cdot 5969$.

Is this a normal curve? Clearly not, if we are to judge by β_2, which should be 3. We should indeed expect a platykurtic and not a normal curve theoretically, for the tetrads must all lie numerically between -1 and $+1$, and thus must give a limited range symmetrical curve, and not a normal curve. Again the median ·0570 should be $\cdot 674,490 \times \cdot 076,117 = \cdot 051,340$ with a probable error ·001,851, or the observed median deviates more than three times its probable error from the value it should take on the basis of a normal curve (cf. xlviii). Again by (xlix) the mean should be $\cdot 060,733 \pm \cdot 002,182$, the actual mean is ·062,341, and this test at least is satisfied.

Lastly we proceeded to apply the (P, χ^2) test for goodness of fit. Now we do not assert that this test is wholly suitable to material such as the present for it is based on random selection from an infinite population, any member of which is equally likely to be drawn. But when we are confronted with a graph and asked to approve the excellency with which the observations fit a normal curve of definite theoretical standard deviation, we naturally ask, failing to judge kurtosis by the eye: Should we consider in the ordinary treatment of observations such a series well fitted by a normal curve?

* The 189th tetrad of the system arranged in order is ·0567 and the 190th tetrad ·0574, giving ·05705 for the mid-value.

SYMMETRICAL DISTRIBUTION OF THE TETRAD DIFFERENCES. *HOLZINGER'S DATA*

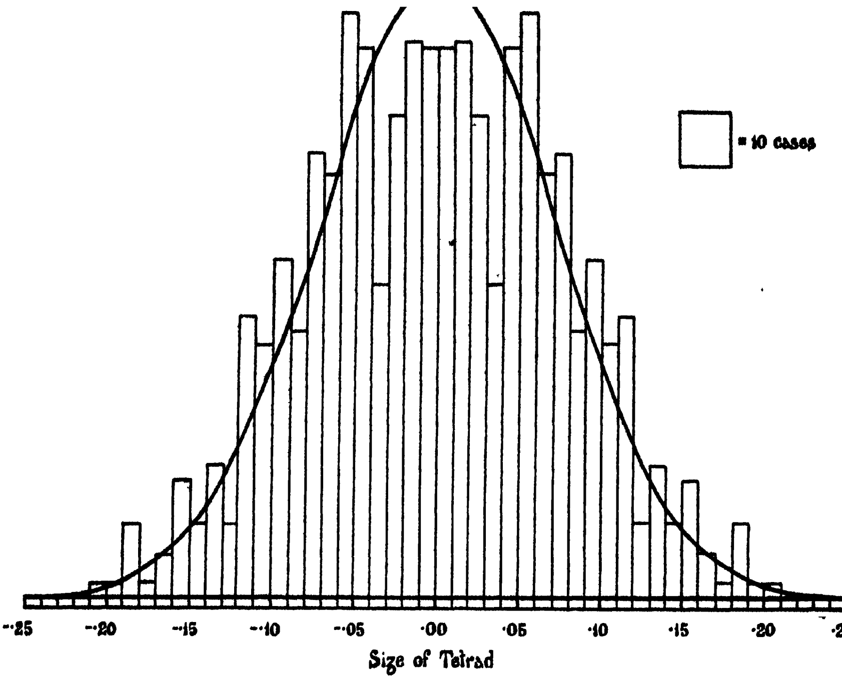


Diagram I.

Now the theoretical value of σ_T is ·071,519 (see p. 274), and we have the following system of values:

Fit of Holzinger's Data Tetrads to a Normal Curve.

Ranges.

Tetrad Values	·00—·01	·01—·02	·02—·03	·03—·04	·04—·05	·05—·06	·06—·07	·07—·08	·08—·09	·09—·10	·10—·11	·11—·12	·12—·13	·13—·14	·14—·15	·15—·16	·16—·18	·18—·21	Totals
Normal Curve m	42·0	41·2	39·7	37·4	34·6	31·4	27·9	24·3	20·8	17·5	14·4	11·6	9·2	7·1	5·4	4·0	5·1	4·5	378·1
Observed Value m'	37	38	33	20	37	41·5	29·5	30	16	24	17	17	6	8	5	8	4	7	378
$\frac{(m'-m)^2}{m}$	·60	·25	1·13	8·09	·17	3·25	·09	1·34	1·11	2·41	·47	2·51	1·11	·11	·03	1·00	·24	1·39	$\chi^2 = 25·30$

Here $\chi^2 = 25·30$ for 18 groups, giving a probability of $P = ·0886$. It must be noted that the chief contributions to χ^2 do not arise from large values of the tetrads. Now the value of P does not represent a great improbability, one sample

in 11 would give a fit as bad or worse to this normal curve, but we do think that the phrase "evidently the agreement could hardly be bettered" is unjustified either by the goodness of fit of the normal curve, or the correspondence between the observed median ·057* and the theoretical median, i.e. ·67449 × theoretical σ_T (·071,519†) = ·048, the difference being about 4·7 times the probable error.

Even if these two values, one obtained from observation and the other from an approximate theory, were within the limits of random sampling as indicated by the probable error—and they are not in this case—it would only mean that *one* constant of a very intricate frequency distribution did not contradict an imperfect theory. Some tetrads might be nearly zero, others large, and the averaging process might easily bring the mean square tetrad within the limits of random sampling. Such agreement in a single constant of the distribution would never enable us to assert as Holzinger and Spearman do of the former's data: "that in this case at any rate *every one* of the abilities can be resolved into two independent factors, the one being always specific and the other throughout common‡."

Several illustrations of the danger of this type of argument occur in *The Abilities of Man*. Thus on p. 149, where Magson's data are dealt with, Professor Spearman writes: "Here the tetrad differences gave a probable error of ·03 and an observed median of ·04 so that again agreement leaves nothing to be desired."

As a matter of fact, taking these values as correct, ·01 may mean serious disagreement when, owing to the large number of individuals tested (149), the probable error of the median is found to be sensible only in the third or fourth decimal place.

Again, take the results of Bonser with 757 subjects and 20 correlations (*The Abilities of Man*, p. 147). Here Professor Spearman says that "the tetrad differences have a probable error of ·011 and an observed median of ·013, or almost exactly the same." Whether these numbers are "almost exactly the same" cannot be determined by inspection; it all depends on the probable error of their difference, and whether that error has its first significant figure in the third or fourth place of decimals. We will illustrate the validity of this criticism by giving the constants for Bonser's data. The standard deviation of the tetrads, σ_T ·019,942, the median as computed from this is ·013,451, while the quartile or observed median is ·019,056. The correlations give

$$\bar{r} = \cdot 3423,\quad \sigma_r = \cdot 083{,}826,\quad \mu_2 = \cdot 0070{,}2681,\quad \mu_3 = -\cdot 0000{,}8836,\quad \mu_4 = \cdot 0001{,}0313;$$

whence $\beta_1 = \cdot 0023$, and $\beta_2 = 2\cdot 0887$, and a reasonable value of β_4 would be 5, say.

Hence we find $\quad \sigma_{\bar r}^2 = \cdot 0003{,}2198,\quad \sigma^2_{\mu_2} = \cdot 0000{,}5945$

and $\quad \{\delta\bar{r}\,\delta\mu_2\} = -\cdot 0000{,}0193;$

leading to $\quad \sigma_T = \cdot 017{,}523,\quad \sigma_{\sigma_T^2} = \cdot 0000{,}4861$

and $\quad \sigma_{\sigma_T} = \cdot 001{,}387.$

* Spearman, *The Abilities of Man* p. 149, gives the theoretical median which he terms "the probable error" as ·051 and the observed median as ·052. We are unable to agree with these values.

† The true value; the value ·076,117 used on the previous page to test normality is the σ_T of the symmetrical distribution of tetrads.

‡ *British Journal of Psychology*, Vol. XVI. p. 88.

If we express our results in the somewhat unusual terminology of Professor Spearman, the observed median is ·013,451 and the theoretical probable error ·011,819 ± ·000,631. He gives these as ·013 and ·011; saying that they are "almost exactly the same." The actual difference is ·001,632, almost exactly 2·6 times the probable error of the difference.

We do not delay to deal with Magson's data, where the probable error will be significant only in the third or fourth place of decimals, while Spearman places before the reader values only to two places of decimals, remarking that their agreement leaves nothing to be desired.

(*b*) We next formed the distribution of square tetrads. This was done in subranges of ·001, and is reproduced in the following table:

Distribution of Square Tetrads for Holzinger's Data.

Value of Squared Tetrad	Frequency	Value of Squared Tetrad	Frequency	Value of Squared Tetrad	Frequency
·000—·001	109	*·014—·015*	6	*·028—·029*	2
·001—·002	40	*·015—·016*	5	*·029—·030*	1
·002—·003	35	*·016—·017*	2	*·030—·031*	1
·003—·004	31	*·017—·018*	5	*·031—·032*	—
·004—·005	22	*·018—·019*	3	*·032—·033*	—
·005—·006	22	*·019—·020*	1	*·033—·034*	—
·006—·007	9	*·020—·021*	2	*·034—·035*	3
·007—·008	12	*·021—·022*	1	*·035—·036*	2
·008—·009	15	*·022—·023*	1	*·036—·037*	—
·009—·010	10	*·023—·024*	2	*·037—·038*	—
·010—·011	9	*·024—·025*	3	*·038—·039*	1
·011—·012	8	*·025—·026*	4	*·039—·040*	—
·012—·013	4	*·026—·027*	—	*·040—·041*	1
·013—·014	6	*·027—·028*	—	*·041—·042*	—

The constants of this distribution are as follows:

$$\text{Mean} = \sigma_T{}^2 = \cdot 005{,}968, \quad \Sigma = \text{Standard Deviation} = \cdot 007{,}465,$$

$$\mu_2 = \cdot 0000{,}5571{,}89, \quad \mu_3 = \cdot 000{,}000{,}876{,}172, \quad \mu_4 = \cdot 000{,}000{,}023{,}904{,}601;$$

whence $\beta_1 = 4\cdot 4376$, and $\beta_2 = 7\cdot 6995$.

If the distribution of the simple tetrad were a normal curve, the values of β_1 and β_2 should be by (xlvii) 8 and 15 respectively. The observed values show how far this is from being the fact. Further we ought to have by (xlvi)

$$\Sigma = \sqrt{2}\,\sigma_T{}^2 = \sqrt{2} \times \cdot 005{,}968 = \cdot 008{,}440,$$

but the observed value is ·007,465.

We think these divergences, taken in conjunction with the previous results, are sufficient to indicate that the tetrads of the Holzinger data do not follow a normal curve. This does not disprove the Spearman hypothesis, because theoretical reasoning is all against such a distribution. But it does show that merely placing a normal curve on top of the tetrad histogram contributes nothing to the proof of the hypothesis.

(c) Lastly, we added the 378 squared tetrads together and took their mean, thus obtaining what should be the exact value of σ_T^2. We found

$$\sigma_T^2 = \cdot 0059{,}4094.$$

Unfortunately in squaring our tetrads, which were themselves only available to four decimal places, the correlations only being given to two, we tabled the squares to *six* decimal places only and not to eight. It is possible therefore that this result is not correct to more than six decimal places.

We have now obtained three distinct values for σ_T^2 and of course for σ_T, and we see that they are by no means identical. Let us examine them.

Values σ_T^2 and σ_T by various processes.

Method	σ_T^2	σ_T
(a) From distribution of tetrads with subranges ·01	·0057,9392	·076,118
(b) From distribution of Squared tetrads with subranges ·001	·0059,6825	·077,254
(c) From addition of Squared tetrads to six decimal places ...	·0059,4094	·077,077
(d) From Formula (xvi) with Probable Error from (xxvii) ...	·0051,1500 ± ·0003,9117	·071,519 ± ·002,736
(e) From Formula (xxv) " " " " ...	·0053,2495 ± ·0003,9117	·072,972 ± ·002,736

Now it will be clear from this table that the process by which we find the value of σ_T^2 is of vital importance, having regard to its probable error. The difference between the theoretical value of σ_T^2, as found by the full Formula (xvi) and as found by the Spearman-Holzinger approximation (xxv), is of the order of the probable error itself and this accordingly may make a considerable difference in the interpretation of our results. Again, the observed value of the mean squared tetrad differs according to whether we obtain it from the distribution of tetrads in ·01 subranges, or of squared tetrads in ·001 subranges, or by the simple adding together of the actual squared tetrads and dividing by their number. The third should give the most accurate value, the first being too coarse a grouping even when the standard deviation is corrected by Sheppard, and the second, although nearly in agreement with the third, probably owing to the *J*-shaped form of the distribution, requiring correction for abruptness. Taking (d) as the best theoretical value, and (c) as the best observed value, we have their difference = 2·11 times its probable error, while had we compared (b) and (d) the deviation would have been 2·18 times the probable error. If we consider the probable error in its ordinary sense, this is neither a very good nor a very bad accordance between theory and observation. What it certainly does not justify is the remark of Spearman and Holzinger that "in this case at any rate every one of the abilities can be resolved into two independent factors, the one being always specific and the other throughout common*."

* *Loc. cit.* p. 88. We have not considered it worth while to deal with the asymmetrical distribution of tetrads given in the paper, as it has been dropped by Spearman in *The Abilities of Man*, and clearly depends entirely on the order in which the variates are arranged in the table.

The results are about as doubtful if we consider σ_T instead of σ_T^2, for the difference between (c) and (d) is then 2·03 times its probable error. If we multiply all the quantities in the σ_T column by ·67449, we shall of course still have the differences as given by (c) and (d) 2·03 times its probable error. We do not get rid of this dubious value by using the less exact forms (a) and (e), and cutting them down to three figures and asking the reader to observe how close are ·052 and ·049!

Reduced to its simplest terms all we can say is that the difference between the mean squared tetrad as observed and as deduced on the hypothesis that the tetrads in the sampled population are really all zero is a little over twice its probable error. This is no convincing proof of the two-factor hypothesis, but it certainly does not disprove it, and this the more so as we really do not know what is the nature of the curve of distribution of σ_T^2 and whether the "probable error" has any meaning for that distribution at all. Nothing whatever is gained in perspicuity by reducing σ_T^2 first to σ_T and then multiplying it by ·67449, unless we assume the distribution of tetrads to be normal and wish to compare the theoretical median tetrad from (d), i.e. ·048,239 ± ·001,845, with the observed median tetrad ·057005, when the difference will be found to be about 4·8 times the probable error.

That Formula (xxv) gives an unsatisfactory result as compared with the completer Formula (xvi) can be elucidated by comparing the terms of each order of magnitude in the two results. We have the following values:

Holzinger's Data.

Order of Terms*	Holzinger-Spearman Approximation Formula (xxv)	Value from General Expression (xvi)
Square ...	·0108,3956	·0108,3956
Cubic ...	− ·0066,6080	− ·0069,0538
Quartic ...	·0011,4616	·0011,8082
Totals ...	·0053,2492	·0051,1500

It will be seen that the Holzinger-Spearman approximation fails in the second significant figure in the cubic, and in the third significant figure in the quartic terms. That formula therefore tends to emphasise an accordance which does not really exist. This will be still more the case if the erroneous term $2(1-\bar{r}^2)^4/n^2$ be added in. The origin of this divergence of (xxv) from (xvi) lies in the erroneous estimates of the mean products involved: see our pp. 254—257.

* Considering the sources of (xxvi) we may break it up into

$$\frac{4}{n}(\bar{\rho}^2+\sigma_\rho^2), \quad -\frac{8\bar{\rho}}{n}\left(\bar{\rho}^3-\frac{3}{m-1}\sigma_\rho^2\right) \text{ and } \frac{4}{n}\left(\bar{\rho}^4-\frac{6\bar{\rho}^2\sigma_\rho^2}{m-1}\right),$$

representing the square, cubic and fourth order terms, where, of course, we must replace $\bar{\rho}$ and σ_ρ by $\bar{r}$ and σ_r, the observed quantities. For the Holzinger data, $\bar{r}=$ ·347,778, $\sigma_r^2=$ ·0145,4500, $n=50$ and $m=36$.

While the Holzinger data give no such triumphant confirmation of the theory as they have been supposed to do,—for they only give a somewhat dubious answer to the needful, but far from all-sufficient condition, namely, the question whether the mean squared tetrad is compatible with selection from a population in which all the tetrads are zero,—the arithmetical analysis has brought out several important points. It shows us in the first place that at present an adequate expression to replace (xvi) by a briefer labour of computation does not exist; (xxv) introduces errors of the order of the probable error. In the second place, with the view of getting an adequately small error of random sampling even for the single condition involved, it is essential that n should be far larger than the value provided by the Holzinger, or indeed by the Simpson data. And the larger n is, the less attention need be paid to the unknown terms in $1/(n\sqrt{n})$ and $1/n^2$. Further, the Holzinger data show that the assumption that the tetrad distribution for the two-factor hypothesis must follow a normal curve is of no validity from the observational as well as from the theoretical standpoint, and thus to make it only obscures and does not strengthen the proof. We simply do not know what is the character of the distribution, even if the mental abilities concerned followed a normal law. Lastly, and this is a most vital point, it must be now clear that working to two or three decimal places, or at least stating final results to this number of places, is wholly inadequate, having regard to the decimal places in which the first significant figures of the probable errors appear. It may even be doubted whether correlations tabled to two decimal places only are really adequate, for the different processes which may be used to obtain the observed mean squared tetrad, σ_T^2, show to what a high degree of accuracy we must work, should we desire to obtain this constant correct to the fourth decimal place where its probable error begins to be appreciable*. This is again an argument in favour of a high value for n. We do not believe that anything like enough stress has thus far been laid on the need for an adequate number of decimal places in the computation of these data.

(12) *Illustration II. Simpson's Data.*

We now turn to Simpson's data, which involve 14 tests leading to 3003 tetrad differences. Unfortunately these tests were only applied to 37 persons, so that the probable errors of all the correlations are exceedingly large and as some 45 out of the 91 correlations are greater than ·50 and 24 greater than ·60, it is not easy to consider the correlations as following normal distributions, far less to assert that the individual tetrad differences would follow such distributions. However, the table of correlations is provided on p. 283.

We have treated these data precisely as we have treated those of Holzinger. We proceeded in the same manner to evaluate first the general Formula (lv), the work

* Suppose two correlations ·334 and ·754 involved in a tetrad are cut down to ·33 and ·75, their product in the latter case to six figures is ·172,500, whereas their real product is ·176,486. Thus the final tetrad may be modified in the third decimal place, and the squared tetrad quite possibly in this place also.

being in this case extremely long and laborious as there were 13 products for each of the 1001 triplets of tetrads. This led to

$$\Sigma S(\rho^2) = 1684{\cdot}0890,$$

$$\Sigma(P_3) = 551{\cdot}619{,}216, \quad \Sigma(P_4) = 204{\cdot}743{,}107.$$

Hence

$$\{\psi_r^2\} = \sigma_T^2 = \frac{877{\cdot}435{,}132}{3003 \times 37} = {\cdot}0078{,}9692,$$

and

$$\sigma_T = {\cdot}0888{,}6463.$$

*Simpson's Correlations**.

	a	b	c	d	e	f	g	h	i	j	k	l	m	n
a	—	·98	·94	·79	·62	·91	·71	·54	·78	·88	·55	·42	·33	·25
b	·98	—	·84	·80	·64	·81	·79	·70	·73	·74	·52	·43	·26	·25
c	·94	·84	—	·62	·55	·82	·49	·56	·73	·71	·53	·40	·28	·21
d	·79	·80	·62	—	·57	·52	·68	·53	·42	·56	·45	·29	·38	·48
e	·62	·64	·55	·57	—	·55	·54	·73	·39	·51	·39	·59	·25	·22
f	·91	·81	·82	·52	·55	—	·53	·57	·59	·66	·54	·31	·28	·19
g	·71	·79	·49	·68	·54	·53	—	·45	·39	·47	·51	·57	·17	·25
h	·54	·70	·56	·53	·73	·57	·45	—	·35	·49	·34	·56	·25	·25
i	·78	·73	·73	·42	·39	·59	·39	·35	—	·69	·36	·29	·26	·09
j	·88	·74	·71	·56	·51	·66	·47	·49	·69	—	·44	·37	·34	·28
k	·55	·52	·53	·45	·39	·54	·51	·34	·36	·44	—	·31	·19	·27
l	·42	·43	·40	·29	·59	·31	·57	·56	·29	·37	·31	—	·21	·07
m	·33	·26	·28	·38	·25	·28	·17	·25	·26	·34	·19	·21	—	·24
n	·25	·25	·21	·48	·22	·19	·25	·25	·09	·28	·27	·07	·24	—

To find the probable error of σ_T we need first to consider the distribution of the correlation coefficients, 91 in number. The constants obtained by direct summation without grouping are as follows:

$$\bar{r} = {\cdot}486{,}703, \quad \sigma_r = {\cdot}208{,}617,$$

$$\mu_2 = {\cdot}0435{,}2100, \quad \mu_3 = {\cdot}0024{,}2837, \quad \mu_4 = {\cdot}0043{,}7599,$$

$$\beta_1 = {\cdot}0715, \quad \beta_2 = 2{\cdot}3104,$$

a slightly skew, but very platykurtic distribution.

Accordingly from (xxv):

$$\sigma_T^2 = \tfrac{4}{37}\left[{\cdot}0624{,}1163 + {\cdot}0442{,}4584 - {\cdot}0000{,}0004 + {\cdot}0000{,}0070\right],$$

where the magnitude of the several terms is indicated. Hence

$$\sigma_T^2 = {\cdot}0115{,}3047.$$

If we had omitted the last two terms as of small importance, the value would have been ·0115,3054†. The former yields $\sigma_T = {\cdot}1073{,}8003$.

* *Psychological Review*, 1914, pp. 101 *et seq.* The hk correlation is in one instance given as ·54 and in the other as ·34. We have followed Spearman, who gives them both as ·34 in *The Abilities of Man*, p. 145.

† This is the value given by the Spearman-Holzinger formula when we discard their erroneous term in $1/n^2$.

We are now in a position to compute the auxiliary quantities on which the standard deviations of σ_T^2 and σ_T depend. We find, remembering $p=14$, $m=91$, and $n=37$, that

$$\sigma_{\bar{r}}^2 = \cdot 0041{,}96325, \quad \sigma^2_{\mu_2} = \cdot 0000{,}37195^*,$$

$$\{\delta\bar{r}\,\delta\mu_2\} = -\cdot 0000{,}34181.$$

Further:
$$\bar{r}(1-\bar{r})(1-2\bar{r}) + \frac{3(1-2\bar{r})}{m-1}\mu_2 = \cdot 0066{,}8238,$$

$$1 + \frac{6\bar{r}(1-\bar{r})}{m-1} = 1\cdot 0166{,}5488.$$

Whence
$$\sigma_{\sigma_T^2} = \cdot 0006{,}68744,$$

and
$$\sigma_{\sigma_T} = \sigma_{\sigma_T^2}/(2\sigma_T) = \cdot 0037{,}62712,$$

the corresponding probable errors being

$$\text{P.E. of } \sigma_{\sigma_T^2} = 0004{,}5106, \quad \text{P.E. of } \sigma_{\sigma_T} = \cdot 0025{,}3791.$$

We have now to compare the three quantities σ_T^2 as found from (lv), σ_T^2 as found from (xxv), and the observed values as found from the distributions of simple and squared tetrads. As in the case of Holzinger's data we provide first a table showing the contributions of (lv) and (xxv) to the terms of square, cubic, and quartic order in σ_T^2.

Simpson's Data.

Order of Terms	Holzinger-Spearman Approximation Formula (xxv)	Value from General Expression (xvi) or (lv)
Square ...	·0303,1361	·0303,1363†
Cubic ...	– ·0247,7493	– ·0297,8747
Quartic ...	·0059,9187	·0073,7076
Totals ...	·0115,3055‡	·0078,9692

These results, while sufficiently startling, really only confirm the conclusion reached on the Holzinger data, namely, that the Holzinger-Spearman approximate formula fails to provide the cubic and quartic terms with anything like the required degree of accuracy. It may fail in the second or even in the first significant figure. This is the more important as in this case the cubic terms are practically as large as the square terms. It would accordingly seem absolutely necessary to use the full Formula (xvi) or (lv), although our painful experience of

* Adopting the value $\beta_4 = 7\cdot 8$: see footnote, p. 260.

† This should agree absolutely with figure in first column; the difference of two in the eighth decimal place is due to number of figures retained in different stages of the working.

‡ Agrees with the exception of one unit in last figure with value found on p. 283.

the tremendous computing labour involved makes us indeed reluctant to accept this as a final conclusion. What appears certain is that the Holzinger-Spearman formula will not give the required result, and the explanation of this no doubt lies in their assumption that $[X_1 X_2 X_3 \ldots]$ is the same as $\{X_1 X_2 X_3 \ldots\}$: see our pp. 254—7.

We have then for our theoretical values:

$$\sigma_T{}^2 = {\cdot}0078{,}9692 \pm {\cdot}0004{,}5106,$$

$$\sigma_T = {\cdot}0888{,}6463 \pm {\cdot}0025{,}3791.$$

It is clear that the Holzinger-Spearman formula gives a result for $\sigma_T{}^2$ differing more than eight times the probable error from the exacter value of Formula (lv)!

We now pass to the consideration of the frequency distributions of the tetrads.

For the distribution of tetrads treated as positive we find:

Simpson's Data.

Range of Tetrad	·00—·02	·02—·04	·04—·06	·06—·08	·08—·10	·10—·12	·12—·14	·14—·16	·16—·18	·18—·20	·20—·22	·22—·24	·24—·26	·26—·28	·28—·30	·30—·32	·32—·34	·34—·36	·36—·38	·38—·40	Total
Frequency	580·5	590·5	456·5	354·5	310	162·5	145·5	98	58	44	49	40	37	19	26	17	5	6	3	1	3003

The constants of this distribution are:

Mean Tetrad = ·073,493, Standard Deviation, $\sigma_T = {\cdot}099{,}090$,

$$\mu_2 = {\cdot}0098{,}1874^*, \quad \mu_4 = {\cdot}0004{,}2416{,}8640,$$

leading to $\beta_2 = 4{\cdot}2778.$

It is thus already clear that the distribution is far from normal ($\beta_2 = 3$). Also we have in this case to deal with a leptokurtic and not a platykurtic curve. This result confirms a conclusion forced on us by the examination of all the material used to discuss the two-factor hypothesis, namely, that the actual nature of the tetrad distribution curve depends on the special mental characters chosen to provide the material, and there is no universal rule that the tetrad distribution will form a curve of a particular character. We conceive it quite possible that with a special selection of mental abilities a tetrad distribution might be bimodal.

With a β_2 as divergent as these data indicate, there is small probability of the tetrad distribution showing any goodness of fit to a normal curve. Taking the theoretical standard deviation as ·08886, we have worked out the frequencies in 18 groups of the corresponding normal curve. This curve is shown superposed on

* Without Sheppard's correction $\mu_2 = {\cdot}0098{,}2707$ and $\sigma_T = {\cdot}099{,}182$; the differences are scarcely of importance with 20 groups.

the observations in Diagram II, and the following table provides the corresponding frequencies.

SYMMETRICAL DISTRIBUTION OF THE TETRAD DIFFERENCES. SIMPSON'S DATA

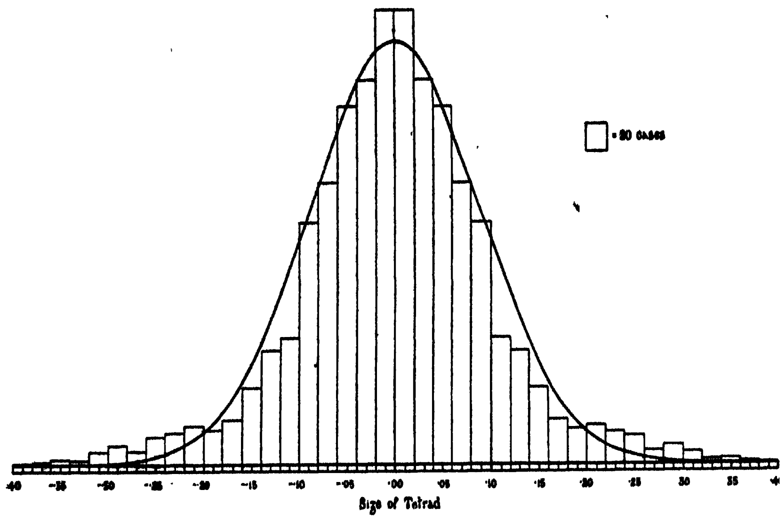


Diagram II.

Now we are quite certain that it is not possible to appreciate goodness of fit by examining ocularly the above diagram, but we can say that if the tetrad distribution were a series of ordinary observations obtained by random selection from a normal curve, not once in 10,000,000 trials should we obtain as bad or worse a fit.

Fit of Simpson's Data Tetrads to a Normal Curve.

Ranges.

Tetrad Values	·00—·02	·02—·04	·04—·06	·06—·08	·08—·10	·10—·12	·12—·14	·14—·16	·16—·18	·18—·20	·20—·22	·22—·24	·24—·26	·26—·28	·28—·30	·30—·32	·32—·34	·34—·40	Totals
Normal Curve m	534·8	508·1	459·9	395·2	322·9	251·5	185·4	130·3	87·0	55·3	33·3	19·1	10·5	5·4	2·7	1·3	0·5	0·4	3003·6
Observed Value m'	580·5	590·5	456·5	354·5	310	162·5	145·5	98	58	44	49	40	37	19	26	17	5	10	3003
$\frac{(m'-m)^2}{m}$	3·91	13·36	·03	4·19	·51	31·50	8·59	8·01	·97	2·31	7·40	22·87	52·12	34·25	201·07	189·61	40·50	230·40	851·60

And this result does not flow from the large tetrads at the tail only. If we take only tetrads from ·00 to ·16, or eight groups with $\chi^2 = 90{\cdot}10$, then not once in a million trials should we have such a bad fit! Now these probabilities might to some extent be modified could we take account of the intercorrelation of the variants. But it is clear that no existing means, and with the highest degree of probability no revised process, could lead to anything more than the statement that it is very unreasonable to suppose these tetrads follow in their distribution a normal curve of errors.

If further reasoning be necessary on this point we can refer to the following constants.

The actually observed median tetrad is ·0540; it should be 059,938 on the theoretical standard deviation and ·066,835 on the observed standard deviation. The deviation of the observed median is about 3·5 times its probable error in the first case and about 7·5 times its probable error in the second case.

The observed mean tetrad is ·073,493; deduced from the theoretical standard deviation it is ·070,904 with a probable error ± 002,024; deduced from the observed standard deviation it is ·079,062. The observed mean tetrad differs from its theoretical value only 1·2 times the probable error, indicating that the actual curve might be a normal curve with the theoretical standard deviation; but the observed mean tetrad differs from the mean deduced from the observed standard deviation by more than four times this probable error, indicating that the actual curve is not a normal curve at all. No result from the distribution of positive tetrads seems compatible with any real approach to normality.

We next take the Distribution of Squared Tetrads given in the following table:

Distribution of Square Tetrad Differences in the Correlations of Simpson's Observations.

Value	Freq.	Value	Freq.	Value	Freq.	Value	Freq.	Value	Freq.
·000—·001	932	*·031—·032*	1	*·062—·063*	6	*·093—·094*	2	*·124—·125*	1
·001—·002	351	*·032—·033*	8	*·063—·064*	2	*·094—·095*	2	*·125—·126*	—
·002—·003	235	*·033—·034*	8	*·064—·065*	4	*·095—·096*	2	*·126—·127*	—
·003—·004	176	*·034—·035*	8	*·065—·066*	3	*·096—·097*	2	*·127—·128*	1
·004—·005	144	*·035—·036*	3	*·066—·067*	3	*·097—·098*	2	*·128—·129*	—
·005—·006	99	*·036—·037*	8	*·067—·068*	4	*·098—·099*	2	*·129—·130*	—
·006—·007	104	*·037—·038*	—	*·068—·069*	2	*·099—·100*	1	*·130—·131*	—
·007—·008	106	*·038—·039*	5	*·069—·070*	3	*·100—·101*	1	*·131—·132*	—
·008—·009	83	*·039—·040*	9	*·070—·071*	1	*·101—·102*	—	*·132—·133*	—
·009—·010	61	*·040—·041*	7	*·071—·072*	1	*·102—·103*	—	*·133—·134*	—
·010—·011	46	*·041—·042*	9	*·072—·073*	3	*·103—·104*	—	*·134—·135*	2
·011—·012	46	*·042—·043*	5	*·073—·074*	4	*·104—·105*	1	*·135—·136*	—
·012—·013	40	*·043—·044*	6	*·074—·075*	1	*·105—·106*	1	*·136—·137*	1
·013—·014	19	*·044—·045*	4	*·075—·076*	2	*·106—·107*	—	*·137—·138*	—
·014—·015	34	*·045—·046*	3	*·076—·077*	—	*·107—·108*	—	*·138—·139*	—
·015—·016	29	*·046—·047*	6	*·077—·078*	1	*·108—·109*	1	*·139—·140*	—
·016—·017	29	*·047—·048*	5	*·078—·079*	5	*·109—·110*	—	*·140—·141*	—
·017—·018	25	*·048—·049*	5	*·079—·080*	2	*·110—·111*	—	*·141—·142*	—
·018—·019	25	*·049—·050*	4	*·080—·081*	—	*·111—·112*	—	*·142—·143*	—
·019—·020	22	*·050—·051*	1	*·081—·082*	5	*·112—·113*	—	*·143—·144*	—
·020—·021	18	*·051—·052*	10	*·082—·083*	3	*·113—·114*	—	*·144—·145*	—
·021—·022	21	*·052—·053*	3	*·083—·084*	3	*·114—·115*	2	*·145—·146*	—
·022—·023	15	*·053—·054*	6	*·084—·085*	3	*·115—·116*	1	*·146—·147*	—
·023—·024	15	*·054—·055*	6	*·085—·086*	1	*·116—·117*	—	*·147—·148*	—
·024—·025	14	*·055—·056*	4	*·086—·087*	1	*·117—·118*	1	*·148—·149*	—
·025—·026	10	*·056—·057*	4	*·087—·088*	3	*·118—·119*	1	*·149—·150*	—
·026—·027	10	*·057—·058*	4	*·088—·089*	—	*·119—·120*	—	*·150—·151*	—
·027—·028	12	*·058—·059*	2	*·089—·090*	1	*·120—·121*	—	*·151—·152*	1
·028—·029	9	*·059—·060*	5	*·090—·091*	—	*·121—·122*	—		
·029—·030	8	*·060—·061*	4	*·091—·092*	2	*·122—·123*	—		
·030—·031	9	*·061—·062*	5	*·092—·093*	1	*·123—·124*	—	Total ...	3003

The constants of this table are as follows:

Mean Square Tetrad = ·009,866, Standard Deviation, $\sigma_T{}^2$ = ·018,002,

μ_2 = ·000,324,066,286, μ_3 = ·000,019,282,816, μ_4 = ·000,001,651,732.

These give $\beta_1 = 10{\cdot}9255$, $\beta_2 = 15{\cdot}7279$.

Constants to be expected for the Distribution of Mean Square Tetrads had the Distribution of Positive Tetrads been a Normal Curve with the Theoretical Standard Deviation:

	Expected	*Observed*
Mean:	·007,897	·009,866
Standard Deviation:	·011,168	·018,002
β_1:	8	10·93
β_2:	15	15·73

We are again forced to the conclusion that the observed values are not such as we should expect to arise were the distribution of tetrads a normal curve of the theoretical standard deviation.

Thus the conclusion already drawn is again confirmed, the observations do not support the view that a normal curve of tetrad distribution is in any way an approximation in the case of Simpson's data.

Having dismissed the normal curve we are left with a *single* test of whether Simpson's data are compatible with the General and Specific Factors Hypothesis. That test is summed up in asking whether the difference between the theoretical and observed values of the Mean Squared Tetrad falls within the limits of random sampling. The Theoretical Mean Squared Tetrad is

$$\cdot0078{,}9692 \pm \cdot0004{,}5106.$$

The Observed Mean Squared Tetrad as deduced from the standard deviation of grouped positive tetrads is ·0098,1874, and as deduced from the mean of grouped squared tetrads ·0098,6630. We have not so far had the courage to add up the 3003 squared tetrads and find their mean exactly, but we believe the mean obtained by grouping in ranges of ·001 must be fairly approximate, and the deviation from the observed value is then about 4·26 times the probable error, or the odds are about 250 to 1 against such a deviation. Notwithstanding that we know nothing of the distribution curve for Mean Squared Tetrads* it is fairly safe to assert that a deviation of this magnitude is wholly incompatible with the mean squared tetrad being zero in the sampled population—in other words it is quite incompatible with the two-factor hypothesis.

How then does it come about that Professor Spearman can write (*The Abilities of Man*, p. 146): "This time, the two distributions, curve and rectangles, far from

* If this material were of the ordinary kind, i.e. a sample of 3003 individuals made from a curve with constants similar to those of the distribution of squared tetrads, then the betas of the curve of means would be $B_1 = \frac{\beta_1}{3000}$, $B_2 - 3 = \frac{\beta_2 - 3}{3003}$, and notwithstanding the high values of β_1 and β_2, i.e. 10·93 and 15·73, we should have B_1 = ·0036 and B_2 = 3·0042, values compatible with a normal distribution of the means of mean squared tetrads, and so giving the usual significance to "probable error."

being totally discrepant as before, display instead one of the most striking agreements between theory and observation ever recorded in psychology. Indeed it would not easily be matched in any other science. The divisibility then is indicated more decisively than ever."

We venture to think that he has been misled by one or two unappreciated facts:

(*a*) The general symmetry is produced artificially by treating each tetrad as alternately positive and negative.

(*b*) That he lays weight on ocular judgment of goodness of fit where in every case the more certain analytical tests show no approach whatever to a normal distribution.

(*c*) That so far as theory is concerned it undoubtedly indicates that normality of distribution is highly improbable. (Professor Spearman gives no proof for it, nor does he state why he anticipates it.)

(*d*) A spurious appearance of the whole series of tetrads being close to zero is produced by an erroneous scale of one-tenth the true value of the tetrads being accidentally placed at the foot of the diagram.

(*e*) The results reached by him and printed underneath the diagram are:

Probable Error = ·061.

Observed Median = ·062.

Besides the unacceptable cutting down to three figures we believe these results to be both erroneous.

If by "Probable Error" is meant the theoretical standard deviation, σ_T, multiplied by ·674,490, then its value is ·0888,6463 × ·674,490 = ·059,938, or to three decimals ·060, so that the first result is somewhat in error. If by *Observed Median* Professor Spearman understands as in the usual terminology the value of the middle tetrad, this is ·0540, and differs widely from ·062. If by *Observed Median* Professor Spearman understands contrary to the usual terminology the value he would obtain from the observed σ_T × ·674,490, he is assuming that his distribution is truly normal, for which all proof fails, but its value is then

$$\cdot 099{,}090 \times \cdot 674{,}490 = \cdot 066{,}835,$$

or cut down to three figures ·067, whereas Professor Spearman gives ·062 as his *Observed Median*. Whether we take ·054 or ·067 as the *Observed Median* tetrad, the latter being obtained by unproven assumption, the difference from ·060 is some four times the probable error ·001,712 of the result. Thus the one point on which "one of the most striking agreements between theory and observation ever recorded in psychology" turns appears to be due to some oversight in arithmetic, and to the unappreciated fact that we cannot talk of agreement between two numbers in statistical science until we have some idea of the magnitude of the probable error of their difference.

Had indeed the difference in the *Observed Median* and the *Probable Error* been in the cases dealt with by Professor Spearman of the order of random sampling, it would not in the least have proved that the divisibility into two factors was decisively established. All it would have achieved would have been to show that in respect of one constant—an undoubtedly important but not all-sufficient constant, namely the mean square tetrad—the data were *compatible* with the hypothesis. Compatibility of data for *one* condition only with an hypothesis does not demonstrate the truth of the hypothesis; other hypotheses might show compatibility in this or other important conditions. As a matter of fact Holzinger's data only show dubious compatibility, and Simpson's data incompatibility; both, we hold, have been erroneously interpreted.

(13) *General Conclusions.*

We may sum up our conclusions as follows:

(*a*) The theory of general and specific factors is, we are inclined to believe, too narrow a structure to form a frame for the great variety of interrelationship of mental abilities. Still it is a theory which it is well worth while to test adequately.

(*b*) The mathematics of the theory—even if we confine ourselves to the narrow hypothesis of linearity adopted by Professor Spearman—become of surprising complexity, and the fundamental formula for the mean square tetrad involves for adequate data weeks of arithmetic labour to compute the resulting value.

(*c*) The formula provided by Holzinger and Spearman to avoid this labour is in error, and this for two reasons. In the first place, to get over the difficulty of testing only a small number of persons, they have endeavoured to evaluate the terms of the second order; their expression for these terms is in error because they are still using in the larger portion of their formula terms only computed to the first order of approximation. Secondly, their evaluation of the mean products of correlation coefficients fails because they have overlooked the fact that their values for mean products involve *all* correlation products, whereas actually only certain of these products occur in the theoretical mean square tetrad formula. We have illustrated how great may be the divergence from the true value found by computing the correlation products by their formula and by the full formula.

(*d*) The actual deduction of the probable error of the full mean squared tetrad formula presents great algebraic complexity, and if this could be overcome. it would almost certainly involve more serious arithmetic work than even that necessary to determine the mean square tetrad. The roughly approximate formula for the mean square tetrad of Holzinger and Spearman does, however, provide a basis upon which some estimate of the probable error of the mean square tetrad can be established. The process involves considerable algebraic and arithmetic labour, and one or two theoretical assumptions not yet justified. But in this way we reach the only method yet suggested for testing the compatibility of theory and observation. Applied, we find dubious compatibility in the case of Holzinger's data, even more dubious results in the case of Bonser's data, and what looks like

complete incompatibility for Simpson's data. Thus the claim of Professor Spearman to have effected a Copernican revolution in psychology seems at present premature.

(*e*) A satisfactory demonstration of his hypothesis would really require a knowledge of the theoretical values of the constants of the tetrad distribution curve, and the comparison of these by aid of their probable errors with the observed values. Even after the present investigation we must admit that we only know one constant and its approximate probable error.

Professor Spearman has surmounted the difficulty by assuming without any demonstration whatever that the distribution of tetrads rendered symmetrical follows a Gaussian or normal curve. There are grave theoretical reasons against this assumption. Since the correlation coefficient, especially for small samples, does not follow a normal curve, even when the variates follow a normal surface, there is no likelihood that a tetrad formed of four intercorrelated coefficients will do so. Still less likely is it that a number of correlated tetrads, each single one picked out from a distribution non-normal and with its own standard deviation (to which it is not even reduced!), will do so. The improbability of a tetrad distribution being a normal curve is illustrated on Holzinger's and again on Simpson's data. The former distribution is platykurtic and the latter leptokurtic, and this suggests, if it fails to prove, that each group of mental abilities will have its own peculiar type of distribution of tetrads.

(*f*) Dealing at length with Holzinger's data and Simpson's data we fail to find in them any adequate proof whatever that the agreement of the observed values with those required by theory is "surprisingly close," or that "In general it seems quite as good as, if not better than, that usually reached in determining the mechanical equivalent of heat and thus establishing the law of conservation of physical energy*."

Yet while very seriously doubting the validity of the arguments, both the accordances said to exist between theory and observation and the mathematical treatment of Professor Spearman, we believe that the hypothesis of a general factor should be further tested on new material. The nature of the non-overlapping mental abilities should be selected by psychological consensus before the tests are made, the number of persons tested should be far greater than in the case of Holzinger's or Simpson's data, trained computers should be employed, and if possible a more adequate mathematical theory of the whole subject developed. Then we might have a better chance either of dismissing the whole theory, or showing that it was worth while spending further energy in developing it. At present we can only return, but return definitely, a verdict of non-proven.

* *The Abilities of Man*, p. 160.

GOITRE IN ADOLESCENCE; AN ANTHROPOMETRIC STUDY OF THE RELATION BETWEEN THE SIZE OF THE THYROID GLAND AND PHYSICAL AND MENTAL DEVELOPMENT.

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CONTENTS.

(1) Foreword.

This research was undertaken in an endeavour to throw some light on the following problems:

(i) What is the best method of measuring the thyroid gland?

(ii) How does the size of the gland change about the period of adolescence?

(iii) What significance can be attached to the so-called "physiological enlargement" often noticed at this period?

(iv) Is there any real distinction between these slight enlargements and so-called "goitre" or is the difference merely one of degree?

(v) What relation exists between the size of the gland and physical and mental development about the adolescent period?

(vi) What association, if any, has the size of the gland with blood pressure and pulse rate?

I do not think that the voluminous literature on the thyroid gland supplies a really satisfactory answer to any of these questions. There is no doubt that too

little attention has been paid to the behaviour of the normal thyroid during growth and to the initial stages of goitre. There is a vague idea that a small degree of enlargement is to be regarded as a natural physiological reaction, in girls at least, during adolescence, and it is generally believed that most of these enlargements are evanescent. A medical man who has worked all his life in parts of Devonshire, Somersetshire, Lancashire or Cheshire will have naturally come to regard most adolescent thyroid enlargements in girls as physiological because he sees them so commonly. And yet one has only to go into a London school and search for enlarged thyroids and then into a school at Stockport or Exeter to become immediately convinced that this physiological reaction, unless it can be accounted for by racial differences, must be a reaction to some external *local* factor and not to factors inherent in the organism itself.

If we admit that, it behoves us to determine whether the presence or absence of this reaction denotes some fault in metabolism, arising from local causes, which may have some influence on normal development, for although we may be satisfied that small enlargements are not associated with symptoms of ill-health we also need to satisfy ourselves that their presence or absence does not imply retardation of full physical or mental development. This is of more than academic interest because, leaving the vexed question of the causation of goitre on one side, we have at our disposal, if necessary, a method of largely reducing the incidence of thyroid enlargement in children, and evidence of the influence of this prophylactic administration of iodine on physical growth also has been provided in a recent paper (1).

I wish here to express my thanks to the Medical Department of the Board of Education, to the County Medical Officers of Health of Cheshire, Devonshire, Somersetshire, Denbighshire and Surrey, to the Medical Officers of Health of Stockport, Taunton, Bridgwater, Exeter, Barnstaple, Torquay, Plymouth, Bath and Tiverton and to the School Medical Department of the London County Council for kindly giving me facilities for this work, and also to the Head Teachers of the many schools visited for their ungrudging assistance in the carrying out of the actual measurements. The work of measuring and testing 440 girls in Cheshire schools was shared by my brother, Dr A. Vernon Stocks, whose name therefore appears at the head of this paper. I am also indebted to Professor Karl Pearson for many fruitful suggestions and to Miss McLearn for drawing the diagrams.

(2) METHODS OF MEASUREMENT.

(*a*) *Introductory.* The first problem lay in the choice of a suitable and convenient method of measuring the size of the thyroid gland. The Swiss Goitre Commission lay down in their *Instruction concernant la Mensuration du corps thyroïde* (Bern, 1922) a division into five categories which may be roughly defined as follows: Category 0, gland not palpable; Category I, can be palpated but outlines are not visible in any position of the neck; Category II, outlines of gland readily seen with head thrown back or on swallowing but enlargement insufficient

to seriously alter the profile of the neck; Category III, sufficient enlargement to produce an evidently "full neck"; Category IV, undoubted goitre. Such a scheme, as they pointed out, has certain advantages over actual mensuration because (i) there is a fraction of cases in which the gland cannot be felt at all, or so imperfectly that its boundaries are difficult to define with accuracy, and (ii) when the gland is soft in consistence measurements depend to some extent on the pressure exerted by the measuring instrument.

The first objection to mensuration is not a serious one for our purpose because only a small proportion of children at ages over 10, in this country at any rate, have impalpable thyroids; in my own English series of 850 girls aged 11 and upwards actually measured I met with 20 such. At earlier ages it would be a more serious objection. In forming any distribution on a scale of measurement this small group can be suitably disposed of at the beginning of the scale. The second objection does not in my experience lead to serious error, particularly if the linear measurement of thyroid breadth be used; it may render difficult the measuring of height of the lobes, but as I shall presently show the use of this latter measurement is generally unnecessary.

The possible methods of actual measurement are:

(i) the circumference at the base of the neck, passing the tape over the spinous process of the 7th cervical vertebra behind and the hollow above the sternum in front;

(ii) the maximum antero-posterior diameter measuring from the spinous process of the 7th cervical vertebra;

(iii) the maximum breadth of the gland, measured by callipers;

(iv) the height of each lobe;

(v) Hunziker (2) has made use of the product of the last two measurements as a measure of the superficial area covered by the gland, taking the mean height of the two lobes as the vertical measurement when these are unequal.

I have used all these methods during this research in order to determine which is the most satisfactory for general use and for establishing standards. The results of the comparison of methods will now be discussed.

(*b*) *Estimation of thyroid size by visual judgment.* In a survey of school children aged 12 made by the School Medical Department of the Board of Education in 1924 the following definition of thyroid enlargement was used: "children in whom the thyroid is sufficiently enlarged for the increase in the size of the neck to be noticed on casual inspection without measurement or palpation." In order to correlate the present work with an analysis of the results of that survey, which will be published elsewhere (9), it was essential to make one of the boundaries between the categories I used in this work coincide with my own idea of the beginning of enlargement according to the above definition. I therefore used four categories as follows:

Category 0, comprising thyroids which could not be palpated, identical with Category 0 of the Swiss Goitre Commission.

Category 1, comprising thyroids which were palpable but presented no visible enlargement on inspection.

Category 2, comprising thyroids which showed visible enlargement, but not sufficient to produce gross deformity in the shape of the neck.

Category 3, comprising thyroids enlarged sufficiently to be termed "goitre" beyond question.

The boundary between 0 and 1 is definite enough, but the numbers found in Category 0 were too small to deal with separately, so the first two categories are usually referred to together as "Category 01." The boundary between 1 and 2 coincides with the limit set for the Board of Education's Medical Officers, and I have proved it to be subject to a large personal equation for different observers; my own interpretation of it coincides with my interpretation of the boundary between Categories I and II of the Swiss Goitre Commission. The boundary between Categories 2 and 3 is necessarily vague and would be subject to a still larger personal equation; the same applies however to the boundaries between II, III and IV of the Swiss Goitre Commission which I find difficult to interpret in practice. To summarize, we have the following scheme of comparison:

Categories used by author	Definition used in Board of Education's Survey, 1924	Swiss Goitre Commission's Categories
01 { 0	"Normal"	0
01 { 1		I
2	"Enlarged"	II
		III
3		IV

My own division into three groups corresponds to what I should at the outset of this work have regarded as "Normal" thyroids, "Physiological enlargements," and "Goitres," without attaching any real significance to the distinction between the last two.

A possible advantage of a method of visual judgment over actual measurements would be that the size of the child might be taken into consideration in forming a judgment; that is to say, it might be expected that the eye would judge of the apparent size of the gland not as an absolute measure but in comparison with the size of the neck, and that on the average it would be found that the limiting size of thyroid at which enlargement appeared to begin would increase

in proportion to the child's height. If this were the case the method would offer distinct advantages since correction for height would be unnecessary. Although my personal impression was that I was unconsciously correcting for the size of the child in this way when classifying by categories, analysis of the results showed that this impression was fallacious. The mean heights of the groups were as follows:

TABLE I.

	Mean Height	Standard Deviation	Probable Error
Boys. Category 01 group ...	139·27 cm.	7·501 cm.	± ·29
Categories 2—3 group ...	141·26 „	7 318 „	± ·35
Difference	1·99 „	—	± ·45
Girls. Category 01 group ...	140·88 cm.	7·842 cm.	± ·62
„ 2 „ ...	142·19 „	6·868 „	± ·30
„ 3 „ ...	144·79 „	6·022 „	± ·49
Difference 01 and 3 ...	3·91 „	—	± ·80

These differences are significant and the correlation coefficients* with height in boys and girls respectively are ·1658 ± ·0373 and ·1137 ± ·0302. In Section (5) it is shown that thyroid breadth has a correlation of ·1528 ± ·0293 in boys and ·2141 ± ·0265 in girls with height after age correction. This suffices to show that, as a matter of fact, height is unconsciously either not at all or only partially corrected for in judging enlargement by the category method, and to what degree it is allowed for it is impossible to say. The usefulness of any method of categories based on visual judgment is therefore definitely limited by the following considerations: (i) comparison between results from different observers using, however carefully, the same definition proves the presence of a very large personal equation, as will be shown in another paper†; (ii) unless an observer is working continuously he cannot be sure that he is himself maintaining the same standards after an interval of even a few months; (iii) it is impossible to make adequate correction for height. In the present research the method proved invaluable for a special purpose, but for future work it cannot be recommended except for the purposes of a *rapid and continuous survey to be made by a single worker.*

(c) *Estimation of thyroid size by circumference or diameter of the neck.* These measurements were made on girls only. Those measured in Cheshire were from Elementary Schools for the most part whilst those measured in London were taken from Central Schools. Since the second group were perhaps selected physically to some extent the effect of age on the neck circumference and diameter has been determined separately. For the purposes of obtaining curves for age correction only girls classified into Category 01 could be used and the means for

* By the biserial method. † See Reference, No. 9.

age-groups of these girls were fitted with straight lines over the limited range of ages except in the case of neck circumference in Cheshire girls, where a parabola gave a better fit. The curve formulae and smoothed means are shown in Table II.

TABLE II.

	Age in years exactly					
	11	*12*	*13*	*14*	*15*	*16*
Neck Diameter (cm.)						
151 Cheshire girls ($d = 4{\cdot}254 + {\cdot}3481a$)	8·08	8·43	8·78	9·13	9·47	—
68 London girls ($d = 5{\cdot}887 + {\cdot}2349a$)	—	—	8·94	9·18	9·41	9·65
Neck Circumference (cm.)						
151 Cheshire girls ($c = 6{\cdot}192 + 2{\cdot}955a - {\cdot}08815a^2$)	28·0	29·0	29·7	30·3	30·7	—
68 London girls ($c = 23{\cdot}092 + {\cdot}6283a$)	—	—	31·3	31·9	32·5	33·1

The mean heights, weights, neck and thyroid measurements of these two groups of "normal" girls, all corrected to age 13 by means of their separate growth curves (see Sections 5, 6), were as follows:

TABLE III.

	Height	Weight	Neck Diameter	Neck Circumference	Thyroid Breadth
Cheshire girls ...	146·8 ± ·4	37·7 ± ·3	8·78 ± ·03	29·71 ± ·09	3·42 ± ·03
London girls ...	148·0 ± ·6	38·5 ± ·5	8·94 ± ·04	31·26 ± ·14	3·28 ± ·04

Although the means were slightly higher for the London girls the differences were insignificant for height and weight and barely significant for neck diameter and thyroid measurements, but for neck circumference the difference of 1·55 ± ·16 was almost ten times its probable error. Moreover the ratios of circumference to "diameter" are respectively 3·38 and 3·49, which differ to the extent of 3 °/₀. The London measurements were carried out at an interval of some months from the Cheshire series, and I believe these differences to be accounted for by the impossibility of maintaining a constant technique in the measurement of neck circumference, where so much depends on the levels at which the tape is kept. This experience strengthened a view I already held that neck circumference is entirely useless as a measure of thyroid enlargement, except perhaps to note the progress of a large goitre in the same individual from time to time by the same observer. This is further confirmed by the following correlation coefficients which have been calculated for the whole series of 540 girls after age correction.

TABLE IV.

		Neck Diameter	Neck Circumference
With thyroid breadth	Category 01 only ...	·2220 ± ·0475	·2989 ± ·0455
	Categories 2—3 only ...	·6536 ± ·0336	·5378 ± ·0450
	Complete sample ...	·4279 ± ·0246	·2951 ± ·0275
With height, Category 01 only		·5168 ± ·0368	·5607 ± ·0346

If the thyroid and neck measurements be corrected for height by the method described in Section 5 (*e*) the resulting coefficients with thyroid breadth are:

TABLE V.

	Neck Diameter	Neck Circumference
Category 01 only ...	$r = \cdot 0876$	$r = \cdot 1714$
Categories 2—3 only ...	$r = \cdot 6557$	$r = \cdot 5393$
Complete sample ...	$r = \cdot 4151$	$r = \cdot 2740$

Of the two neck measurements the antero-posterior diameter gives the better index of thyroid size, but the magnitude of these coefficients is such that in dealing with any but large goitres, neither measurement is of much value for this purpose, being inferior to visual estimation when used by a single observer (see Table VIII).

(*d*) *Actual measurements of the thyroid gland.* The instrument used consisted of one fixed and one sliding arm, each carrying ivory points, the distance between the points being indicated on a scale on the rod carrying the sliding arm, the latter having a vernier attachment. The boundaries of the gland having been defined by palpation with the thumb and forefinger of the left hand without undue pressure, the points of the instrument were brought into apposition with the boundaries and the distance between them read off. Any accurate pair of callipers would suffice for such measurement.

The maximum breadth, and the height of each lobe, were thus measured. Hünziker's surface area measure was also determined by multiplying the breadth by the arithmetic mean of the heights of the lobes; usually the heights of the lobes did not differ*. In my experience there was little difficulty in measuring breadth, but often considerable difficulty in defining the upper limits of the lobes with certainty. It seemed doubtful therefore whether any advantage was gained by introducing the additional vertical measurements over the simple measurement

* In 62 girls the right lobe was larger than the left, whereas in only nine cases was the left larger; there is some tendency for the right to enlarge first.

of breadth alone. The latter has the advantage of being easier to deal with in computation and undoubtedly is subject to a smaller personal equation than the area measurement. It is possible to decide which is the better measure to use from an examination of the distributions of the same children on both scales and their relation to age and visual estimates of size.

TABLE VI.

Mean Thyroid Measurements on Girls.

	Age	Thyroid Breadth; cm. Smoothed Values			Thyroid Area; sq. cm. Smoothed Values		
		Mean	S.D.	Coefficient of Variation	Mean	S.D.	Coefficient of Variation
Normal Thyroids, Category 01	*10*	3·29	·400	12·2	7·50	1·60	21·3
	11	3·29	·405	12·3	7·60	1·66	21·9
	12	3·30	·433	13·1	7·66	1·75	22·9
	13	3·30	·482	14·6	7·75	1·95	25·2
	14	3·31	·508	15·4	8·20	3·20	39·0
	15	3·92	·517	13·2	12·00	3·70	30·8
	16	4·03	·518	12·8	13·10	2·90	22·1
Slightly Enlarged Thyroids, Category 2	*10*	4·15	·59	14·2	11·58	3·50	30·2
	11	4·18	·61	14·6	13·44	4·10	29·7
	12	4·47	·73	16·3	15·09	5·00	33·1
	13	4·71	·93	19·8	16·51	5·80	35·1
	14	4·89	·93	19·0	17·71	5·82	32·9
	15	4·96	·78	15·7	18·68	5·70	30·5
	16	5·00	·62	12·4	19·44	5·50	28·3
Definite Goitres, Category 3	*10*	7·12	·72	10·1	36·37	8·79	24·2
	11	7·12	·76	10·7	36·00	8·79	24·4
	12	7·12	·90	12·6	35·62	8·79	24·7
	13	7·09	1·10	15·5	35·24	8·79	24·9
	14	6·96	1·17	16·8	34·87	8·79	25·2
	15	6·90	1·03	14·9	34·89	8·79	25·5
	16	6·90	·82	11·9	34·12	8·79	25·7

The effect of age on thyroid growth is described in Section (4); it will suffice here to tabulate the smoothed means and standard deviations and the coefficients of variation at each age for comparison (Table VI)*. It appears that whereas the coefficient of variation of thyroid breadth averages about 15, that of thyroid area averages about 27; for both measures it has a maximal value about ages 13—14. Theoretically the coefficient of variation of the *square of the breadth* would be 29†.

* The data from which these are derived are set out in Table XV.

† [If V_{x^2} be the coefficient of variation of x^2 and V_x of x, then

$$V_{x^2} = 2V_x \sqrt{1+\tfrac{1}{2}(V_x/100)^2}/(1+(V_x/100)^2) = 2V_x \text{ approximately. E.S.P.]}$$

The relations of the two measures with Category are shown in Table VII after correcting to age 12.

TABLE VII.

Thyroid Breadth, corrected to age 12.

Central values in cm.

	2·5	*3·0*	*3·5*	*4·0*	*4·5*	*5·0*	*5·5*	*6·0*	*6·5*	*7·0*	*7·5*	*8·0*	*8·5*	*9·0*	Totals
Boys. Category 01 ...	17	82	144	65	2	—	—	—	—	—	—	—	—	—	310
" 2 ...	—	—	3	61	67	40	10	5	1	—	—	—	—	—	187
" 3 ...	—	—	—	—	—	—	—	—	4	1	1	2	—	1	9
Girls. Category 01 ...	28	68	91	29	5	—	—	—	—	—	—	—	—	—	221
" 2 ...	—	2	22	67	65	42	20	20	9	2	1	1	—	—	251
" 3 ...	—	—	—	—	1	3	5	11	3	11	17	6	8	3	68

Thyroid Area, corrected to age 12.

Central values in sq. cm.

	5	*10*	*15*	*20*	*25*	*30*	*35*	*40*	*45*	*50*	*55*	*60*	*65*	Totals
Girls. Category 01 ...	102	110	9	—	—	—	—	—	—	—	—	—	—	221
" 2 ...	3	70	96	44	26	9	2	1	—	—	—	—	—	251
" 3 ...	—	—	3	4	10	11	13	14	6	2	4	—	1	68

The resulting correlation coefficients* computed by the triserial method, obtaining the means of the general categories from the probability integral†, were as follows:

TABLE VIII.

	Boys	Girls
Thyroid Breadth, age corrected, with Category:		
With class index correction $r=$	·8182 ± ·0099	·9118 ± ·0049
Without " " $r=$	·6746 ± ·0163	·6825 ± ·0155
Thyroid Area, age corrected, with Category:		
With class index correction $r=$	—	·9040 ± ·0053
Without " " $r=$	—	·6746 ± ·0157

* In the case of the girls the actual numbers in Category 01 were raised to bear the correct ratio to the other categories before computing r, since the 221 who were measured represented a random sample of girls in this category in the schools visited; this is explained in Section (5).

† An alternative method was used for girls by finding the slope of the line fitted by least squares to the three means and comparing with the regression line; this gave $r = ·9050$ for thyroid breadth with Category. We only need to determine the order of magnitude of these coefficients; owing to heterogeneity their values are somewhat dependent on the prevalence of goitre in the district sampled.

The general conclusion is that thyroid "area" offers no advantages over thyroid breadth. The two variables are very highly correlated, and the latter being very simple to measure is recommended as the best index of size to use in biometric work, or for comparative purposes.

(3) FORM OF FREQUENCY DISTRIBUTION OF THYROID SIZE AMONGST CHILDREN AGED 12.

This must obviously be different in a goitre district to a non-goitre district, and the manner in which it changes as goitre increases ought to throw some light on the relation between the so-called "physiological" enlargements and goitres. In order to attack the problem directly and completely it would be necessary for a single observer to measure the thyroids of, say, 500 children of each sex in a dozen districts of differing goitre prevalence, a large undertaking which was not practicable. For the present an approximate method was used which, though open to objections, was believed to be sufficiently accurate for the immediate problems it was designed to meet. The general method of attack for all the purposes in view was as follows:

(α) 440 girls aged 10—15 in schools in the neighbourhoods of Runcorn, Stockton Heath, Lymm, Northwich and Knutsford in Cheshire, where goitre incidence is considerable, and at Altrincham where it is lower, were measured as regards the thyroid and physical and mental factors, and in addition 100 girls in London County Council schools. In the Cheshire group of schools about 15 °/$_{\circ}$ of the girls had thyroid enlargement according to my own standard, and therefore since the time available was limited all girls showing any enlargement (Categories 2 and 3) in the school visited were first measured and a random sample was then taken from the normals (Category 01) who remained as time permitted. No conscious selection was exercised within the limits of each of these groups, and it was therefore possible to work out mathematical expressions for the frequency distributions on a scale of thyroid breadth of each of the categories of girls separately.

(β) In the course of an attempt to assess the extent of the personal equation of different School Medical Officers I was able to rapidly survey unselected samples of 150 boys and 150 girls in ten different districts of Somersetshire, Devonshire and Surrey and to record the numbers of each age 11, 12 and 13 who came within the three categories, using the same standards as before. The complete details of this survey will be published elsewhere (9).

(γ) By making the approximate assumption that within the boundaries of each category the distribution of thyroid sizes would not differ seriously from district to district it was possible, by combining the three frequency distributions ascertained in (α) in the proportions observed in the three categories in a locality as in (β), to roughly construct the distribution of thyroid sizes for that locality.

(δ) 533 boys aged 11—14 were measured in rural schools in the neighbourhood of Dartmoor and the S. Devonshire coast, and here the goitre incidence was so high

that it was practicable to measure *every* boy of these ages in the schools visited. Separate frequency distributions for the categories were fitted with curves as for the girls, and the complete distribution resulting from the approximate method (γ) compared with the total observed distribution in a sample area.

(ϵ) A similar check on the validity of the approximate method for girls was secured later by rapidly measuring the thyroid breadths alone of a complete school; this was carried out in Stockport by the measurement of 372 girls.

It will suffice here to reduce to a mathematical form the frequency distributions in the three categories on a scale of thyroid breadth, and to give one or two examples of the estimated composite curves. For obvious reasons it is difficult in rural schools to obtain large enough numbers by confining attention to a single year of age, hence the method used has been to measure children aged 11, 12 and 13 last birthday, calculate separate moments for each age-group and combine them, giving equal weight to each of the three ages; the resultant moments and curves are taken as applicable to the age-group "12."

Category 01. (Normal thyroids according to my judgment.)

Boys. The frequencies and resulting constants are given in Table IX.

TABLE IX.

Thyroid breadth. Central values	Age last Birthday			Percentage frequency. Ages equally weighted
	11	*12*	*13*	
2·37 cm.	1	—	—	6·40
2·62 "	8	3	7	
2·87 "	8	10	9	8·93
3·12 "	20	19	19	19·25
3·37 "	19	26	18	20·76
3·62 "	21	26	24	23·48
3·87 "	13	18	13	14·50
4·12 "	7	5	7	6·68
4·37 "	—	—	1	
Totals	97	107	98	100·00

Taking 3·37 group as origin and the class interval as unit, for age 11, $\nu_1' = \cdot 04124$, $\nu_2' = 2\cdot 84536$; age 12, $\nu_1' = \cdot 27103$, $\nu_2' = 2\cdot 14019$; age 13, $\nu_1' = \cdot 17347$, $\nu_2' = 2\cdot 78571$. Combining the ages equally $\nu_1' = \cdot 16258$ and transferring to the mean as origin and applying Sheppard's correction, $\mu_2 = 2\cdot 48066$. Hence mean breadth $= 3\cdot 4106$ cm., $\sigma = \cdot 39375$, and fitting a normal curve this gives for unit total frequency

$$y_1 = 1\cdot 01319 e^{-\frac{(x - 3\cdot 4106)^2}{\cdot 31008}} \qquad \text{.........................(1 a)},$$

where x is the thyroid breadth in cm.

This curve when tested for goodness of fit with the frequencies in the last column of Table IX raised to the total number of boys measured (302) gives for seven groups $\chi^2 = 5{\cdot}66$ and $P = {\cdot}46$, so the fit is satisfactory as shown in Figure 1 *a*.

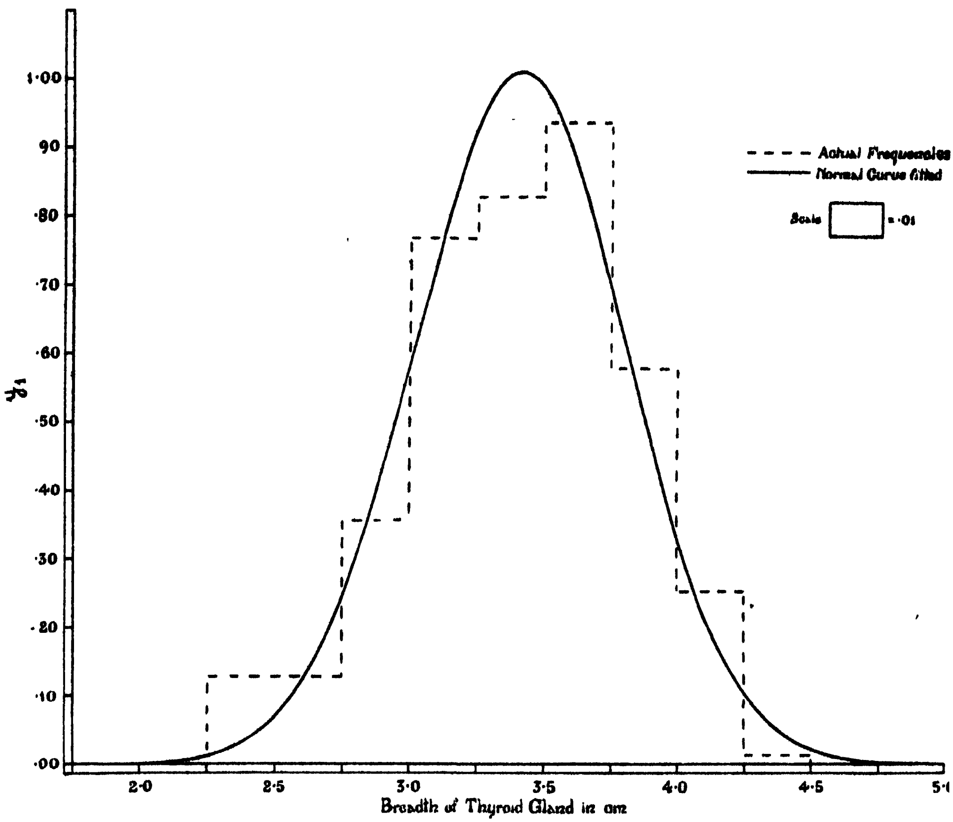


Fig. 1 *a*. Frequency distributions of Thyroid Breadth in Boys aged 11—18 whose thyroids were considered by author to be normal or subnormal (Category 01).

Girls. Different class intervals were used owing to the difficulty of disposing of the group of impalpable thyroids. See Table X.

Taking 3·495 group as origin and the class interval as unit and proceeding as above, for ages 11—13 we have $\nu_1' = -{\cdot}35462$, and transferring to the mean as origin and applying Sheppard's correction, $\mu_2 = 65453$. Hence mean breadth $= 3{\cdot}3177$ cm., $\sigma = {\cdot}40452$ cm., and fitting a normal curve this gives for unit total frequency

$$y_1 = {\cdot}9\ 621 e^{-\frac{(x-3{\cdot}3177)^2}{{\cdot}32726}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1\ b).$$

Testing for goodness of fit* with smaller class intervals as shown in Figure 1 *b*, for six groups $\chi^2 = 5{\cdot}55$ and $P = {\cdot}36$, and for five groups $\chi^2 = 3{\cdot}54$, $P = {\cdot}48$. The

* The total frequency 161 was used. This total is perhaps not strictly applicable, owing to the method of equally weighting the age-groups. Hence this and some of the other values of χ^2 calculated in this section must be treated as approximations only.

TABLE X.

Thyroid breadth. Central values	Age last Birthday			Percentage frequency. Ages equally weighted
	11	*12*	*13*	
2·495 cm.	4	2	14	10·95
2·995 „	10	7	28	27·20
3·495 „	17	16	41	48·55
3·995 „	3	5	13	12·96
4·495 „	—	—	1	·34
Totals	34	30	97	100·00

misfit at the start of this curve can be explained by the disposal of all the impalpable thyroids into the first group whereas some of them doubtless belonged to the second.

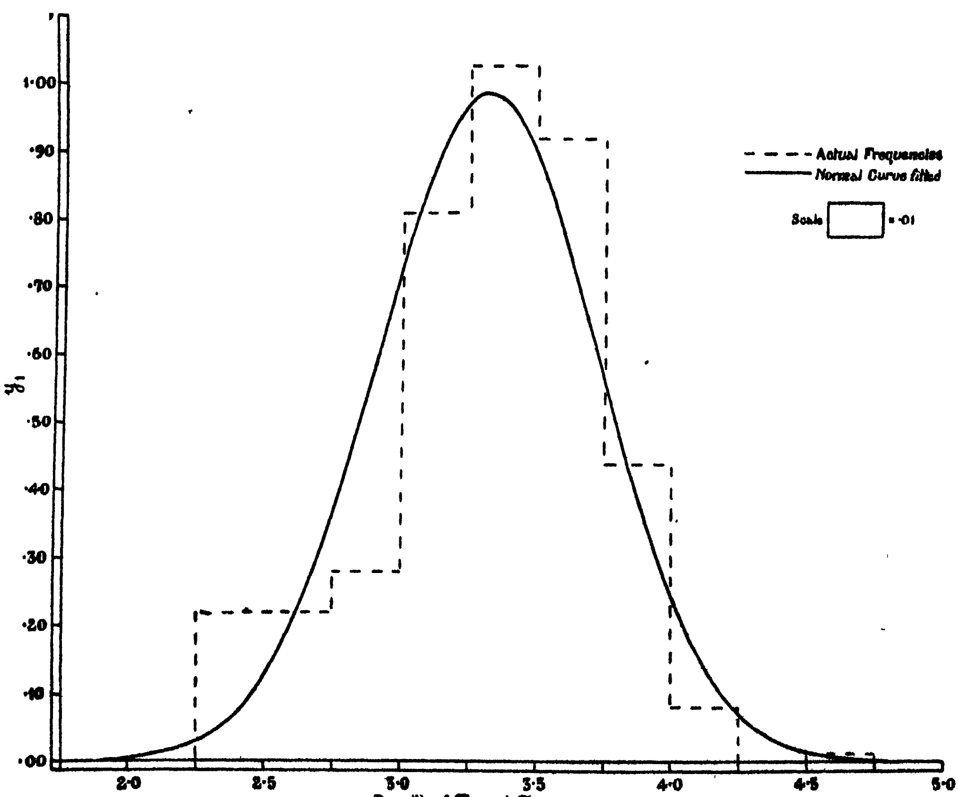


Fig. 1 *b*. Frequency distributions of Thyroid Breadth in Girls aged 11—13 whose thyroids were considered by author to be normal or subnormal (Category 01).

Category 2. (Thyroids slightly or moderately enlarged.) See Table XI.

Boys. Combining the three ages equally the resulting moments without applying Sheppard's correction are, from 4·495 as origin $\nu_1' = \cdot 12231$, and referred to

TABLE XI.

Thyroid breadth Central values	Boys			Girls			Percentage frequencies. Ages equally weighted	
	Age last Birthday			Age last Birthday				
	11	*12*	*13*	*11*	*12*	*13*	Boys	Girls
3·495 cm.	—	2	1	7	7	6	1·45	11·30
3·995 "	17	27	18	16	12	27	31·35	28·25
4·495 "	23	19	29	9	20	26	36·30	26·35
4·995 "	14	13	12	4	13	12	20·10	13·95
5·495 "	3	7	4	3	4	10	7·00	7·85
5·995 "	—	2	4	2	7	9	2·85	8·25
6·495 "	—	—	2	—	2	6	·95	4·05
6·995 "	—	—	—	—	—	2	—	
7·495 "	—	—	—	—	—	1	—	
Totals	57	70	70	41	65	99	100·00	100·00

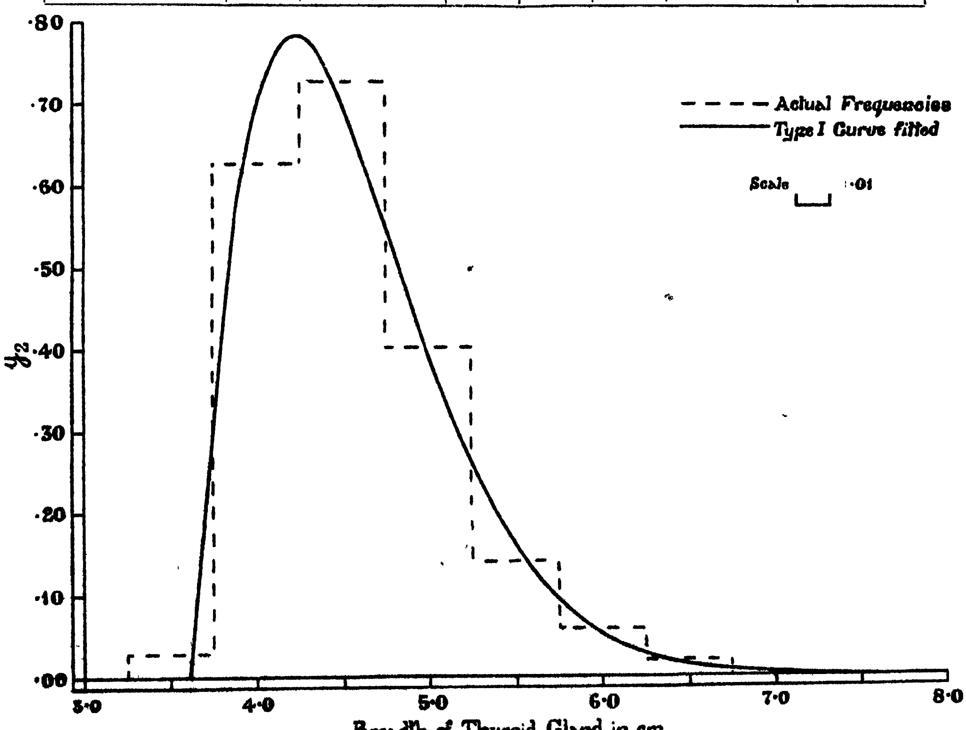


Fig. 2 a. Frequency distribution of Thyroid Breadth in Devonshire Boys aged 11—13 whose thyroids were considered by author to exhibit slight or moderate enlargement (Category 2).

the mean $\nu_2 = 1{\cdot}24603$, $\nu_3 = 1{\cdot}25426$, $\nu_4 = 5{\cdot}88869$, giving $\beta_1 = {\cdot}81319$, $\beta_2 = 3{\cdot}7928$. Hence a Type I Pearsonian skew curve can be fitted. Proceeding to fit such a curve with the above constants without attempting to apply correction for abruptness we have in the usual notation $r = 13{\cdot}9102$, $b = 5{\cdot}8830$ cm., $a_2 = 5{\cdot}2786$ cm., $a_1 = {\cdot}6044$ cm., $m_2 = 10{\cdot}6886$, $m_1 = 1{\cdot}2216$. Hence the mode is at $4{\cdot}2200$ cm., the mean at $4{\cdot}5561$ cm., the curve starts at $3{\cdot}6156$ cm. and ends at $9{\cdot}4986$ cm. and has the formula, for unit total frequency,

$$y_2 = {\cdot}78140 \left(\frac{x - 3{\cdot}6156}{{\cdot}6044}\right)^{1{\cdot}2216} \left(\frac{9{\cdot}4986 - x}{5{\cdot}2786}\right)^{10{\cdot}6886} \quad \ldots\ldots\ldots\ldots (2\,a).$$

Testing for goodness of fit as before, for seven groups $\chi^2 = 1{\cdot}02$, $P = {\cdot}98$, so the fit is excellent as shown by Figure 2 *a*.

Girls. Combining the ages equally the resulting moments without Sheppard's corrections and without abruptness corrections are, from 4·495 as origin $\nu_1' = {\cdot}21128$, and referred to the mean $\nu_2 = 2{\cdot}58080$, $\nu_3 = 3{\cdot}33804$, $\nu_4 = 21{\cdot}6968$. Thus without high contact or abruptness corrections $\beta_1 = {\cdot}64822$, $\beta_2 = 3{\cdot}25752$ and a Type I curve can be fitted as for the boys. The constants of this are $r = 6{\cdot}7541$, $b = 5{\cdot}2938$ cm., $a_2 = 4{\cdot}6076$ cm., $a_1 = {\cdot}6362$ cm., $m_2 = 4{\cdot}1828$, $m_1 = {\cdot}5713$. Hence the mode is at $4{\cdot}0052$ cm., the mean at $4{\cdot}6006$ cm., the curve starts at $3{\cdot}3691$ cm. and ends at $8{\cdot}6129$ cm., and its formula for unit total frequency is

$$y_2 = {\cdot}53367 \left(\frac{x - 3{\cdot}3691}{{\cdot}6362}\right)^{{\cdot}5713} \left(\frac{8{\cdot}6129 - x}{4{\cdot}6076}\right)^{4{\cdot}1828} \quad \ldots\ldots\ldots\ldots\ldots (2\,b).$$

Testing for goodness of fit as before, for seven groups $\chi^2 = 6{\cdot}95$, $P = {\cdot}33$, and for six groups $\chi^2 = 4{\cdot}12$, $P = {\cdot}53$, which is satisfactory, as shown in Figure 2 *b*.

Category 3. (Large goitres.) See Table XII.

TABLE XII.

Thyroid breadth. Central values	Boys	Girls	
	Ages 11—14	Ages 11—14	
4·495	—	2	2
4·995	—	3	6
5·495	—	3	
5·995	—	12	15
6·495	3	3	
6·995	2	12	29
7·495	1	17	
7·995	1	6	14
8·495	—	8	
8·995	1	3	3
Totals	8	69	

The means and standard deviations for separate ages from 11 to 14 showed little or no change with age in this group so they have been combined. In boys the number in this category was naturally too small to give any indication of the form of frequency distribution, but since the corresponding distribution for girls can be satisfactorily represented by a normal curve, this type of curve has also been fitted to the scanty data for boys (8 only!) for the sake of uniformity in the two sexes, using the actual measurements to calculate the moments. Thyroids of this size are so infrequent in boys that the precise method of their disposal along the tail of the compound curve makes little difference in any calculations.

Boys. Mean breadth 7·2612 cm. $\sigma = {\cdot}7754$ cm.

$$y_3 = {\cdot}51451 e^{-\frac{(x-7\cdot 2612)^2}{1\cdot 2024}} \qquad \text{.........................(3 a).}$$

Girls. Mean breadth 7·0457 cm. $\sigma = 1{\cdot}0943$ cm.

$$y_3 = {\cdot}36455 e^{-\frac{(x-7\cdot 0457)^2}{2\cdot 3952}} \qquad \text{.........................(3 b).}$$

This last curve tested for goodness of fit gives for six groups $\chi^2 = 2{\cdot}80$, $P = {\cdot}73$.

It was found impossible after many attempts to fit any simpler form of mathematical curve satisfactorily to the combined data for Categories 2 and 3; the separate treatment of these two categories must not be taken to imply any difference in nature or causation between them—in fact I can find no evidence to

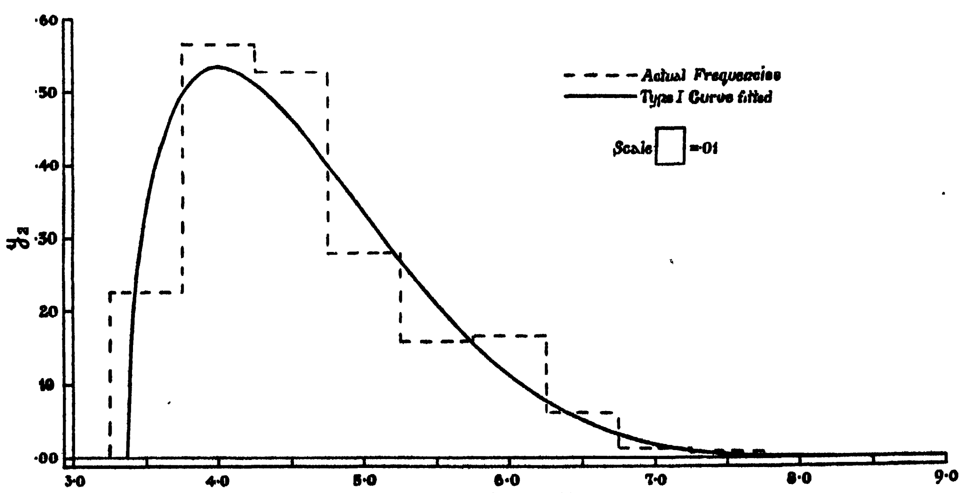


Fig. 2 *b*. Frequency distribution of Thyroid Breadth in Cheshire Girls aged 11—13 whose thyroids were considered by author to exhibit slight or moderate enlargement (Category 2).

suggest that the distinction between small "physiological" enlargements and the larger goitres as usually met with in this country is more than one of degree—the method here used is merely an expedient to obtain a nearer approximation to the form of the total frequency distribution in mathematical form. As a matter

of fact a collection of the available evidence bearing on the question, which will be published elsewhere (9), leads to the conclusion that in comparing districts of differing goitre prevalence all degrees of enlargement above the average estimate of normality in children aged 12 are found to be present in approximately constant proportions within the limits of sampling errors; that is to say each large goitre is attended by a fairly constant proportion of small goitres. For my own delimitation of Categories 2 and 3 it appears that the relative proportions in which they occur would on this supposition approximate to 9 to 2 in girls and 49 to 1 in boys if large enough numbers were examined.

If γ_1, γ_2, γ_3 be the proportions per unit total frequency in the Categories 01, 2, 3 respectively met with in a given district, an approximation to the complete frequency distribution in that district is arrived at by

$$y = \gamma_1 y_1 + \gamma_2 y_2 + \gamma_3 y_3 \quad \text{.............................(4)},$$

where y_1, y_2, y_3 are given by equations (1), (2), (3). In Figure 3 this has been applied as an example to a fairly homogeneous area bordering the S. Devonshire

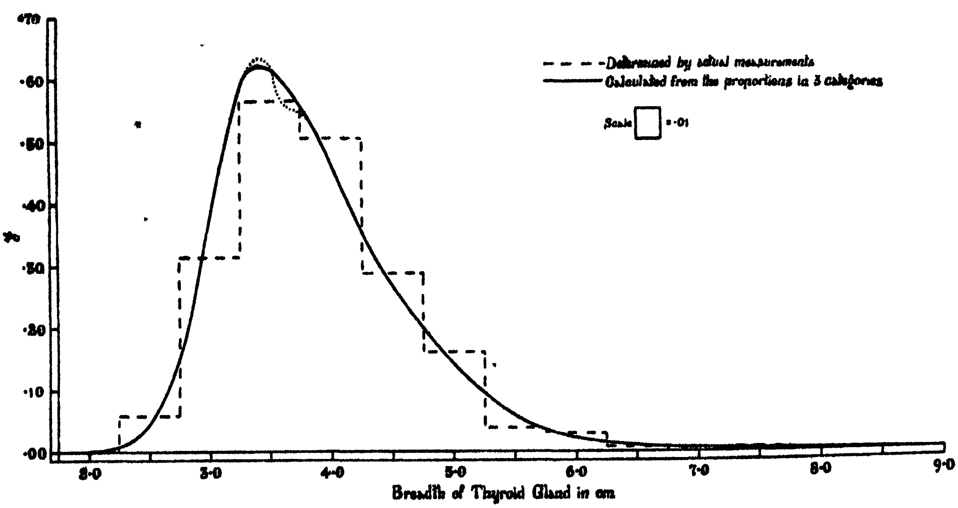


Fig. 3. Frequency distributions of thyroid breadth in 249 boys aged 11—13 at schools near the S. Devonshire coast.

coast in which an examination of 249 unselected boys aged 11—13 gave $\gamma_1 = \cdot 624$, $\gamma_2 = \cdot 365$, $\gamma_3 = \cdot 011$. The calculated curve* is drawn for comparison with the observed frequencies, which were:

Thyroid breadth. Central values	*2·5*	*3*	*3·5*	*4*	*4·5*	*5*	*5·5*	*6*	*6·5*	*7*	*7·5*
Frequency per unit total	·029	·157	·283	·253	·144	·092	·019	·015	·004	—	·004

* Slight smoothing is called for at the crest of the curve, but this has no appreciable effect on calculations for goitre rates below 50 %.

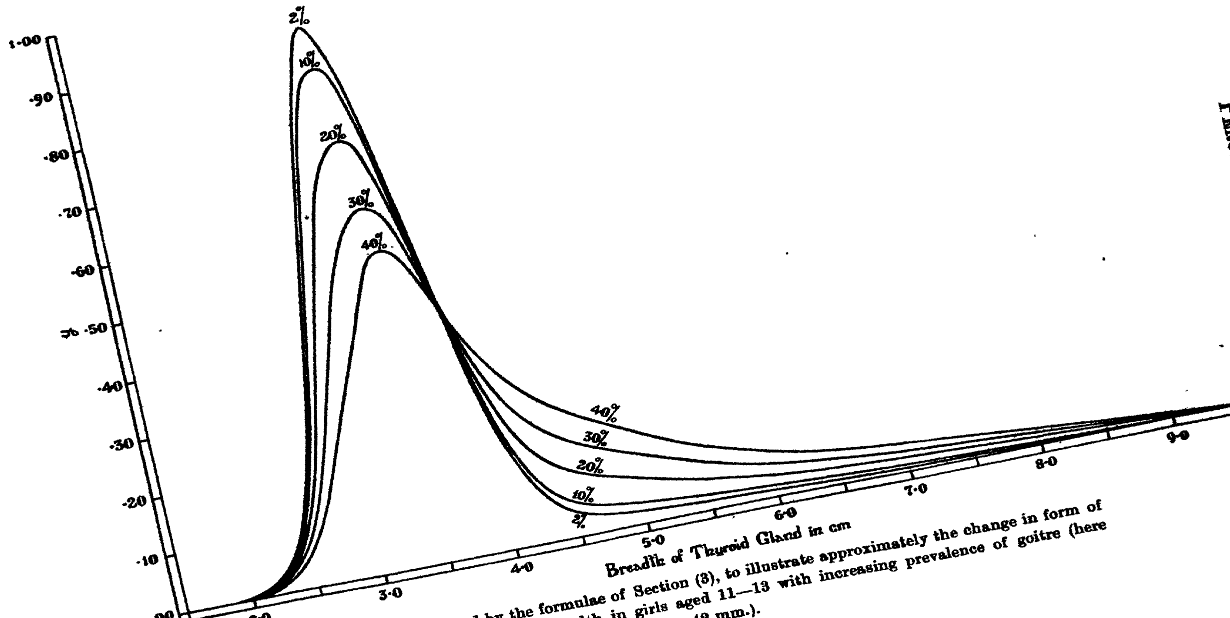


Fig. 4. Diagram constructed by the formulae of Section (3), to illustrate approximately the change in form of the frequency distribution of thyroid breadth in girls aged 11—13 with increasing prevalence of goitre (here defined as percentage having a thyroid breadth exceeding 42 mm.).

When tested for goodness of fit, raising the total to 249, with eight groups $\chi^2 = 4{\cdot}75$, $P = {\cdot}69$.

If only the total rate of thyroid enlargement, g_0, according to my own standard be known, $g_0 = \gamma_2 + \gamma_3$, $\gamma_1 = 1 - g_0$, and on the assumption of constancy in the ratio $\gamma_2 : \gamma_3$ as estimated above the complete distributions would be

$$\text{for boys} \quad y = (1 - g_0)\, y_1 + {\cdot}98 \; g_0 y_2 + {\cdot}02 \; g_0 y_3 \quad \text{.....................(5 a)},$$

$$\text{for girls} \quad y = (1 - g_0)\, y_1 + {\cdot}818 g_0 y_2 + {\cdot}182 g_0 y_3 \quad \text{.....................(5 b)}.$$

As a result of a standardizing survey which I carried out in conjunction with 19 School Medical Officers in Devonshire, Somersetshire and Surrey I found that g_0 might be related to the average observer's estimate, $\bar{g}_r$, of the rate of thyroid enlargement in the same sample of children aged 11—13 by the formulae

$$\text{for boys} \quad g_0 = 1{\cdot}1047 \bar{g}_r + {\cdot}0327 \quad \text{...........................(6 a)},$$

$$\text{for girls} \quad g_0 = 1{\cdot}0357 \bar{g}_r + {\cdot}0521 \quad \text{...........................(6 b)},$$

or if it was desired to express the rate more usefully in terms of a suitable limiting thyroid breadth such as 42 mm., if g_s be the proportion of boys or girls aged 11—13 with thyroid breadth exceeding the standard 42 mm., then

$$\text{for boys} \quad g_s = {\cdot}680 g_0 + {\cdot}0225 \quad \text{.............................(7 a)},$$

$$\text{for girls} \quad g_s = {\cdot}673 g_0 + {\cdot}0147 \quad \text{.............................(7 b)}.$$

On the basis of these last equations and the preceding assumptions the curves in Figure 4 have been drawn to show the kind of way in which the distribution of sizes changes with increasing goitre prevalence. As a check on their validity I have used the measurements made on unselected Stockport girls after these curves had been constructed; these are given in Table XVII. The mean percentage of girls, aged 11, 12, 13, whose thyroid breadths exceeded 42 mm. was 19·12. Hence from equation (7 *b*) $g_0 = {\cdot}262$ and from equation (5 *b*) the resulting curve is given by $y = {\cdot}738 y_1 + {\cdot}214 y_2 + {\cdot}048 y_3$, which is drawn in Figure 6 for comparison with the observed frequencies in girls aged $11\frac{1}{2}$ to $13\frac{1}{2}$. The test for goodness of fit to this series of 131 girls (see Table XIX) gives for six groups $\chi^2 = 4{\cdot}68$, $P = {\cdot}46$.

The above method of treatment will no doubt seem clumsy or even indefensible to some biometricians but the nature of the problem presents unique features which have not as yet been explored by exact methods, and the present exploration of it is merely tentative to prepare the ground for further work.

(4) Relation of thyroid size to age.

The construction of a growth curve for the thyroid gland is a matter of some difficulty owing to (i) rapid changes in goitre incidence with age about the period of puberty, (ii) the fact that practically all measurements of the gland have been made in areas where goitre is prevalent, (iii) uncertainty as to what limits should be set to "normality" in the size of the gland at each age, (iv) difficulty in measuring the gland at ages before nine owing to its smallness. It is first necessary to be clear as to the *changes in goitre incidence with age*. In Table XIII data from various sources have been collected and made comparable by expressing the frequency at each age of "goitre" or "thyroid enlargement" in terms of the

TABLE XIII. *Incidence of Thyroid Enlargement at various ages expressed in terms of the frequency at age 12 measured by the same observer in the same locality.*

Locality	Observer	Criterion	Age last Birthday													Total examined
			5—6	*7*	*8*	*9*	*10*	*11*	*12*	*13*	*14*	*15*	*16*	*17—19*	*20—25*	
Girls																
Bern	Kerzmann and	Categories	—	·537	·495	·766	·756	·832	1	1·066	1·057	1·178	1·171	—	—	4038
Neuchatel, etc.	de Quervain (3)	III—IV	—	·528	·553	·765	·796	·913	1	1·228	1·344	—	—	—	—	981
Lancashire, 1924	Holroyd ...	Visibly	·086	·228	·300	·471	·612	·632	1	·938	1·125	—	—	—	—	3346
Derbyshire, Heanor ...	Turton ...	enlarged	·738	·800	·890	·880	·738	·880	1	1·143	1·472	1·348	1·363	1·291	1·122	2047
Cheshire, N.E., 1924 ...	A. V. Stocks ...	thyroids	—	—	·192	—	—	1·0	1	1·231	—	—	—	—	—	—
Stockport, 1926	P. Stocks ...		—	—	—	—	·729	·838	1	1·045	1·059	1·099	1·229	—	—	362
Somerset and Devon ...	P. Stocks ...	Cat. 2—3	—	—	—	—	—	1·008	1	1·159	—	—	—	—	—	1440
Shropshire, 1924			·110	—	·252	—	—	—	1	—	—	—	—	—	—	5610
Gloucestershire, 1911—21	School Medical	Visibly	·097	—	—	—	—	—	1	—	—	—	—	—	—	40,932
Yorkshire, E. Riding, 1924	Officers	enlarged	—	—	·430	—	—	—	1	1·208	—	—	—	—	—	3044
Isle of Wight, 1924 ...		thyroids	—	—	—	—	—	—	1	1·060	—	—	—	—	—	741
Glossop, 1924—25 ...	Milligan ...		—	—	·057	—	—	—	1	—	—	—	—	—	—	403
Cincinnati, U.S.A., 1924 ...	Olesen (6) ...	Enlarge-	·407	·496	·593	·673	·764	·837	1	1·074	1·082	1·028	1·109	—	—	21,009
Minnesota, U.S.A., 1924 ...	,, (4) ...	ment of	—	—	—	—	—	·818	1	1·064	1·078	1·034	1·057	·989	—	2260
Connecticut, U.S.A., 1925	,, (5) ...	any degree	—	—	—	—	—	·809	1	1·220	1·212	1·295	1·334	1·375	—	6603
Mean rate	—	—	·288	·518	·418	·708	·732	·857	1	1·120	1·181	1·164	1·210	1·218	1·122	—
Boys																
Bern	Kerzmann and	Categories	—	·579	·655	·869	·874	·922	1	1·041	1·227	1·327	1·388	—	—	3527
Neuchatel, etc. (1) ...	de Quervain (3)	III—IV	—	·318	·830	·630	·925	·857	1	1·101	1·040	—	—	—	—	1001
Lancashire, 1924	Holroyd ...	Visibly enlarged	·210	·249	·712	·531	·783	·634	1	·992	1·048	—	—	—	—	3247
Cheshire, N.E., 1924 ...	A. V. Stocks ...	thyroids	—	—	·548	—	—	·809	1	·952	—	—	—	—	—	—
Somerset and Devon ...	P. Stocks ...	Cat. 2—3	—	—	—	—	—	·944	1	·921	—	—	—	—	—	2016
Shropshire, 1924			—	—	·536	—	—	—	1	—	—	—	—	—	—	5534
Gloucestershire, 1911—21	School Medical	Visibly	·300	—	—	—	—	—	1	—	—	—	—	—	—	41,131
Yorkshire, E. Riding, 1924	Officers	enlarged	—	—	·537	—	—	—	1	·774	—	—	—	—	—	3239
Isle of Wight, 1924 ...		thyroids	—	—	—	—	—	—	1	1·368	—	—	—	—	—	754
Glossop, 1924—25 ...	Milligan ...		—	—	·273	—	—	—	1	—	—	—	—	—	—	358
Cincinnati, U.S.A., 1924 ...	Olesen (6) ...	Enlarge-	·505	·559	·653	·752	·825	·876	1	1·042	·943	·822	·737	—	—	21,299
Minnesota, U.S.A., 1924 ...	,, (4) ...	ment of	—	—	—	—	—	1·023	1	·942	1·010	·725	·721	·508	—	1691
Connecticut, U.S.A., 1925	,, (5) ...	any degree	—	—	—	—	—	1·757	1	1·171	1·286	·914	·700	·500	—	5787
Mean rate	—	—	·338	·426	·593	·695	·852	·978	1	1·030	1·092	·947	·887	·504	—	—

frequency at age 12—13 measured by the same observer in the same locality. It must be admitted that these figures show great variation at the early ages even when so treated, but nevertheless the mean rates obtained by combining all the observations form a tolerably smooth series and have been fitted in Figure 5 with parabolae passing through unit rate at age 12·5 for boys and girls respectively. The equations of these curves are:

$$\text{Boys} \quad z = -1{\cdot}6924 + {\cdot}40065a - {\cdot}014821a^2 \quad \ldots\ldots\ldots\ldots\ldots (8\,a),$$
$$\text{Girls} \quad z = -1{\cdot}1855 + {\cdot}26754a - {\cdot}007416a^2 \quad \ldots\ldots\ldots\ldots\ldots (8\,b),$$

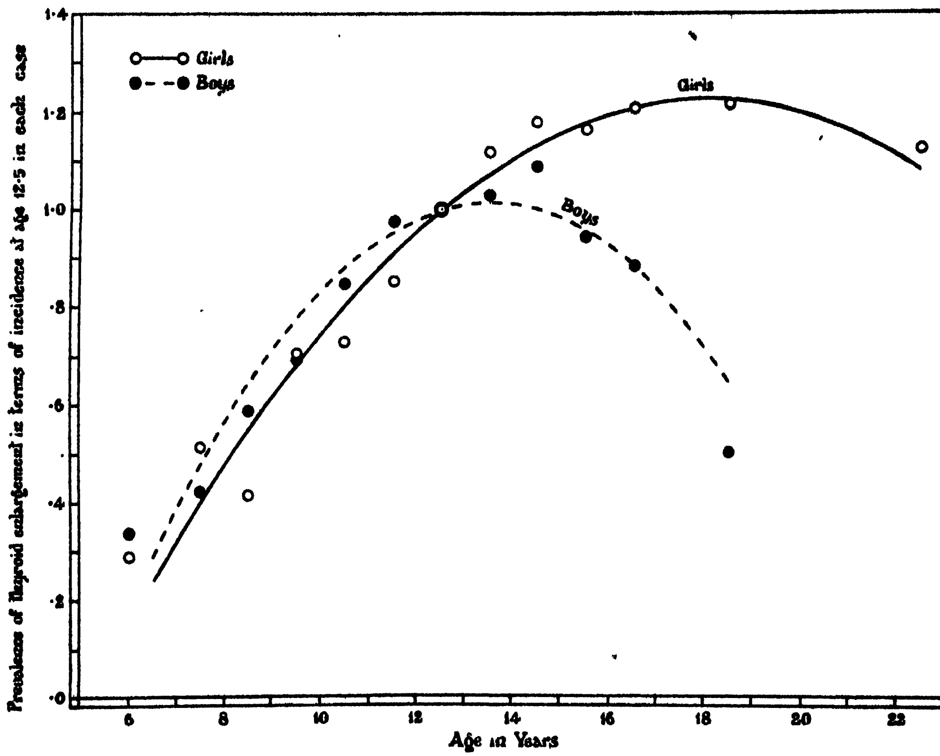


Fig. 5. Curves showing average effect of age on the frequency of thyroid enlargement.

where z is the prevalence in terms of that at age $12\frac{1}{2}$ and a is the age in years. The values of z at ages from 9·5 to 16·5 are shown in Table XIV.

TABLE XIV.

Age in Years	*9·5*	*10·5*	*11·5*	*12·5*	*13·5*	*14·5*	*15·5*	*16·5*
Boys ...	·776	·881	·955	1	1·015	1·001	·957	·883
Girls ...	·687	·806	·911	1	1·075	1·135	1·180	1·210

After these curves had been drawn, their close similarity with a pair of age curves relating to children at Timaru given by Hercus, Benson and Carter in their paper on goitre in New Zealand (7) was noticed; their maxima however

TABLE XV. *Breadth of Thyroid Gland in 581 Girls, and 645 Boys, classified by Categories.*

Category 01

	Central Age	Thyroid Breadth* 2·25–	2·75–	3·25–	3·75–	4·25–	4·75–	Totals	†Mean breadth, cm.
Girls	9·5‡	1	9	11	2	—	—	23	3·29
	10·75	—	1	—	1	—	—	2	3·29
	11·25	4	3	11	2	—	—	20	
	11·75	7	6	1	—	—	—	14	3·30
	12·25	1	4	8	1	—	—	14	
	12·75	1	3	8	4	—	—	16	
	13·25	7	7	15	5	1	—	35	3·30
	13·75	7	21	26	8	—	—	62	
	14·25	5	11	9	5	3	—	33	3·35
	14·75	—	2	4	3	1	1	11	3·80
	15·25	—	—	2	2	—	2	6	
	15·75	—	—	2	2	2	—	6	4·03
	16·25	—	—	—	—	1	—	1	
	20§	—	—	3	4	3	1	11	4·03
Boys	10·75	—	7	4	1	—	—	12	3·22
	11·25	4	9	18	9	—	—	40	3·41
	11·75	5	12	18	10	—	—	45	
	12·25	2	14	26	17	—	—	59	3·43
	12·75	1	15	26	6	—	—	48	
	13·25	3	14	27	12	1	—	57	3·41
	13·75	4	14	15	8	—	—	41	
	14·25	2	17	24	6	1	—	50	3·42
	14·75	—	6	16	6	—	—	28	
	15·25	—	3	9	1	1	—	14	
	15·75	—	1	7	6	—	—	14	3·54
	16·5	—	3	6	2	—	—	11	

Category 2

	Central Age	Thyroid Breadth* 3·25–	3·75–	4·25–	4·75–	5·25–	5·75–	6·25–	6·75–	7·25–	7·75–	Totals	†Mean breadth, cm.
Girls	9·5‡	3	3	4	—	—	—	—	—	—	—	10	4·12
	10·75	—	1	—	1	1	—	—	—	—	—	3	4·19
	11·25	4	7	1	1	1	—	—	—	—	—	14	
	11·75	3	9	8	3	2	2	—	—	—	—	27	4·47
	12·25	5	7	6	6	2	2	—	—	—	—	28	
	12·75	2	5	14	7	2	5	2	—	—	—	37	4·71
	13·25	3	14	11	7	4	2	1	1	—	—	43	
	13·75	3	13	15	5	6	7	5	1	1	—	56	4·89
	14·25	—	7	10	7	1	1	2	—	—	1	29	
	14·75	1	—	3	—	3	—	—	—	—	—	7	4·97
	15·25	—	—	1	—	1	—	1	—	—	—	3	
	15·75	—	1	2	—	1	1	—	—	—	—	5	
	16·25	—	—	—	—	—	—	—	—	—	—	—	—
	20§	—	—	—	—	—	—	—	—	—	—	—	—
Boys	10·75	—	—	2	—	—	—	—	—	—	—	2	4·49
	11·25	—	9	13	5	2	—	—	—	—	—	29	
	11·75	—	8	8	9	1	—	—	—	—	—	26	4·51
	12·25	1	15	9	5	3	2	—	—	—	—	35	
	12·75	1	12	10	8	4	—	—	—	—	—	35	4·53
	13·25	1	11	13	7	1	—	—	—	—	—	33	4·60
	13·75	—	7	16	5	3	4	2	—	—	—	37	
	14·25	—	3	2	1	—	—	—	—	—	—	6	
	14·75	—	1	5	2	2	—	—	—	—	—	10	4·55
	15·25	—	1	—	—	—	1	—	—	—	—	2	
	15·75	—	—	—	1	—	—	—	—	—	—	1	
	16·5	—	1	—	—	—	—	—	—	—	—	1	

Category 3

	Central Age	Thyroid Breadth* 4·25–	4·75–	5·25–	5·75–	6·25–	6·75–	7·25–	7·75–	8·25–	8·75–	Totals	†Mean breadth, cm.
Girls	9·5‡	—	—	—	—	—	—	—	—	—	—	—	—
	10·75	—	—	—	—	—	—	—	—	—	—	—	—
	11·25	—	—	—	1	—	—	1	—	—	—	2	7·12
	11·75	—	—	—	1	—	—	1	—	—	—	2	
	12·25	—	1	—	—	1	1	1	1	1	1	7	
	12·75	—	—	—	1	1	2	2	2	1	—	9	7·07
	13·25	1	1	1	3	1	4	5	2	1	2	21	
	13·75	—	1	1	3	—	1	4	1	2	—	13	
	14·25	1	—	1	1	—	3	3	—	3	—	12	6·90
	14·75	—	—	—	2	—	—	—	—	—	—	2	
	15·25	—	—	—	—	—	1	—	—	—	—	1	
	15·75	—	—	—	—	—	—	—	—	—	—	—	—
	16·25	—	—	—	—	—	—	—	—	—	—	—	—
	20§	—	—	—	—	—	—	—	—	—	—	—	—
Boys	10·75	—	—	—	—	—	—	—	—	—	—	—	—
	11·25	—	—	—	—	—	—	—	—	—	—	—	—
	11·75	—	—	—	—	1	—	—	—	—	—	1	
	12·25	—	—	—	—	1	—	—	—	—	—	1	
	12·75	—	—	—	—	—	—	—	—	—	—	—	7·27
	13·25	—	—	—	—	—	—	—	1	—	1	2	
	13·75	—	—	—	—	2	1	1	—	—	—	4	
	14·25	—	—	—	—	—	1	—	1	—	—	2	
	14·75	—	—	—	—	—	—	—	—	—	—	—	—
	15·25	—	—	—	—	—	—	—	—	—	—	—	—
	15·75	—	—	—	—	—	—	—	—	—	—	—	—
	16·5	—	—	—	—	—	—	—	—	—	—	—	—

* Measured to nearest $\frac{1}{10}$ mm. † Calculated from tables with class intervals of ·25 cm. ‡ Stockport girls. § College students.

are reached a year or two earlier in both sexes. Some other relevant data are also available which have not been included in Table XIII; thus Dr F. M. Fonseca, Medical Officer of Health of Ebbw Vale, gives in his annual report for 1924 rates of prevalence at various ages suggesting a falling rate in girls after age 12, which if significant is exceptional.

Figure 5 shows that in girls there is a steady increase in goitre incidence till about the 16th year, little change from 16 to 19 and then a tendency to decrease after 19, as is indicated by Dr Turton's observations at Heanor, Derbyshire, which are the only continuous data of that kind I have found. In boys it seems certain that the maximum is reached earlier in most localities, though in Professor de Quervain's observations at Bern it was still apparently increasing at 16. My own observations in Somersetshire and Devonshire on some 2000 boys indicated a maximal incidence at age 12, and this is supported also by Dr Holroyd's observations on 3247 boys in Lancashire, by Dr A. V. Stocks' observations in Cheshire, by the findings of the School Medical Staff of the East Riding of Yorkshire, as well as by the Timaru figures referred to above. In the Isle of Wight however a significantly higher rate was found in boys at 13 than at 12.

All the observations used in constructing these curves are based on visual observation without measurement or palpation, that is by the same method essentially as was used by myself in classification by categories. Having obtained the mean breadth of the gland for each of my categories at various ages it is therefore possible, knowing the prevalence of enlargement according to my own standard at age 12 in a locality, to construct the mean curve of growth to be expected for thyroids in that locality. Thus if g_0 be the goitre rate according to my standard at age 12·5, $\bar{B}_1$, $\bar{B}_2$, $\bar{B}_3$ the mean thyroid breadths for Categories 01, 2 and 3, at a given age a, then the mean breadths in that locality at age a are given approximately by:

$$\text{Boys}\quad \bar{B} = (1 - zg_0)\,\bar{B}_1 + \cdot 98\ zg_0\bar{B}_2 + \cdot 02\ zg_0\bar{B}_3 \ldots\ldots\ldots\ldots\ldots (9\,a),$$

$$\text{Girls}\quad \bar{B} = (1 - zg_0)\,\bar{B}_1 + \cdot 818zg_0\bar{B}_2 + \cdot 182zg_0\bar{B}_3 \ldots\ldots\ldots\ldots\ldots (9\,b),$$

where the respective values of z are given by equations (8 *a*), (8 *b*) above. In Table XV are shown the thyroid breadth distributions at different ages, separating the three categories, and the mean breadths are shown in the same table. After smoothing by drawing freehand curves through these points the resulting values of $\bar{B}_1$, $\bar{B}_2$ and $\bar{B}_3$ are as shown in Table XVI.

To test the validity of the formulae (9) I have used the measurements made on the Stockport girls aged 9—17. These are set out in Table XVII, and for comparison a series of unselected Devonshire Boys* of ages 11—14 and Stockport boys aged 14—16 have been given in the same table. As shown at the end of Section (3) for this sample $g_0 = \cdot 262$, and substituting this value in equation (9 *b*) we obtain

$$\bar{B} = (1 - \cdot 262z)\,\bar{B}_1 + \cdot 214z\bar{B}_2 + \cdot 048z\bar{B}_3,$$

* The only locality where any selection was used in the Devonshire series was Chagford; hence the Chagford boys have been omitted from the complete table, which otherwise consists of the three category tables combined.

TABLE XVI.

Central Age		9·5	10·5	11·5	12·5	13·5	14·5	15·5	16·5
Boys	$\bar{B}_1$ cm.	—	—	3·42	3·42	3·42	3·43	3·54	3·54
	$\bar{B}_2$ "	—	—	4·49	4·52	4·59	4·57	4·55	4·55
	$\bar{B}_3$ "	—	—	7·27	7·27	7·27	7·27	7·27	7·27
Girls	$\bar{B}_1$ "	3·29	3·29	3·29	3·30	3·30	3·60	4·01	4·03
	$\bar{B}_2$ "	4·12	4·17	4·25	4·60	4·80	4·92	4·98	5·00
	$\bar{B}_3$ "	7·12	7·12	7·12	7·10	7·06	6·90	6·90	6·90

TABLE XVII.

Breadth of Thyroid Gland in Random Sample Populations of Girls and Boys in Goitre Districts.

Central Age	Thyroid Breadth, cm.														Totals	Mean Breadth*	No. with breadth over 42 mm.
	2·25–	2·75–	3·25–	3·75–	4·25–	4·75–	5·25–	5·75–	6·25–	6·75–	7·25–	7·75–	8·25–	8·75–			
	Stockport Girls																
9·5	1	9	14	5	4	—	—	—	—	—	—	—	—	—	33	3·544	4
10·5	—	6	16	5	2	—	—	1	—	—	—	—	—	—	30	3·612	3
11·5	—	12	19	5	4	—	1	—	—	—	1	—	1	—	43	3·760	7
12·5	—	8	23	16	3	—	—	—	1	—	2	—	—	1	54	3·907	8
13·5	—	14	47	13	10	4	1	1	5	2	—	1	1	—	99	3·996	26
14·5	—	6	24	12	4	1	1	1	1	—	—	—	1	—	51	3·924	9
15·5	—	3	15	13	1	—	1	2	1	2	—	2	—	1	41	4·419	13
16·5	—	—	2	5	1	—	1	1	—	—	—	1	—	—	11	4·620	4
17·5	—	—	4	—	3	1	—	—	—	1	—	1	—	—	10	4·695	6
	Devonshire and Denbighshire Boys																
10·75	—	7	4	1	2	—	—	—	—	—	—	—	—	—	14	3·461	2
11·25	4	9	18	18	11	5	2	—	—	—	—	—	—	—	67	3·844	39
11·75	5	12	18	18	7	9	1	—	1	—	—	—	—	—	71	3·835	
12·25	2	14	27	32	9	5	3	1	1	—	—	—	—	—	94	3·859	45
12·75	1	15	27	18	9	7	4	—	—	—	—	—	—	—	81	3·855	
13·25	3	14	28	23	14	7	1	—	—	—	—	1	—	1	92	3·916	59
13·75	4	14	15	15	16	5	2	4	4	1	1	—	—	—	81	4·157	
14·25	2	4	11	6	—	1	—	—	—	—	—	1	—	—	25	3·749	2
14·75	—	—	4	2	2	—	—	—	—	—	—	—	—	—	8		2
	Stockport Boys																
14·5	—	19	25	11	4	—	2	—	—	1	—	—	—	—	62	3·616	—
15·5	—	4	16	8	1	1	—	1	—	—	—	—	—	—	31	3·661	—
16·5	—	3	6	3	—	—	—	—	—	—	—	—	—	—	12		—

* Calculated from tables with class intervals of ·25 cm.

and further substituting the appropriate values of z from Table XIV at ages 9 to 16, this leads to the calculated breadths in Table XVIII.

TABLE XVIII.

Age	*9·5*	*10·5*	*11·5*	*12·5*	*13·5*	*14·5*	*15·5*	*16·5*
Mean thyroid breadth calculated	3·54	3·59	3·65	3·76	3·84	4·10	4·42	4·45
Actual mean breadth found ...	3·54 ± ·13	3·61 ± ·14	3·76 ± ·11	3·91 ± ·10	4·00 ± ·08	3·92 ± ·11	4·42 ± ·12	4·66 ± ·16

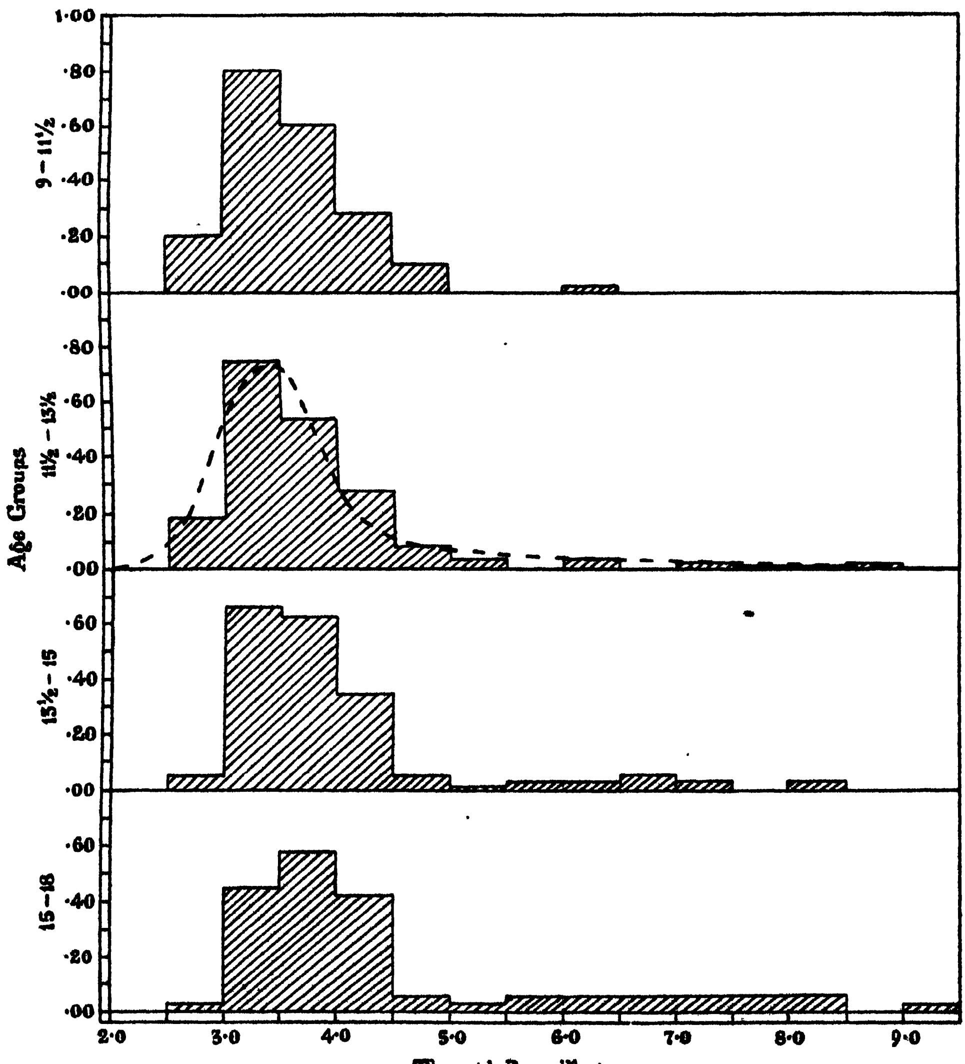


Fig. 6. Frequency distributions of thyroid breadth in Stockport girls at four age-groups. Distribution for ages 11½—18½ compared with frequency curve calculated from percentage at these ages having a thyroid breadth over 42 mm.

There is a reasonable measure of agreement between the calculated and actual means, which do not differ significantly at any age. The calculated curve is shown in Figure 7.

In the case of boys I have selected the town of Tiverton where I found in 154 boys, $\gamma_2 = \cdot 225$, $\gamma_3 = \cdot 011$, $g_0 = \cdot 236$, at ages 11—13. Hence

$$\bar{B} = (1 - \cdot 236z)\,\bar{B}_1 + \cdot 225z\bar{B}_2 + \cdot 011z\bar{B}_3,$$

and the resulting curve is shown in the lower part of Figure 7.

Hunziker (2) has tabulated the thyroid "superficial areas" of 353 untreated girls from a goitre region at ages 6—14, but these figures give us little information as to the *normal* growth of the thyroid, since both their actual mean values and the slope of the curve with age are functions of the local goitre prevalence and vary from one district to another. Thus the growth curves in Figure 7 for Stockport and Tiverton are only characteristic of those places or others with a similar goitre prevalence. The change in the form of frequency distribution with advancing age is illustrated in Figure 6, constructed from the Stockport data. The frequencies have been plotted for four age-groups, viz. 9—11½, 11½—13½, 13½—15, and 15—18, the actual distributions being as in Table XIX.

TABLE XIX.

Central thyroid breadths in cm.

Age-group	*2·75*	*3·25*	*3·75*	*4·25*	*4·75*	*5·25*	*5·75*	*6·25*	*6·75*	*7·25*	*7·75*	*8·25*	*8·75*	*9·25*	Totals
9—11½	8	32	24	11	4	—	—	1	—	—	—	—	—	—	80
11½—13½	12	49	35	18	5	3	—	3	—	2	1	1	2	—	131
13½—15	3	33	31	17	3	1	2	2	3	2	—	2	—	—	99
15—18	1	14	18	13	2	1	2	2	2	2	2	2	—	1	62

In the diagram these have been reduced to a scale of unit total frequency in each case.

The next question is, *what constitutes normality* in the thyroid growth curves? No one who has examined schools in London and Stockport would contend for example that the Stockport mean curve should be taken as a criterion for normality. In contrast to this curve I have computed the curves to be expected in two other large towns, Bath and Liverpool.

At *Bath* I examined samples of 242 boys and 254 girls aged 11—13 and found for boys $\gamma_2 = \cdot 080$, $\gamma_3 = \cdot 003$, and for girls $\gamma_2 = \cdot 208$, $\gamma_3 = \cdot 011$, from which by equations (9), putting $g_0 = \gamma_2 + \gamma_3$, the approximate growth curves drawn in Figure 7 are obtained.

For *Liverpool* the following figures resulted from the combined observations of 14 medical officers:

Boys aged 12; examined 4914, thyroid enlarged 19, rate ·0039,
Girls aged 12; „ 4679, „ „ 70, „ ·0149.

It can be shown by calculation from the range of the individual personal equation that such rates based on 14 observers are liable to a considerable error. Regarding these rates as due to an average observer, however, and transferring to my own standard by means of equations (6), the rates would be $g_0 = {\cdot}037$ for boys and ·067 for girls. The curve calculated on this basis for girls shows no appreciable increase in size of the gland from the 11th year to the 14th (see Figure 7) but a rapid increase from 14 to 16; the curve for boys does not differ sensibly from the "normal" curve and is not shown separately.

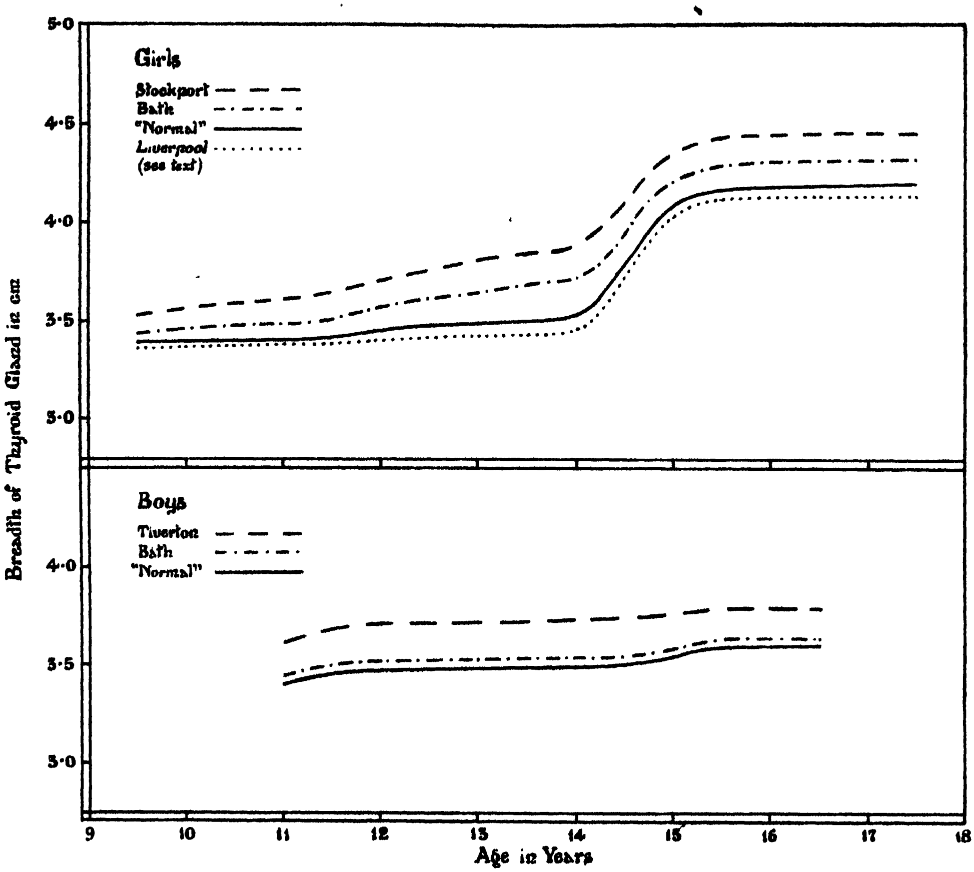


Fig. 7. Estimated mean growth curves of the thyroid gland in several towns, and "normal" growth curves for regions where goitre is not endemic.

Now Bath is situated in a region of rather high goitre prevalence and the Liverpool rates are subject to a considerable error. A better criterion on which to found a hypothetical "normal" growth curve is the mean rate for the whole of England and Wales outside the goitre area. After excluding all areas in which the goitre rate as determined by the 1924 survey was significantly high, we are left with a residual area in which 266,531 children were examined by 404 medical officers with resulting mean rates ·0149 for boys and ·0441 for girls. These

figures for the whole area, which have small probable errors, afford the best estimate of the amount of visible enlargement to be normally expected in a population of children aged about 12 and examined by an "average" observer. Transferring from the average standard to my own standard by equations (6) these rates are equivalent to $g_0 = \cdot 0493$ for boys and $\cdot 0977$ for girls, and on the same assumptions as before the *normal* curves of growth in thyroid breadth are approximately given by the equations

$$\text{Boys} \quad B = (1 - \cdot 049z)\,\bar{B}_1 + \cdot 048z\bar{B}_2 + \cdot 001z\bar{B}_3 \quad \ldots\ldots\ldots\ldots\ldots (10\,a),$$

$$\text{Girls} \quad \bar{B} = (1 - \cdot 098z)\,\bar{B}_1 + \cdot 080z\bar{B}_2 + \cdot 018z\bar{B}_3 \quad \ldots\ldots\ldots\ldots\ldots (10\,b).$$

These curves have been drawn in Figure 7 and the values at half-yearly intervals are given below:

TABLE XX.

Age	*9·5*	*10*	*10·5*	*11*	*11·5*	*12*	*12·5*	*13*	*13·5*	*14*	*14·5*	*15*	*15·5*	*16*	*16·5*
Boys	—	—	—	3·40	3·47	3·47	3·48	3·48	3·48	3·48	3·48	3·53	3·59	3·59	3·59 cm.
Girls	3·38	3·39	3·40	3·41	3·42	3·45	3·48	3·49	3·50	3·52	3·78	4·08	4·16	4·18	4·18 cm.

The conclusion is that under normal conditions the thyroid gland scarcely changes at all in size in *boys* from 11½ to 14, shows a slight increase from 14 to 16, and then remains stationary: from this it appears that the maximum size of the gland in boys has been almost attained by the 12th year, the relative increase in breadth from 11 to 15½ being only 5·6 °/ₒ as compared with 10 °/ₒ for stature. In *girls* under normal conditions the thyroid undergoes very slight increase in size from 10 to 13½ years but increases rapidly from the end of the 14th year to the middle of the 16th in the average girl, reaching the adult level about 16. To contrast its percentage rate of growth with the corresponding growth in stature during these periods, we have

	10—13½	*13½—15½*	*10—15½*
Per cent. increase in thyroid breadth	3·3	19	22·7
Per cent. increase in stature ...	14	7	21·5

The relative growth from 10 to 15½ years is therefore about the same as for height, but whereas growth in height is roughly uniform during the period, that of the thyroid gland shows a pronounced retardation preceding puberty compensated for by a phenomenally rapid growth during puberty. As far as I am aware no other linear measurement hitherto taken on the body has revealed such remarkable prepubescent retardation.

(5) Relation to height and dimensions of head and neck.

(*a*) *Material.* In Cheshire the measurements of height, weight and grip, including the extraction of all previous records available, and the entire arrangements for the school visits, were carried out by Dr A. V. Stocks. In Devonshire, where annual height and weight records are kept of all children, the last of these records was used, the date being recorded; since all measurements had first to be corrected to age 12 from the age at which they were taken, any error involved by such a procedure would be trifling after age correction. The annual records extending back to age 4 or 5 were also extracted for use in Section (7).

The number of girls measured in Cheshire was 442, and this total was augmented to 540 by girls measured in London schools. Each series was corrected to age 12 from its own cubic age curve, and the resultant means and standard deviations did not differ significantly for any measurement except neck circumference to which reference has been made in Section (2), so there was no objection to combining them. The boys measured in Devonshire numbered 461, this total being augmented to 533 by boys from rural schools at Llanrwst, Llangerniew and Eglwysbach in Denbighshire.

The measurements taken were—in girls: height, weight, neck diameter and circumference, thyroid breadth and height of lobes, eye and hair colour, strength of grip, pulse rate, systolic and diastolic pressures and proficiency in class; in boys: height, weight, horizontal circumference of head, thyroid breadth and strength of grip. The boys measured were unselected but only random samples of the girls in the normal category were measured, as already explained in Section 3 (*a*). The totals measured in Category 01 and Categories 2 + 3 were finally in the proportions ·56 to 1. The actual proportion in which these categories occurred in the aggregate school population drawn upon was approximately 5·7 to 1, which implied that every 10th girl of category 01 had been measured. To ensure that a complete sample of the population was being employed for correlation, separate tables were made for these two category groups, the table for Category 01 being then multiplied by 10 and combined with the table for Categories 2 + 3. The probable errors of the correlation coefficients were calculated on the basis of the total girls measured and not on the augmented totals. It is to be noted that both for boys and girls the group consisting of Categories 2 + 3 is more homogeneous than the completed sample in the sense that all the children in it had been exposed to the external goitre-producing factor whereas in the complete population this may not have been so and all may not have had the same chance of developing thyroid enlargement. In any case the complete sample is heterogeneous in the sense that it consists of a mixture of "normals" with "abnormals," and since the proportions in which these are mixed would differ in samples from different districts, the correlation coefficients for the complete samples would also vary, and those here calculated are only applicable to the districts sampled in this particular instance. For this and other reasons correlation coefficients have been worked out separately in all the succeeding sections (i) for Category

01 group, (ii) for Categories 2 + 3, (iii) for the complete sample, the last of these being of only limited value owing to its dependence on goitre prevalence.

(*b*) *Correction to age* 12. The correction of the thyroid breadth for age was first carried out by means of the age curves for the Category 01 group, the assumption being that since this represented its growth when not visibly enlarged, it was the best base line from which to compute degrees of enlargement. The points on this curve are tabulated along the lines $\bar{B}_1$ in Table XVI in Section (4). It was later realised that a better curve from which to calculate age corrections would be the "normal" growth curve deduced in Figure 7. For boys the resulting age corrections were so nearly identical that it made no difference which curve was used, but for girls the curves were appreciably different and hence all the series of coefficients with thyroid breadth corrected from the "normal" growth curve were calculated and are shown in Table XXVI; the coefficients do not differ significantly from the others. For brevity the Category 01 growth curve will be referred to as $B1$ and the "normal" growth curve as BN. The two series of age corrections at half-year periods were as follows, a rule representing zero and a dot no data:

TABLE XXI.

Corrections to age 12 *exactly, in cm.*

		10	10·5	11	11·5	12	12·5	13	13·5	14	14·5	15	15·5	16
Boys	$B1$	·	+ ·32	+ ·09	—	—	—	—	—	—	—	—	·	·
	BN	·	+ ·32	+ ·08	+ ·01	—	—	—	—	—	—	− ·02	·	·
Girls	$B1$	+ ·02	+ ·02	+ ·01	+ ·01	—	—	—	—	− ·01	− ·27	− ·62	− ·73	− ·75
	BN	+ ·06	+ ·05	+ ·04	+ ·03	—	− ·03	− ·04	− ·05	− ·07	− ·33	− ·63	− ·71	− ·73

(*c*) *Correlation with Height.* The growth curves for height and weight were constructed from the complete records of the children measured, going back in some cases to three years of age. For boys the total entries of height were 2438, of weight 2494; in girls the total entries were 1401 and 1395 respectively. The complete series of mean heights and weights are shown in Table XXII and to these cubic curves were fitted with the formulae

$$\text{Boys}\quad H = 72{\cdot}635 + 7{\cdot}3096a - {\cdot}23739a^2 + {\cdot}008092a^3 \ldots\ldots\ldots\ldots\ldots(11\,a),$$

$$\text{Girls}\quad H = 70{\cdot}857 + 6{\cdot}6641a - {\cdot}07158a^2 + {\cdot}000633a^3 \ldots\ldots\ldots\ldots\ldots(11\,b),$$

where H = height in cm., a = age in years.

These curves are drawn in Figure 8, the scale unit being the height at age 11 exactly. The actual mean heights reduced from Table XXII to this scale are shown also in Figure 8 as black dots in order to demonstrate that the cubic curves closely fit the data and have not had the effect of smoothing out pubescent irregularities to any appreciable extent. The cubic curves have been carried only to age 15, after which the curves have been continued in a freehand manner to

age 20 by means of other data in the laboratory. On the same diagram are shown the thyroid breadth curves, the scale unit being the breadth at age 11;

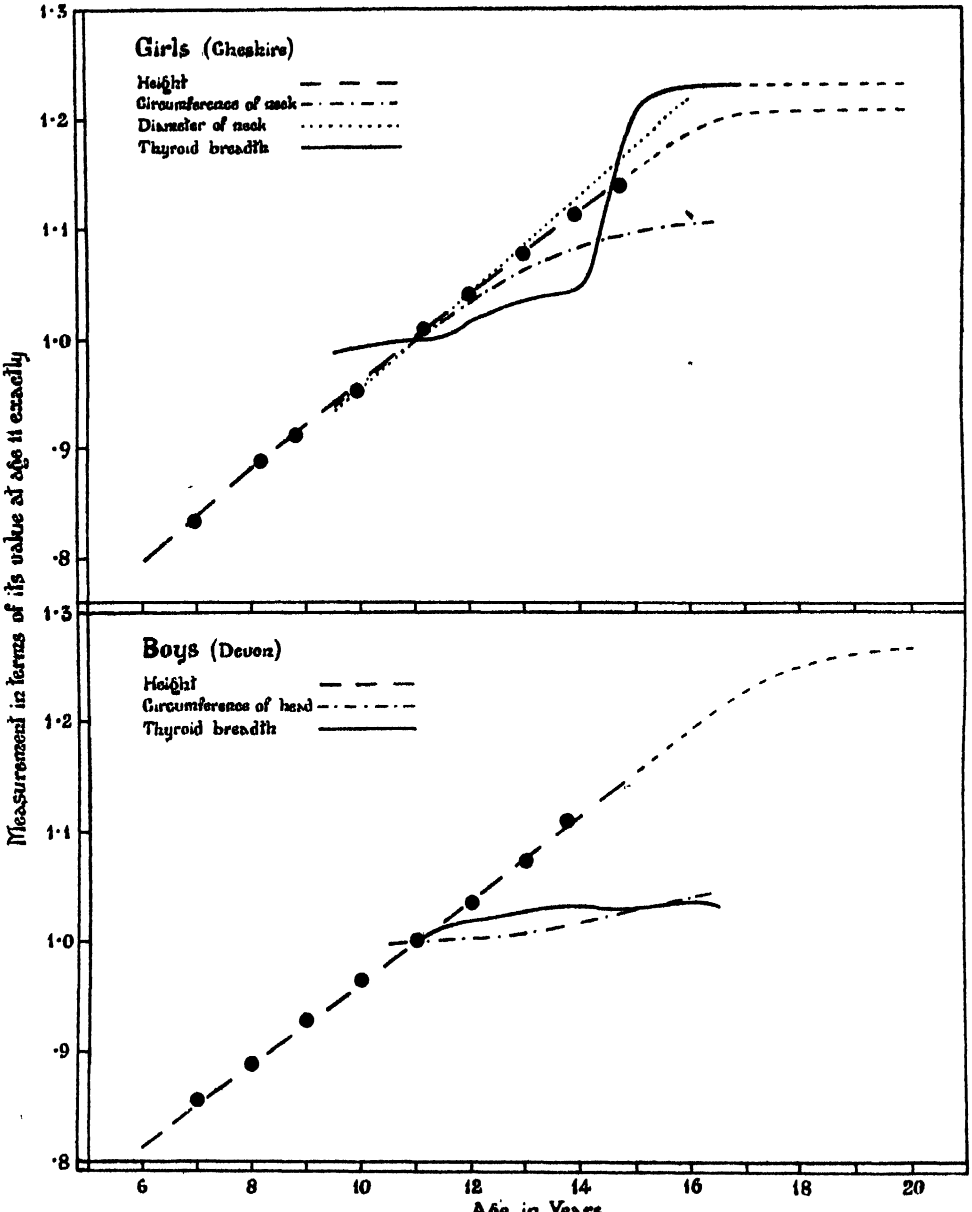


Fig. 8. Growth curves of thyroid gland contrasted with growth curves for other linear measurements in the same districts, the mean measurements at age 11 exactly being taken as scale unit in each case.

these curves are arrived at in the same way as the curves for Bath in Figure 7 and Section (4), and represent the average growth curves for the thyroid in the

school populations which were sampled for the purpose of this and the following sections. Their formulae, before converting to the scale unit of breadth at age 11, were

$$\text{Boys} \quad \bar{B} = (1 - \cdot 386z)\,\bar{B}_1 + \cdot 378z\bar{B}_2 + \cdot 008z\bar{B}_3,$$

$$\text{Girls} \quad \bar{B} = (1 - \cdot 150z)\,\bar{B}_1 + \cdot 123z\bar{B}_2 + \cdot 027z\bar{B}_3,$$

where the values of $\bar{B}_1$, $\bar{B}_2$, $\bar{B}_3$, z are tabulated in Tables XVI and XIV. The contrast with growth in height is readily seen when the curves are reduced to the same relative scale as in Figure 8.

Correction of height for age was carried out from curves (11) above; points on these curves from 10 to 16 are given in Table XXIII. After age correction the correlation coefficients between thyroid breadth and stature were

$$\text{Boys} \quad r = \cdot 1371 \pm \cdot 0294. \quad \text{Girls} \begin{cases} B1 \text{ curve } r = \cdot 2141 \pm \cdot 0298, \\ BN \quad \text{,,} \quad r = \cdot 2222 \pm \cdot 0291. \end{cases}$$

The values of η for stature on breadth were for boys $\eta = \cdot 2227$, girls $\eta = \cdot 2787$, these values being significant as are also the correlation coefficients above; thus for boys $\eta^2 - \bar{\eta}^2 = \cdot 02392$ and the probable error of $\bar{\eta}^2 = \cdot 0067$; the corresponding values for girls being $\cdot 07151$ and $\cdot 0034$. The mean heights at intervals of thyroid breadth are shown in Table XXIV.

Cubic curves have been fitted as regression lines of height on thyroid size, having the equations

$$\text{Boys} \quad H' = 91\cdot 854 + 27\cdot 7527B' - 4\cdot 98536B'^2 + \cdot 281859B'^3,$$

$$\text{Girls} \quad H' = 106\cdot 445 + 19\cdot 4829B' - 3\cdot 35291B'^2 + \cdot 191280B'^3,$$

where H' and B' are statures and thyroid breadths in cm. corrected to age 12. These are drawn in Figure 9 and it is evident that the slope of the regression lines is pronounced whilst the thyroid is not sensibly enlarged, but with moderate enlargement it becomes horizontal; for girls it again turns upwards when enlargement becomes considerable, but this last relation is doubtful in boys, the last point in the figure representing only one case. The correlations within the separate category groups—including respectively 310 and 196 boys and 180 and 306 girls—are as follows:

Category 01.	Boys $r = \cdot 2045 \pm \cdot 0367$,	Girls ($B1$) $r = \cdot 2909 \pm \cdot 0485$,
Categories 2 + 3.	,, $r = \cdot 0031 \pm \cdot 0482$,	,, $r = \cdot 2251 \pm \cdot 0366$.

If the size of the thyroid has no relation to height other than that arising from simultaneous growth in all linear measurements we should expect to find a positive correlation amongst the Category 01 or normal groups, and that this would become almost obliterated in passing to the group of goitrous children since the small variations in thyroid size due to height would become lost in the much larger variations due to the goitre factor. This is what actually occurs in boys, the coefficient being reduced from $\cdot 2045$ to $\cdot 0031$, but in girls it is initially larger, viz. $\cdot 2909$, and is only reduced to $\cdot 2251$ in the goitre group. This suggests that

whereas in boys the correlation is entirely due to the natural relationship existing between any two linear measurements in groups of children of the same age, in girls there is an additional relation involved which becomes more pronounced as the thyroid becomes larger.

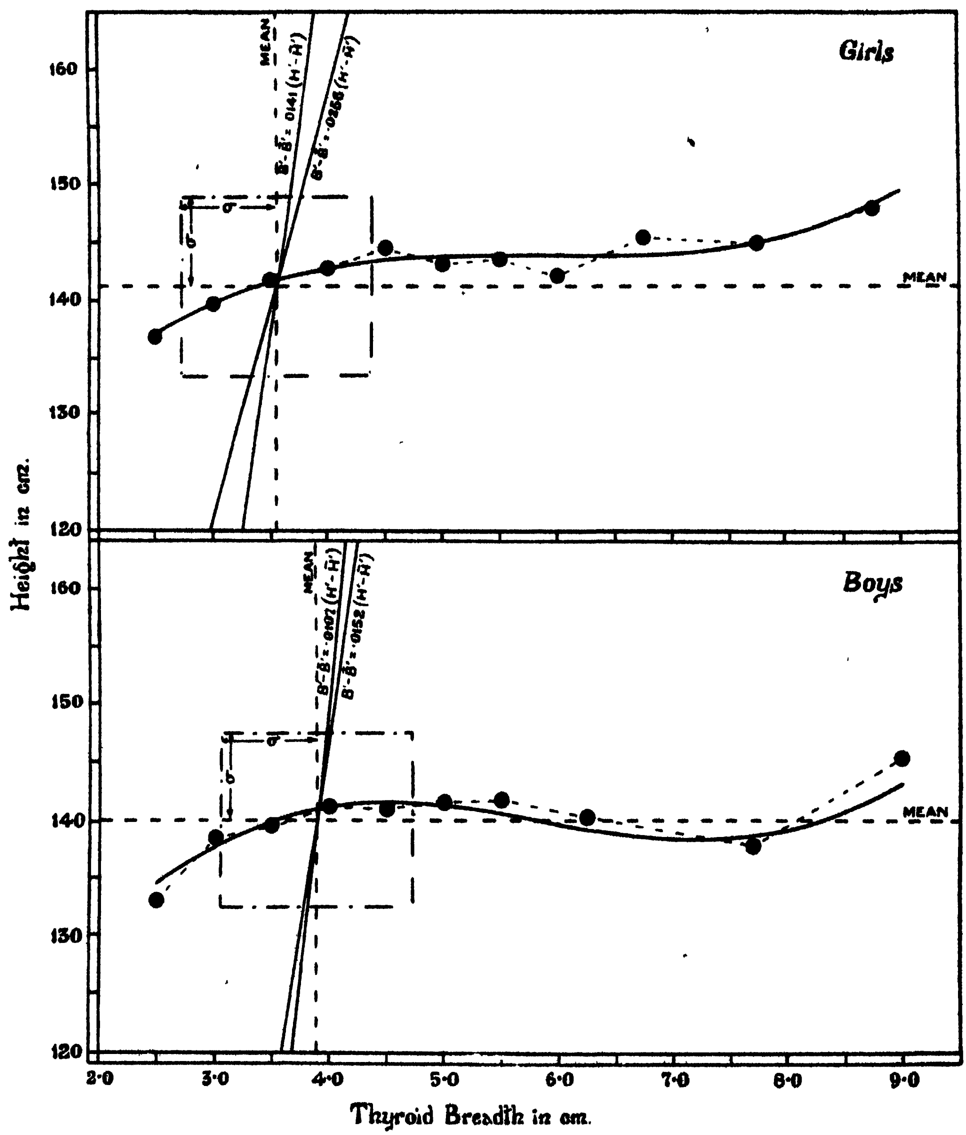


Fig. 9. Relations between thyroid breadth and height, both corrected to age 12.

It is this additional relation that we are concerned in discovering in this research; that is, *whether enlargement of the thyroid gland beyond the dimensions expected for the age and height of the child measured tends to be associated with a height or other physical measurement above or below average.*

What breadth of thyroid is to be expected at age 12 for a given stature? This can only be ascertained by obtaining a measure of the relation between thyroid breadth and stature in children whose thyroid glands are functioning normally. It is obvious that the regression line of thyroid breadth on height for

TABLE XXII.

Records of Height and Weight in Devonshire Boys and Cheshire Girls whose Thyroid Glands were measured.

Central Age	Boys				Girls			
	Height		Weight		Height		Weight	
	Total entries	Mean height, cm.	Total entries	Mean weight, kgm	Total entries	Mean height, cm.	Total entries	Mean weight, kgm.
3·25	—	—	—	—	14	91·02	15	14·05
3·37	45	95·45	46	15·91	—	—	—	—
3·75	—	—	—	—	9	93·78	9	15·78
4·25	—	—	—	—	25	98·29	25	16·17
4·55	96	101·20	98	17·16	—	—	—	—
4·75	—	—	—	—	29	101·38	30	17·25
5·25	77	105·90	87	18·42	60	104·45	59	17·18
5·75	86	108·12	100	18·69	67	106·05	68	18·23
6·25	95	111·00	96	19·98	53	111·15	53	19·41
6·75	95	113·93	96	20·76	38	112·61	37	19·58
7·25	111	116·45	115	21·89	23	115·06	22	20·36
7·75	126	118·31	130	22·88	20	121·15	21	23·45
8·25	160	121·23	158	23·79	103	120·93	103	23·08
8·75	156	124·08	158	24·96	137	124·19	129	23·87
9·25	160	126·06	159	25·92	37	124·93	38	25·00
9·75	150	129·41	156	27·41	15	128·65	15	25·45
10·25	150	131·23	156	28·45	10	131·65	9	27·78
10·75	174	134·19	179	30·00	12	136·70	11	31·10
11·25	169	136·56	169	31·20	65	137·79	65	30·17
11·75	165	138·25	165	32·94	110	139·91	112	32·16
12·25	161	141·57	161	34·57	165	143·05	166	34·74
12·75	115	142·61	119	34·96	138	145·86	137	37·63
13·25	89	148·01	88	38·53	110	147·90	110	38·88
13·75	58	150·26	43	40·36	92	151·12	92	40·85
14·25	—	—	11	41·36	54	153·11	54	43·06
14·75	—	—	4	45·45	9	155·12	9	43·79
15·25	—	—	—	—	4	152·95	4	46·95
15·75	—	—	—	—	2	152·95	2	43·95
Totals	2438	—	2494	—	1401	—	1395	—

the whole sample, including the goitrous, will not give this information, and correction of breadth from this line would be merely begging the question. The nearest approximation to the relation we require is given by the regression line of thyroid breadth on stature, both corrected to age 12, in the Category 01 group of children who presented no obvious enlargement. In using this line to correct from, the

TABLE XXIII. *Mean Physical Measurements and Smoothed Values at Half-year Periods in Devonshire Boys and Cheshire Girls.*

Boys: Height, cm. from curve	Boys: Weight, kgm. from curve	Boys: Head circumference, mm.: No.	Mean	Smoothed	Boys: Strength of grip, kgm.: No.	Mean	Smoothed	Girls: Height, cm. from curve	Girls: Weight, kgm. from curve	Girls: Strength of grip, kgm.: No.	Mean	Smoothed	Blood pressure: mm. of mercury: Systolic: No.	Mean	Smoothed	Diastolic: No.	Mean	Smoothed	Pulse: Smoothed	No.	Mean	S
—	—	—	—	—	—	—	—	132·32	28·05	1		13·3	1		101·5	1		66·6	34·9	1		·667
133·83	29·87	14	525·6	525	14	14·79	14·7	135·01	29·57	4	14·25	14·1	4	107·0	104·4	3	69·1	68·0	36·4	4	·677	·672
136·34	31·31	68	528·6	526	69	16·44	16·4	137·67	31·19	36		14·9	36		107·6	35		69·3	38·3	36		·676
138·87	32·83	72	526·1	527	72	17·92	17·9	140·30	32·91	43	16·28	15·7	43	112·1	110·5	34	71·3	70·6	39·9	43		·678
141·43	34·43	95	530·5	529	95	19·11	19·1	142·91	34·73	48	16·48	16·5	49		113·4	43		72·0	41·4	49	·680	·679
144·01	36·12	83	527·5	530	83	20·96	20·1	145·50	36·65	61	17·34	17·3	62	115·9	115·8	53	73·4	73·0	42·8	62	·675	·685
146·63	37·91	92	533·1	532	92	20·46	21·3	148·06	38·68	92	18·02	18·0	98	116·6	116·7	85		73·5	43·2	99	·697	·691
149·30	39·79	82	535·5		82	23·84	23·1	150·60	40·83	88	18·89	18·9	131	117·4	117·4	114	73·3	73·3	44·1	131	·696	·689
152·01	41·79	26	533·5	535	26	24·74	25·0	153·12	43·09	54	21·06	21·0	73	119·7	119·7	70	73·1	73·1	46·6	73	·649	·660
154·77	43·87	8			8		27·0	155·61	45·47	10		22·3	20	120·3	120·5	19	73·2	73·2	47·3	20	·655	·652
—	—	—	—	—	—	—	—	158·08	47·97	4	22·87	23·4	10		121·0	10	76·7	74·5	46·5	9	·669	·659
—	—	—	—	—	—	—	—	160·53	50·60	2		24·4	12	122·2	121·3	11		78·0	43·3	12		·679

TABLE XXIV.

Relations of Physical Measurements to Breadth of the Thyroid Gland, both corrected to age 12.—Means and Frequencies.*

Central Thyroid Breadth, cm.	Boys: Height, cm.		Boys: Weight, kgm.		Girls: Height, cm.		Girls: Weight, kgm.		Girls: Rate of growth†, cm. per year		Girls: Rate of increase† in weight, kgm. per year		Girls: Strength of grip, kgm.		Girls: Systolic pressure, mm.		Girls: Diastolic pressure, mm.		Girls: Pulse pressure, mm.		Girls: Pulse interval, seconds		Girls: Proficiency in class	
	No.	Mean	No.	Mean	No.	Mean	No.	Mean	No.	Mean	No.	Mean	No.	Mean	No.	Mean	No.	Mean	No.	Mean	No.	Mean	No.	Mean
2·5	17	133·10	19	28·87	25	136·93	24	29·95	23	− ·217	22	− ·575	28	15·64	28	108·4	27	63·4	27	41·8	28	·672	27	− ·191
3·0	82	138·64	78	33·06	58	139·75	57	32·55	56	+ ·073	56	− ·162	66	16·66	66	109·0	55	70·4	55	39·5	65	·657	59	− ·141
3·5	147	139·71	143	33·22	99	141·95	97	33·60	92	− ·014	92	+ ·072	109	15·97	109	111·1	96	71·1	96	41·3	109	·673	106	+ ·040
4·0	126	141·14	127	34·37	84	142·80	82	33·97	71	+ ·298	68	+ ·607	89	16·12	89	111·4	76	71·1	76	41·7	88	·678	83	+ ·177
4·5	69	140·97	67	33·85	68	144·56	68	36·82	65	− ·543	66	− ·020	71	19·66	70	108·4	64	72·3	64	39·6	70	·691	66	+ ·074
5·0	40	141·55	40	33·60	39	143·07	39	35·30	37	− ·002	38	− ·183	42	17·50	43	112·6	37	71·6	37	42·8	43	·687	38	− ·059
5·5	10	141·85	10	35·25	22	143·68	22	35·50	20	− ·234	20	+ ·407	24	17·75	24	112·8	23	69·0	23	43·9	24		22	+ ·536
6·0	5	141·85	5	33·45	30	142·25	30	34·15	28	+ ·066	29	+ ·142	30	16·57	30	116·7	26	73·7	26	45·5	30	·689	28	+ ·059
6·5	5	138·85	5	34·05	13	144·07	13	35·52	11	+ ·020	12	+ ·526	13	17·88	13	113·1	13	73·9	13	38·9	13	·647	13	− ·499
7·0	1	139·45	1	33·45	14	146·74	14	37·30	12	+ ·233	12	+ ·357	14	18·21	13	111·6	11	72·4	11	41·5	14	·664	14	+1·166
7·5	1	139·45	1	30·45	17	146·16	17	38·21	17	+ ·213	17	+1·078	17	19·21	17	117·9	17	74·9	17	43·2	17	·659	17	+ ·221
8·0	2	136·45	2	33·45	8	142·45	8	36·45	7	+ ·209	7	+1·178	8	18·50	8	117·6	7	78·4	7	42·7	8	·688	8	+ ·809
8·5	—	—	—	—	7	148·88	7	39·45	7		6	+1·278	7	19·17	7	113·4	6	73·7	6	42·0	7		6	+1·059
9·0	1	145·45	1	42·45	2	145·45	2	37·95	2	+ ·566	2		2		2	117·0	2	74·5	2	39·5	2		2	

* In the case of the girls these are the actual numbers measured; for correlation the frequencies have been adjusted as explained in the text.

† In excess or defect of the average rates given by the cubic curves in the same interval.

assumption is that no *causal* association between thyroid size and stature will begin to come into operation until the gland was noticeably enlarged; such an assumption seems reasonable, and at any rate if any causal association is also present in the normal group it means that we are over-correcting for height and reducing all the positive coefficients too far, so that any conclusions drawn as to the significance of positive coefficients between thyroid breadth and physical characters will not be invalidated.

The two regression lines on stature, both for boys and girls, are drawn in Figure 9 for comparison; they scarcely show any significant departure from linearity for the complete samples, and still less for Category 01 alone. For the complete samples of boys $\eta = \cdot 2093$, $\eta^2 - \bar{\eta}^2 = \cdot 01220$, p.e. of $\bar{\eta}^2 = \cdot 00741$, and of girls $\eta = \cdot 3126$, $\eta^2 - \bar{\eta}^2 = \cdot 09057$, p.e. of $\bar{\eta}^2 = \cdot 00363$. Thus η^2 is non-significant for the boys but is significant for the girls. The data for obtaining the regression coefficients are

TABLE XXV.

		Mean Breadth B'	$\sigma_{B'}$	Mean Stature H'	$\sigma_{H'}$
Boys	Complete sample ...	3·8972 cm.	·8327 cm.	140·043 cm.	7·495 cm.
	Category 01 only ...	3·4192 „	·3915 „	139·266 „	7·501 „
Girls	Complete sample ...	3·6922 „	·9129 „	141·487 „	7·550 „
	Category 01 only ...	3·4086 „	·3761 „	141·246 „	7·765 „

and the equations of the four regression lines become

$$\text{Boys}\begin{cases}\text{Complete sample.} & (B' - 3{\cdot}8972) = {\cdot}01523\,(H' - 140{\cdot}043) \text{ cm. per cm.}\\ \text{Category 01 only.} & (B' - 3{\cdot}4192) = {\cdot}01067\,(H' - 139{\cdot}266) \quad ,,\end{cases}$$

$$\text{Girls}\begin{cases}\text{Complete sample.} & (B' - 3{\cdot}6922) = {\cdot}02589\,(H' - 141{\cdot}487) \quad ,,\\ \text{Category 01 only.} & (B' - 3{\cdot}4086) = {\cdot}01409\,(H' - 141{\cdot}246) \quad ,,\end{cases}$$

Correction of thyroid breadths from the regression lines for the complete samples has of course the effect of reducing the correlation with height in the complete samples to zero. Correction from the regression lines for normal children only likewise reduces the correlation with height in the Category 01 group to zero, and what we want to determine is whether, after such correction of every individual thyroid breadth, we are left with any significant correlation in the goitrous and complete samples. The standard deviations of thyroid breadth after correction from the regression lines for Category 01, which may be termed $\sigma_{B''}$, and the standard deviations of height, $\sigma_{H'}$, in these two samples are

$$\text{Boys}\begin{cases}\text{Complete sample.} & \sigma_{B''} = {\cdot}8255 \text{ cm.,} & \sigma_{H'} = 7{\cdot}4947 \text{ cm.}\\ \text{Categories } 2+3. & \sigma_{B''} = {\cdot}7877 \;\; ,, & \sigma_{H'} = 7{\cdot}3185 \;\; ,,\end{cases}$$

$$\text{Girls}\begin{cases}\text{Complete sample.} & \sigma_{B''} = {\cdot}8973 \;\; ,, & \sigma_{H'} = 7{\cdot}5496 \;\; ,,\\ \text{Categories } 2+3. & \sigma_{B''} = 1{\cdot}3250 \;\; ,, & \sigma_{H'} = 6{\cdot}7765 \;\; ,,\end{cases}$$

The correlation coefficient of thyroid breadth, so corrected, with height, is then given by $r_{B''H'} = \frac{1}{\sigma_{B''}}(\sigma_{B'} r_{B'H} - b\sigma_{H'})$*, where b is the regression coefficient found above for B' on H' in the Category 01 group, that is, where $b = \cdot 01067$ for boys and $\cdot 01409$ for girls. The resulting values of $r_{B''H'}$ are

$$\text{Boys}\begin{cases}\text{Complete sample.} & r = \cdot 0414 \pm \cdot 0299. \\ \text{Categories } 2+3. & r = -\cdot 0960 \pm \cdot 0477.\end{cases}$$

$$\text{Girls}\begin{cases}\text{Complete sample.} & r = \cdot 0993 \pm \cdot 0309. \\ \text{Categories } 2+3. & r = \cdot 1561 \pm \cdot 0376.\end{cases}$$

In the *boys* there remains no significant correlation over and above that which is attributable to the normal degree of proportionality between linear measurements at a constant age. If anything the relation becomes negative in the goitrous group, which might be due to over correction by the assumption made above, but it is not significantly negative.

In the *girls* there remains a significant positive correlation over and above that due to proportionality and this becomes more pronounced when attention is confined to the goitrous group alone as might be expected from the regression line on breadth. This indicates that either (i) goitres tend to appear more readily or to become more pronounced relative to the child's size in girls whose height is above average for their age, or else (ii) girls whose thyroids enlarge about puberty tend in consequence to grow more rapidly in height. If the first be the correct interpretation, the natural explanation is that a girl who is taller than average, or is growing more rapidly than average, requires more thyroid secretion than the average girl and more than her size of gland is capable of supplying unless the manufacture of the essential factor in the secretion from external sources can keep up with the demand. Usually the intake of iodine for this purpose would be abundant enough to meet extra demands, but in goitrous regions where evidence suggests that it is barely sufficient, the girls who require most would be the first to exhibit signs of deficiency by compensatory enlargement of the gland.

If the second interpretation be the correct one it would mean that enlargement of the gland about puberty is caused by some infective, nervous or other stimulus which itself also stimulates growth or else that the altered thyroid activity accompanying enlargement is responsible for more rapid growth; such an explanation is highly improbable and is in my view quite disproved by evidence in the succeeding sections. This important question will be further discussed at the end of Section (7).

(*d*) *Partial correlation to eliminate the effect of height.* The calculations used above may be further applied to get rid of the height factor from the correlations

* Since the values of B'' are corrected from the regression line, breadth $= bH'$, for an array at height H' we have $B'' = B' - bH'$; hence from the product-moment formula

$$r_{B''H'}\sigma_{B''}\sigma_{H'} = \frac{1}{N}\Sigma[B'' \times H'] = \frac{1}{N}\Sigma[(B' - bH')H'] = r_{B'H'}\sigma_{B'}\sigma_{H'} - b\sigma^2_{H'}.$$

between thyroid breadth and other physical measurements*. In dealing with the Category 01 group alone the ordinary method of partial correlation for height constant can be used since by so doing we are correcting from the regression lines of both factors on height within the normal group. In dealing with the Categories 2 + 3 group or the complete sample, however, it is desired to correct for only that part of the correlation arising from normal proportionality, that is, to correct thyroid breadth from the regression line on height for the normal group. The regression and correlation coefficients of other physical factors with height do not differ significantly in the three groups as can be seen by reference to Table XXVI. Hence the formula for partial correlation with some factor X' (corrected to age 12) becomes

$$\frac{r_{B'X'} - r_{X'H'} \cdot b\,\dfrac{\sigma_{H'}}{\sigma_{B'}}}{\sqrt{1 - r^2_{B'X'}}\,\sqrt{1 - b^2\,\dfrac{\sigma^2_{H'}}{\sigma^2_{B'}}}},$$

where b is the regression coefficient of B' on H' for the Category 01 group and has in all cases the values ·01067 for boys and ·01409 for girls, whilst $\sigma_{H'}$, $\sigma_{B'}$ and $r_{B'X'}$ are the values for the group being dealt with. For Category 01 group alone $b\dfrac{\sigma_{H'}}{\sigma_{B'}} = r_{B'H'}$ for that group, but for the Categories 2 + 3 group or the complete sample it has somewhat different values as follows:

$$b\frac{\sigma_{H'}}{\sigma_{B'}}\begin{cases}\text{Categories } 2+3. & \text{Boys } \cdot 0996, \quad \text{Girls } \cdot 0711.\\ \text{Complete sample.} & \quad ,, \quad \cdot 0960, \quad\quad ,, \quad \cdot 3165.\end{cases}$$

These coefficients will therefore be used for partial correlation in the following sections to eliminate the normal effect of height.

(*e*) *Correlation with dimensions of neck*. The neck measurements have been described in Section 2 (*c*). Their growth curves are shown in Figure 8 where the measurements are reduced to a proportionate scale; the contrast with the thyroid curve is evident. The correlation coefficients after age correction are shown in Table XXVI. In order to correct for height the regressions on height are required in the normal groups alone, since neck measurements are obviously affected by large goitres, and these are:

Antero-posterior diameter.	·2425 mm. per cm. of height.
Neck circumference.	·8052 ,, ,, ,,

Passing to the goitre groups and complete samples, we have:

Categories 2 + 3. $\sigma_{D'} = 7{\cdot}5251$ mm., $\sigma_{C'} = 20{\cdot}9867$ mm., $\sigma_{H'} = 6{\cdot}7765$ cm.,
Complete sample. $\sigma_{D'} = 5{\cdot}8697$ mm., $\sigma_{C'} = 19{\cdot}2662$ mm., $\sigma_{H'} = 7{\cdot}5496$ cm.,

* It is not suggested that by making age and height constant the effect of size is completely eliminated; it must be admitted that height would not perhaps be naturally so highly correlated with thyroid breadth as would some horizontal measurements on the body such as breadth of neck or shoulders, but as it was not practicable to include such measurements correction for height has been used here, the effect of this being to limit the field to children of equal age and height and to that extent exclude natural correlations due to size.

TABLE XXVI.

Correlation coefficients between Breadth of Thyroid Gland and various Physical and Mental Characters, all corrected for age.

BOYS	With Thyroid Breadth corrected for age*						With height corrected for age		
	Without correction for height			After elimination of normal correlation due to height†					
Factors corrected for age	Category 01	Categories 2+3	All boys	Category 01	Categories 2+3	All Boys	Category 01	Categories 2+3	All boys
Height	·2045 ± ·0367	·0031 ± ·0482	·1528 ± ·0293	—	− ·0960	·0414	—	—	—
Weight	·3048 ± ·0352	·0396 ± ·0482	·1187 ± ·0298	·2394	− ·0863	·0701	·7938 ± ·0143	·8509 ± ·0133	·8069 ± ·0105
Rate of growth	·0527 ± ·0470	− ·0407 ± ·0543	·0176 ± ·0356	− ·0336	− ·0852	− ·0212	·4049 ± ·0394	·3785 ± ·0466	·3868 ± ·0303
Rate of increase in weight	− ·0450 ± ·0457	·0745 ± ·0537	·1004 ± ·0346	·1237	·0356	·0691	·3375 ± ·0406	·4254 ± ·0442	·3843 ± ·0298
Strength of grip	·1675 ± ·0368	·0311 ± ·0481	·1601 ± ·0290	·0619	− ·0288	·1288	·5770 ± ·0256	·5518 ± ·0335	·5732 ± ·0201
Head circumference ...	·0702 ± ·0378	·0568 ± ·0480	·0250 ± ·0298	− ·0212	·0151	− ·0174	·4349 ± ·0311	·4235 ± ·0395	·4242 ± ·0246

GIRLS	With Thyroid Breadth corrected for age from $B1$* growth curve						Thyroid breadth corrected from BN curve*	With height corrected for age	With weight corrected for age
	Without correction for height			Corrected for height as above†					
Factors corrected for age	Category 01	Categories 2+3	All girls	Category 01	Categories 2+3	All girls	All girls	All girls	All girls
Height	·2909 ± ·0485	·2251 ± ·0366	·2141 ± ·0298	—	·1561	·0993	·2222 ± ·0291	—	·7205 ± ·0148
Weight	·3413 ± ·0454	·3305 ± ·0344	·2427 ± ·0290	·1940	·4039	·2305	·2292 ± ·0292	·7205 ± ·0148	—
Rate of growth	·0134 ± ·0527	·1579 ± ·0389	·0254 ± ·0319	− ·0598	·1455	− ·0023	·0135 ± ·0319	·2373 ± ·0301	—
Rate of increase in weight	·2396 ± ·0498	·3695 ± ·0346	·1424 ± ·0313	·1436	·3718	·1062	·1272 ± ·0314	·3880 ± ·0291	—
Strength of grip	·2067 ± ·0480	·2058 ± ·0369	·1509 ± ·0299	·0730	·1972	·1075	·1265 ± ·0291	·5030 ± ·0229	·6376 ± ·0179
Systolic pressure ...	·0422 ± ·0500	·1660 ± ·0376	·1267 ± ·0301	− ·0413	·1527	·0991	·1088 ± ·0293	·2757 ± ·0283	—
Diastolic pressure ...	·1340 ± ·0528	·1730 ± ·0397	·1427 ± ·0319	·1059	·1656	·1310	·1593 ± ·0307	·1149 ± ·0322	—
Pulse pressure	·0793 ± ·0535	·0259 ± ·0409	·0868 ± ·0323	·0317	·0140	·0684	·0363 ± ·0314	·1698 ± ·0317	—
Pulse interval	·1202 ± ·0494	− ·0651 ± ·0384	·0588 ± ·0305	·1139	·0681	·0546	·0338 ± ·0296	·0391 ± ·0306	—
Proficiency in class ...	·1201 ± ·0504	·0820 ± ·0394	·0771 ± ·0312	·0413	·0641	·0766	·0718 ± ·0303	·2081 ± ·0300	—
Diameter of neck ...	·2220 ± ·0475	6536 ± ·0336	·4279 ± ·0246	·0876	·6557	·4151	—	·5168‡ ± ·0368	—
Circumference of neck ...	·2989 ± ·0455	·5378 ± ·0450	·2951 ± ·0275	·1714	·5393	·2740	—	·5607‡ ± ·0346	—

* $B1$ curve means the growth curve for Category 01 alone; BN is the slightly different growth curve deduced for the "non-goitrous" areas; see text. In boys the corrections do not differ sensibly.

† See text.

‡ Category 01 only.

and the coefficients for use in partial correlation are given by multiplying the regression coefficients above by the respective ratios of the standard deviations, giving finally the partial coefficients when the normal effect of height is eliminated from both measurements as follows:

	Category 01	Categories 2+3	Complete sample
Thyroid breadth with neck diameter.	·0876	·6557	·4151
Thyroid breadth with neck circumference.	·1714	·5393	·2740

The implications of these coefficients have been discussed in Section 2 (c).

(*f*) *Correlation with Horizontal Circumference of Head.* In the 540 boys this was measured by a steel tape. The actual means at each half-year age-group are shown in Table XXIII. After combining a few of these groups a freehand curve was drawn through the points and the smoothed values also shown in the same table were obtained. The growth curve, using the head circumference at age 11 as scale unit, is drawn in Figure 8, and it is evident that the relative rate of growth of the head in boys between ages 11 and 16 is very much the same as for the thyroid gland, both being notably less than for height.

After correction to age 12 the means and standard deviations were:

Category 01.	Mean $528{\cdot}30 \pm {\cdot}55$,	$\sigma = 14{\cdot}4421$ mm.
Categories 2 + 3.	„ $527{\cdot}61 \pm {\cdot}75$,	$\sigma = 15{\cdot}5787$ „
Complete sample.	„ $527{\cdot}90 \pm {\cdot}43$,	$\sigma = 14{\cdot}8918$ „

There is no significant difference between the means.

The correlation coefficients with thyroid breadth are given in Table XXVI; they are insignificant for all boys and also for each subgroup, and when the effect of height is eliminated the coefficients become practically zero. We must therefore conclude that there is no relation between thyroid size and size of head in boys apart from that to be normally expected on account of differing size of the boys at the same age. The correlation between head circumference and stature after age correction is of the order ·42.

(6) Relation to Body Weight and Physical Strength.

(*a*) *Body weight.* The manner in which the weights of the children were obtained has been mentioned in Section (5), and the complete data are set out in Table XXII. These were fitted with cubic curves having the formulae:

$$\text{Boys}\quad W = 11{\cdot}145 + 1{\cdot}3127a - {\cdot}017752a^2 + {\cdot}005368a^3,$$

$$\text{Girls}\quad W = 12{\cdot}755 + {\cdot}4845a + {\cdot}05452a^2 + {\cdot}004272a^3,$$

where W = weight in kgm., a = age in years. Points on these curves at half-year intervals are given in Table XXIII and the growth curve with age for girls is

drawn in Figure 10, using the mean weight at age 11 exactly as scale unit. The cubic has been drawn as far as age 15, the rest of the curve being drawn freehand

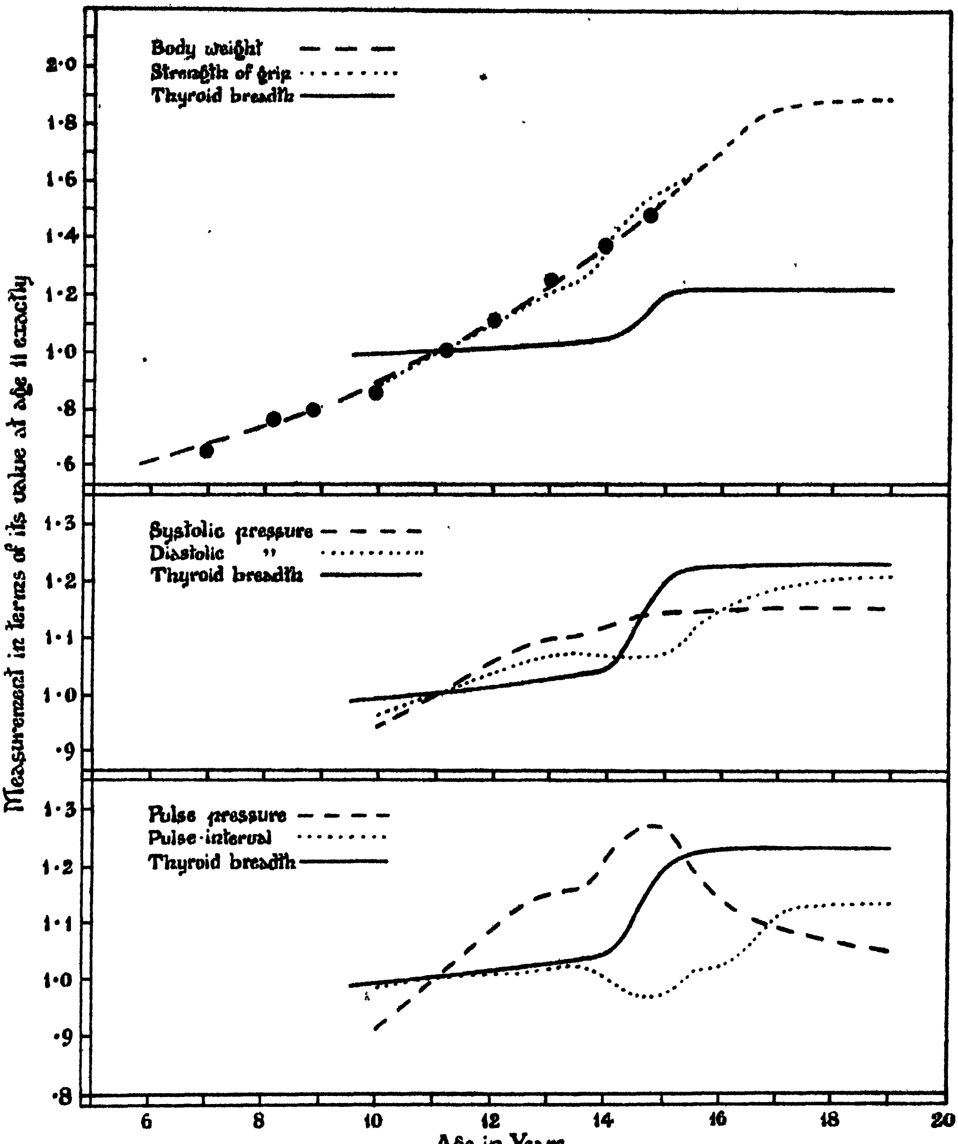


Fig. 10. Growth curves of thyroid gland in girls compared with growth curves for certain physical and physiological characters in the same districts, the mean measurement at age 11 exactly being taken as scale unit in each case.

on the basis of measurements of college students. The points inserted indicate that the cubic fits the data fairly well though possibly it fails to represent a very slight dip about age 10.

After correction to age 12 the means and standard deviations were:

Boys	Category 01.	Mean 33·06 ± ·21,	σ = 5·4520 kgm.
	Categories 2 + 3.	„ 34·16 ± ·25,	5·2046 „
	Complete sample.	„ 33·49 ± ·16,	5·3830 „
Girls	Category 01.	Mean 33·06 ± ·31,	σ = 6·1819 „
	Categories 2 + 3.	„ 34·56 ± ·22,	5·7465 „
	Complete sample.	„ 33·18 ± ·19,	6·2131 „

The correlation coefficients with thyroid breadth after age correction are shown in Table XXVI (p. 331), and there is at once obvious a contrast between the sexes. In normal boys and girls alike there is a correlation exceeding ·3; in boys this is entirely obliterated in the goitrous group and reduced to ·1187 in the complete sample, but in girls it is maintained in the goitrous group and reduced to ·24 in the complete sample. After correction for the effect of height by partial correlation as described in Section 5 (*e*) the coefficients remain significantly positive in the normal groups both for boys and girls; thus:

Category 01.	Boys r = ·2394 ± ·0365,	Girls r = ·1940 ± 0489.
Categories 2 + 3.	„ r = − ·0863 ± ·0479,	„ r = ·4039 ± ·0324.
Complete sample.	„ r = ·0701 ± ·0301,	„ r = ·2305 ± ·0292.

In the goitrous group the correlation has become zero or slightly negative in boys but is positive to the extent of ·4 in girls; in the complete samples it has become insignificant in boys, but in girls there remains a significant relation of the order ·23.

The contrast is clearly shown by the regression lines of weight on thyroid breadth drawn in Figure 11; the data for these are given in Table XXIV and the means have been fitted with cubic and quartic curves as follows:

$$\text{Boys}\quad W' = \cdot 271 + 20\cdot 114B' - 3\cdot 8552B'^2 + \cdot 23585B'^3,$$
$$\text{Girls}\quad W' = -7\cdot 906 + 27\cdot 405B' - 6\cdot 3928B'^2 + \cdot 64386B'^3 - \cdot 022952B'^4,$$

where W' and B' are corrected to age 12 and the units are kgm. and cm. respectively. For boys $\eta = \cdot 2136$, $\eta^2 - \bar{\eta}^2 = \cdot 01956$, p.e. of $\bar{\eta}^2 = \cdot 00890$, so η is barely significant; for girls $\eta = \cdot 2785$, $\eta^2 - \bar{\eta}^2 = \cdot 07125$, p.e. of $\bar{\eta}^2 = \cdot 00344$, so η is undoubtedly significant.

On the same diagram the regression lines of thyroid breadth on weight are also drawn; these do not deviate sensibly from straight lines, and their equations are:

$$\text{Boys}\quad B' = 3\cdot 8998 + \cdot 0185\,(W' - 33\cdot 504),$$
$$\text{Girls}\quad B' = 3\cdot 5623 + \cdot 0350\,(W' - 33\cdot 278).$$

The conclusions which must be drawn are that (i) both in girls and boys thyroid glands which are larger within normal limits seem to be associated with body weights above average for the age in question even after correction for the height of the child, a result which would probably be obtained with any organ of soft consistency; (ii) in boys pronounced enlargement beyond normal limits has no positive relation with weight, the net result being that in a population of boys in a goitre district there is on the whole no significant tendency for goitres to occur

in the heavier boys at a given age; (iii) in girls the positive association between thyroid size and heaviness becomes much more pronounced in goitrous girls than normal girls after correcting for their ages and heights, and this certainly indicates that either (α) the thyroid is more prone to enlarge in well-developed girls than in under-developed girls of the same age and height, or else (β) the gland in its

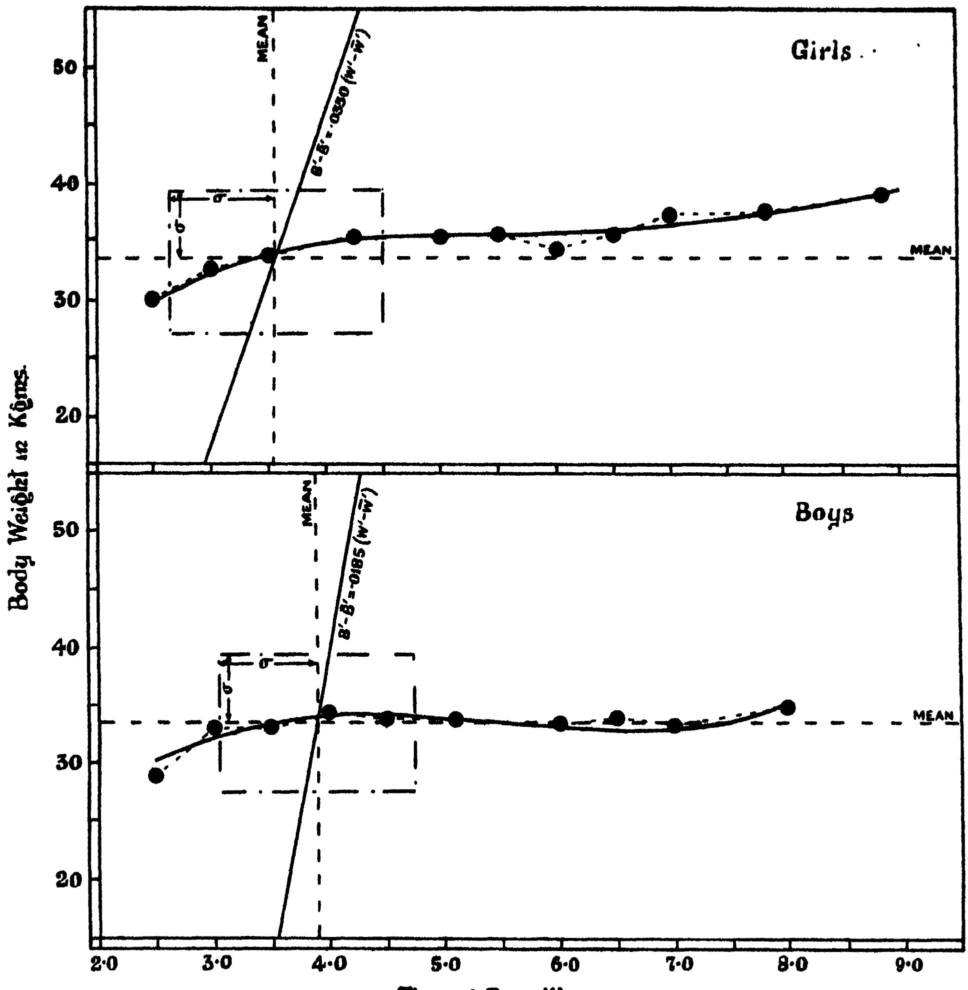


Fig. 11. Relations between thyroid breadth and body weight, both corrected to age 12.

enlarged state has a direct influence in raising the body weight above the average. The existence of a similar though slighter relation with height and further evidence in the following sections make it improbable that any explanation under the heading (β) is the correct one. The additional weight of the gland itself could have only a very slight effect in producing a positive correlation*.

* A rough calculation shows that this could not exceed ·002 in the normal group and ·005 in the other groups.

(*b*) *Strength of grip.* This was measured by means of a Mathieu's dynamometer allowing two grips with the right hand followed by two grips with the left. In 529 boys 325 had a stronger mean grip in the right than the left hand, 78 *vice versâ*, and in 126 the grips were equal; in 516 girls 381 were stronger in the right, 91 in the left and in 44 the grips were equal. The highest of the four readings has been used as the measure of strength of grip in the present work. The means of these maximum grips at half-year intervals are shown in Table XXIII for boys and girls, and these form a sufficiently smooth series after combining one or two age-groups to draw a freehand curve through without difficulty. Points on these curves are also shown in the same table. The curve for girls is drawn in Figure 10 using the grip at age 11 as scale unit, and it closely follows the curve of body weight excepting the presence of a rather more pronounced pubescent dip.

After correction to age 12 from these curves the means and standard deviations expressed in kilograms were as follows:

Boys	Category 01.	Mean 18·595 ± ·152,	σ = 4·0029.
	Categories 2 + 3.	„ 19·816 ± ·217,	4·5080.
	Complete sample.	„ 19·089 ± ·129,	4·2639.
Girls	Category 01.	Mean 16·500 ± ·203,	σ = 4·0465.
	Categories 2 + 3.	„ 17·180 ± ·155,	4·0312.
	Complete sample.	„ 16·598 ± ·129,	4·0513.

The correlation coefficients between grip and thyroid breadth after correction for age and also for height are shown in Table XXVI; the height-corrected values with their probable errors are:

Category 01.	Boys r = ·0619 ± ·0378,	Girls r = ·0730 ± ·0499.
Categories 2 + 3.	„ r = − ·0288 ± ·0481,	„ r = ·1972 ± ·0371.
Complete sample.	„ r = ·1288 ± ·0297,	„ r = ·1075 ± ·0302.

These indicate again an important sex difference; in boys and girls alike the thyroid size within normal limits is positively related to strength of grip in children of the same age, but this relation becomes insignificant when corrected for height as well; in boys the relation is obliterated in the goitre group alone but in girls a pronounced correlation with grip remains, indicating some subtle relation between proneness to goitrous enlargement and the state of physical development.

This might be a secondary result of the association just noticed with body weight; thus in both sexes there is a correlation between grip and weight, even after correction for height, as shown by the values below all corrected for age:

	Boys	Girls
Strength of grip with weight.	r = ·6376 ± ·0179,	·5936 ± ·0199.
„ „ height.	r = ·5732 ± ·0201,	·5030 ± ·0229.
Weight with height.	r = ·8069 ± ·0105,	·7205 ± ·0148.
Grip with weight, height constant.	r = ·3618 ± ·0262,	·3859 ± ·0262.

After partial correlation, making age, height and weight all constant the correlation for the goitrous group of girls between thyroid size and grip becomes $r = \cdot 0489 \pm \cdot 0384$, and this seems to show that the relation between thyroid size and grip is secondary to the relation with general physical development. Proceeding in the same way with the correlation coefficients for the complete samples:

		Boys	Girls
Thyroid breadth with grip, age, height and weight constant	$r =$	$\cdot 1112 \pm \cdot 0298$,	$\cdot 0206 \pm \cdot 0302$.

It may be concluded that all the relations between thyroid size and grip are due to general physical development; but in boys there may be a residual relationship not so explained which is confined to the slighter degrees of thyroid enlargement.

It is of interest to note that in girls, when age, height and grip are all made constant, the correlation coefficients between thyroid breadth and weight still remain considerable; thus:

Category 01, $r = \cdot 1705$; Categories 2 + 3, $r = \cdot 3848$; Complete sample, $r = \cdot 1950$.

From this it appears likely that the relation of the thyroid size with body weight is only to a small extent a relation with *muscular* development.

The regression of grip on thyroid breadth is shown in Table XXIV.

(7) RELATION TO RATE OF GROWTH.

This has been investigated by two different methods applied both to rates of increase in height and weight.

(*a*) Wherever possible the past records of height and weight of every child were extracted giving the age distributions of Table XXII. By separating the Cheshire girls into three thyroid categories and fitting cubic curves to the mean heights and weights of each group, as far back as age 3 or 4 in some cases, the following curves were obtained:

$$
\begin{array}{lll}
\text{Category 01.} & H_1 = 73\cdot 238 + 5\cdot 0991a + \cdot 11973a^2 & - \cdot 006594a^3, \\
\quad\text{,,} \quad 2. & H_2 = 64\cdot 398 + 9\cdot 4739a - \cdot 41182a^2 & + \cdot 013151a^3, \\
\quad\text{,,} \quad 3. & H_3 = 87\cdot 721 + 1\cdot 3962a + \cdot 48515a^2 & - \cdot 017477a^3, \\
\text{Category 01.} & W_1 = 12\cdot 349 + \cdot 5791a + \cdot 031810a^2 & + \cdot 004932a^3, \\
\quad\text{,,} \quad 2. & W_2 = 10\cdot 551 + 1\cdot 4179a - \cdot 053196a^2 & + \cdot 008260a^3, \\
\quad\text{,,} \quad 3. & W_3 = 13\cdot 109 + \cdot 1386a + \cdot 122366a^2 & + \cdot 001840a^3,
\end{array}
$$

—the numbers of observations forming these curves being respectively 481, 709 and 211 for height, and 481, 708 and 206 for weight.

It would be unsafe to place much reliance on these curves outside the limits of ages 8 to 14, but within those limits they should represent with fair accuracy the average manner of growth of girls (1) who do not develop any enlargement of the gland during the period, (2) who develop a moderate degree of enlargement, (3) who develop pronounced goitres. By differentiating with respect to age, a, the proportionate annual rates of growth $\frac{1}{H}\cdot\frac{dH}{da}$ and $\frac{1}{W}\cdot\frac{dW}{da}$ can be found at each year of age, and also the ratios of weight to height, W/H, for each group.

The conclusions suggested by Table XXVII are that:

(i) girls who exhibit *slight* thyroid enlargements before 14 are characterised by a rate of growth below average in height, and an average rate of growth in weight, from 8 to 11, compensated for about puberty by an unusually rapid growth both in height and weight after the 12th year has been reached; their weight per cm. of height is therefore higher than in the normal group of girls at all ages from 8 to 14;

TABLE XXVII.

Girls.

Age	8	9	10	11	12	13	14	Mean Values	
								8—11	12—14
$\frac{1}{H_1}\cdot\frac{dH_1}{da}$	·0486	·0456	·0426	·0395	·0365	·0335	·0305	·0441	·0335
$\frac{1}{H_2}\cdot\frac{dH_2}{da}$	·0449	·0418	·0395	·0381	·0373	·0370	·0345	·0411	·0372
$\frac{1}{H_3}\cdot\frac{dH_3}{da}$	·0480	·0464	·0441	·0413	·0381	·0345	·0305	·0449	·0344
$\frac{1}{W_1}\cdot\frac{dW_1}{da}$	·0945	·0990	·1027	·1054	·1072	·1083	·1086	·1004	·1037
$\frac{1}{W_2}\cdot\frac{dW_2}{da}$	·0948	·0986	·1023	·1057	·1085	·1107	·1122	·1003	·1105
$\frac{1}{W_3}\cdot\frac{dW_3}{da}$	·1066	·1089	·1098	·1097	·1088	·1073	·1054	·1087	·1072
$\frac{W_1}{H_1}$	·1821	·1914	·2026	·2157	·2310	·2484	·2682	·1979	·2492
$\frac{W_2}{H_2}$	·1884	·1988	·2111	·2253	·2415	·2597	·2798	·2059	·2603
$\frac{W_3}{H_3}$	·1900	·2019	·2153	·2302	·2468	·2652	·2855	·2093	·2658

(ii) girls who develop *large* goitres are characterised by rates of growth above average both in height and weight from the 8th to about the 13th year, compensated for afterwards by a less rapid growth; their weight per cm. of stature is higher than for either the normal or Category 2 groups of girls at every age.

How reliable these indications may be there is no satisfactory method of judging; the number of measurements used for the curves was, however, considerable*. The question is further discussed in subsection (*b*).

* Conclusions (i) and (ii) above have been also tested by drawing freehand curves through the points instead of fitting cubics and it is found that they still hold good, the only appreciable modifications being that the value of $\frac{1}{H_1}\cdot\frac{dH_1}{da}$ would be smaller at age 9 and correspondingly larger at age 10, and the values of $\frac{1}{W_1}\cdot\frac{dW_1}{da}$ and $\frac{1}{W_2}\cdot\frac{dW_2}{da}$ would be somewhat larger at age 12 and smaller at 13 than the values in Table XXVII. It must of course be remembered that whatever method of curve fitting be used, individual peculiarities of growth associated with puberty tend to disappear in mean curves for groups of children owing to the variation in chronological age at which puberty occurs.

The indices of weight per cm. of height were worked out also for the boys, using two groups only and taking the ratios of the means without smoothing.

TABLE XXVIII.

Boys.

Age	8	9	10	11	12	13	14	8—11	12—14
W_1/H_1 ...	·1937	·2030	·2138	·2250	·2400	·2490	·2713	·2089	·2534
W_{23}/H_{23} ...	·1965	·2040	·2151	·2273	·2429	·2553	·2686	·2107	·2556

In this case there is again an excess in the ratios for the goitre group at almost every age, but not so pronounced as in the case of the girls.

(*b*) The problem has been attacked by the correlation method, taking into account only the rates of growth during the year or two immediately preceding the measurement of the thyroid. In order to correct this for age the following method was used.

The present (or most recent) height was corrected to age 12 by means of the cubic curve obtained in Section (5) giving a value H_0; the record preceding this at an interval of not less than a year before was similarly corrected from the age at which it was taken to age 12, giving a value H_0'. If growth had been proceeding in the interval between these measurements at the rate given by the cubic curve we should have $H_0 = H_0'$; if growth had been more rapid than the cubic curve, $H_0 > H_0'$, if less rapid then $H_0 < H_0'$. If t = interval in years between the dates of the measurements, the rate of growth in cm. per year in excess or defect of the mean rate during the age period dealt with is given by $h = \frac{H_0 - H_0'}{t}$, which may have a + or − sign.

The same method was used for increase in weight giving rates of growth in excess or defect of the mean rate during the age period covered, $\omega = \frac{W_0 - W_0'}{t}$. The values of h and ω were then correlated with thyroid breadth corrected for age. The mean rates after combining the categories in their correct proportions being slightly in excess of zero, the whole series of means given in Table XXIV and used in drawing the regression lines in Figure 12 have been adjusted by subtracting this small deviation to bring the mean rate for the whole to exactly zero.

The standard deviations were as follows:

$$\text{Boys}\begin{cases}\text{Category 01.} & \sigma_h = 2{\cdot}7773, \quad \sigma_\omega = 1{\cdot}8862.\\ \text{Categories 2 + 3.} & \sigma_h = 2{\cdot}6073, \quad \sigma_\omega = 1{\cdot}8827.\\ \text{Complete sample.} & \sigma_h = 2{\cdot}7066, \quad \sigma_\omega = 1{\cdot}8970.\end{cases}$$

$$\text{Girls}\begin{cases}\text{Category 01.} & \sigma_h = 1{\cdot}9392, \quad \sigma_\omega = 1{\cdot}8027.\\ \text{Categories 2 + 3.} & \sigma_h = 1{\cdot}5865, \quad \sigma_\omega = 1{\cdot}4831.\\ \text{Complete sample.} & \sigma_h = 1{\cdot}8979, \quad \sigma_\omega = 1{\cdot}7588.\end{cases}$$

It is noteworthy that the standard deviations are considerably smaller for h and ω in the goitre groups of girls than in the normal group, but there is little difference for boys.

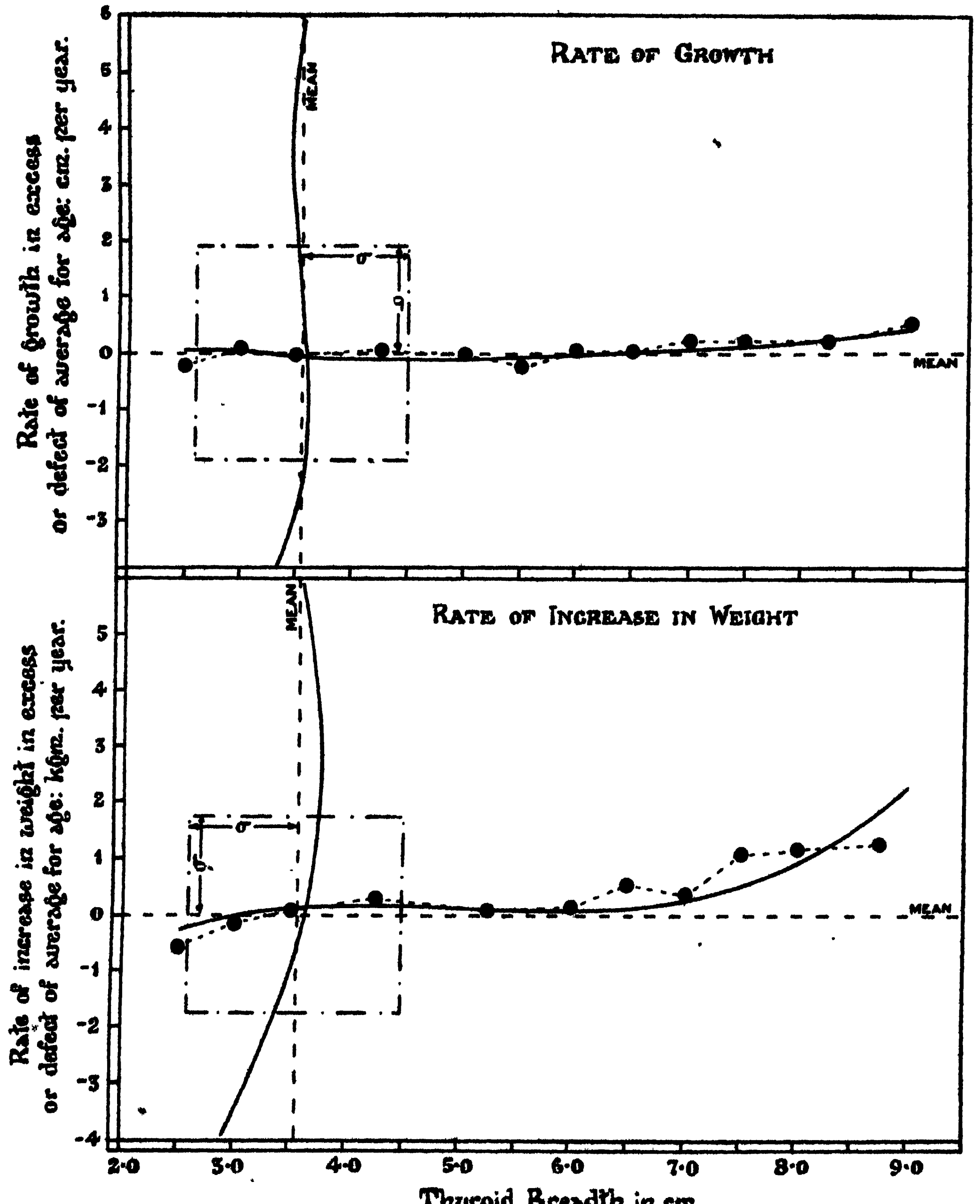


Fig. 12. Relations between thyroid breadth and rates of growth in Cheshire girls, both corrected to age 12.

The correlation coefficients with thyroid breadth corrected for age, and the partial coefficients after correction for height as before, are given in Table XXVI; the partial coefficients with their probable errors were:

Boys	Category 01.	$r_{B'h.H} = -\cdot0336 \pm \cdot0471$,	$r_{B'w.H} = \cdot1237 \pm \cdot0451$.
	Categories 2 + 3.	$-\cdot0852 \pm \cdot0540$,	$\cdot0356 \pm \cdot0539$.
	Complete sample.	$-\cdot0212 \pm \cdot0356$,	$\cdot0691 \pm \cdot0348$.
Girls	Category 01.	$r_{B'h.H} = -\cdot0598 \pm \cdot0525$,	$r_{B'w.H} = \cdot1436 \pm \cdot0517$.
	Categories 2 + 3.	$\cdot1455 \pm \cdot0392$,	$\cdot3718 \pm \cdot0345$.
	Complete sample.	$-\cdot0023 \pm \cdot0319$,	$\cdot1062 \pm \cdot0316$.

The coefficients in boys all become insignificant when the normal effect of size has been corrected for, except a small residual coefficient in the normal group between rate of increase in weight and thyroid breadth; in boys goitrous enlargement is not therefore associated to any measurable extent with rapid growth.

In girls, however, there remains a significant relation between rate of growth in height and thyroid size in the goitre group, though not in the normal or complete groups, indicating that among girls of the same age and height with goitres, the larger goitres tend to occur on the whole in the girls who have recently been growing quickly; with rate of increase in weight there remains a significant correlation in both normal and goitre groups, but this is much more pronounced in the goitre group. These relations are made somewhat clearer by the regression lines in Figure 12; cubic curves have been fitted to all the four regressions, their equations being:

$$h \text{ on } B' \ldots h = \cdot3895 - \cdot16030B' + \cdot006552B'^2 + \cdot0013606B'^3,$$

$$B' \text{ on } h \ldots B' = 3\cdot5821 - \cdot02400h - \cdot011481h^2 + \cdot002580h^3,$$

$$\omega \text{ on } B' \ldots \omega = -4\cdot1263 + 2\cdot71267B' + \cdot560847B'^2 + \cdot037562B'^3,$$

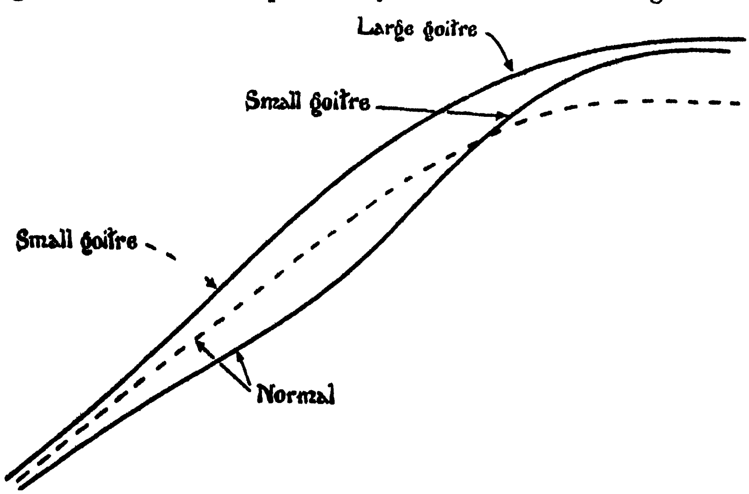
$$B' \text{ on } \omega \ldots B' = 3\cdot6111 + \cdot10099\omega - \cdot020691\omega^2 + \cdot000520\omega^3.$$

Examination of the upper curves in Figure 12 shows a slight tendency for rate of growth in height to be more rapid than the average in girls whose thyroids exceed 5 or 6 cm. in breadth; there seems, however, to be no tendency for girls growing at rates above the average for their age to have on the whole larger thyroids. The lower set of curves shows that very small thyroids are associated with slower growth in weight, and goitres exceeding about 6 cm. in breadth are associated with a notably high rate of growth; moderate degrees of enlargement have no relation with rate of increase in weight. There is also a very definite tendency for girls gaining weight at a rate below average to have small thyroids, an effect which would probably be found for most organs of the body, but little or no tendency is indicated for the thyroid to go on increasing as the rate of growth in weight is more and more excessive.

This result would seem paradoxical were it not that the rate of growth we are here considering is limited to a short period immediately preceding measurement of the gland. Since goitres usually take several years to reach a large size we should expect that a moderately excessive rate of growth dating back over several

years would be more characteristic of the large goitre group than a very excessive rate of growth dating back for only a short time, and therefore since very high rates of growth are of short duration and usually compensated to some extent by a period of slow growth, we should expect those girls who had recently been growing very rapidly to be characterised by small rather than large goitres; in other words that the regression of thyroid size on rate of growth would become zero or negative as the rate of growth reached high values. This is just what is indicated by the regression lines and the results in subsection (*a*) above; moreover it is not incompatible with the regression lines of Figure 10 which indicated a linear relation with *present* height and weight.

It seems therefore that large goitres are more likely to arise in a goitre district in those girls whose rate of growth in height and general development is above the average over a continuous period of years whilst small enlargements are more



likely to be found in those girls whose growth is at first slow and then very rapid for a short period, bringing both groups eventually above the average for their age. This tendency has been represented in diagrammatic form. A comparison of this diagram with the analysed curves of blood pressure for girls with large, small and zero goitres in Figure 14 shows a remarkable similarity, which strengthens these conclusions.

In a previous paper(1) it was concluded that girls in Bern where goitre is severely endemic exhibit a retardation in growth increasing with the size of the goitre, which retardation is counteracted by regular iodine administration. At first sight this effect seems contradictory to the relation found in English girls and this might suggest that the forms of goitre in the two countries are different in character. The present research gives no evidence in support of such an idea but strongly suggests that they only differ as regards the intensity of the factor producing them. No theory of infection alone is capable of explaining the facts on either assumption. The iodine-deprivation theory of causation is however consistent with all the facts, difficult as they seem at first sight; thus in a district

such as Cheshire where there must be a *slight* deficiency in iodine intake on this theory, symptoms of deficiency will chiefly appear, *ceteris paribus*, in those girls needing most thyroid secretion to maintain their growth—namely in the best developed and quickly growing girls—hence we find positive correlations of thyroid size with growth factors due to this selection which obscure any slight retardation in growth which may result in these bigger girls. Where however the deprivation becomes so *severe* and universal that practically all girls whether quickly or slowly growing are affected, as at Bern, the correlation due to selection disappears and the opposite effect due to retardation becomes evident.

(8) RELATION TO BLOOD PRESSURE AND PULSE RATE.

The blood pressures and pulse rates of all the girls were measured in the sitting posture, using the auscultatory method checked by palpation. The index for the diastolic pressure was the point at which sudden dulling of the sound from "clear" to "muffled" occurs, usually termed the 4th point or end of the 3rd phase. The same instrument was used as in a similar work on boys, the results of which were published in 1924(8); this instrument was a Brunton sphygmomanometer with 12 cm. cuff; it had previously been tested as to accuracy against a mercury manometer.

One of the reasons for measuring blood pressures in the present research was that correlations have been found between basal metabolic rate and pulse pressure

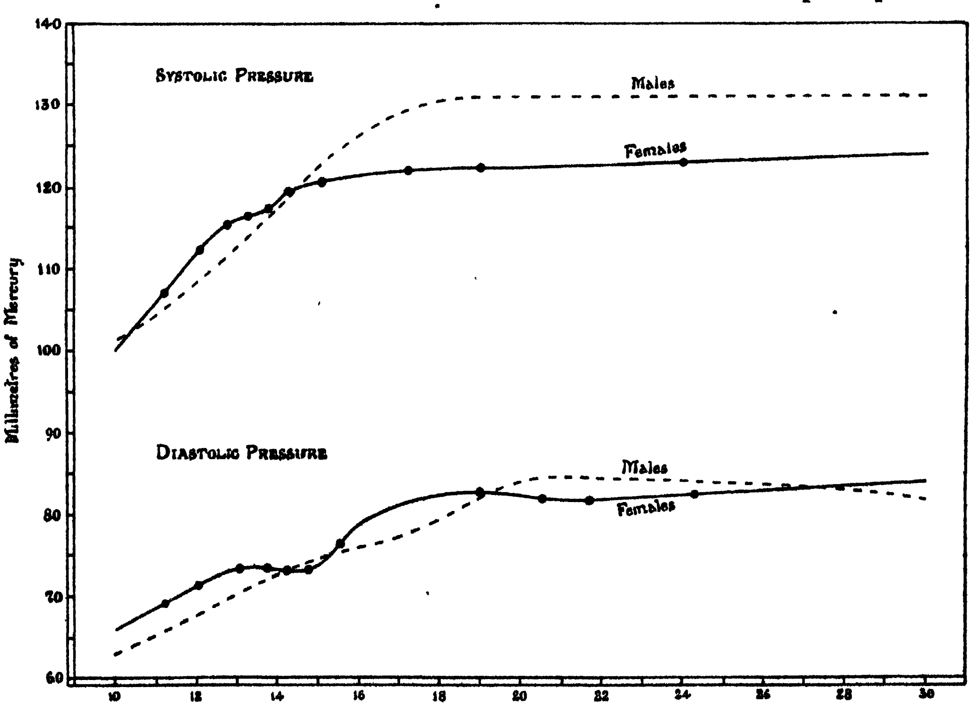


Fig. 18. Effect of age on blood pressure. Cheshire and London girls and students (compared with London boys and students).

as well as pulse rate by Read and others, and though this may be only characteristic of exophthalmic goitre, it suggested that a relation of some kind might be found between pulse pressure and the size of the thyroid in adolescence.

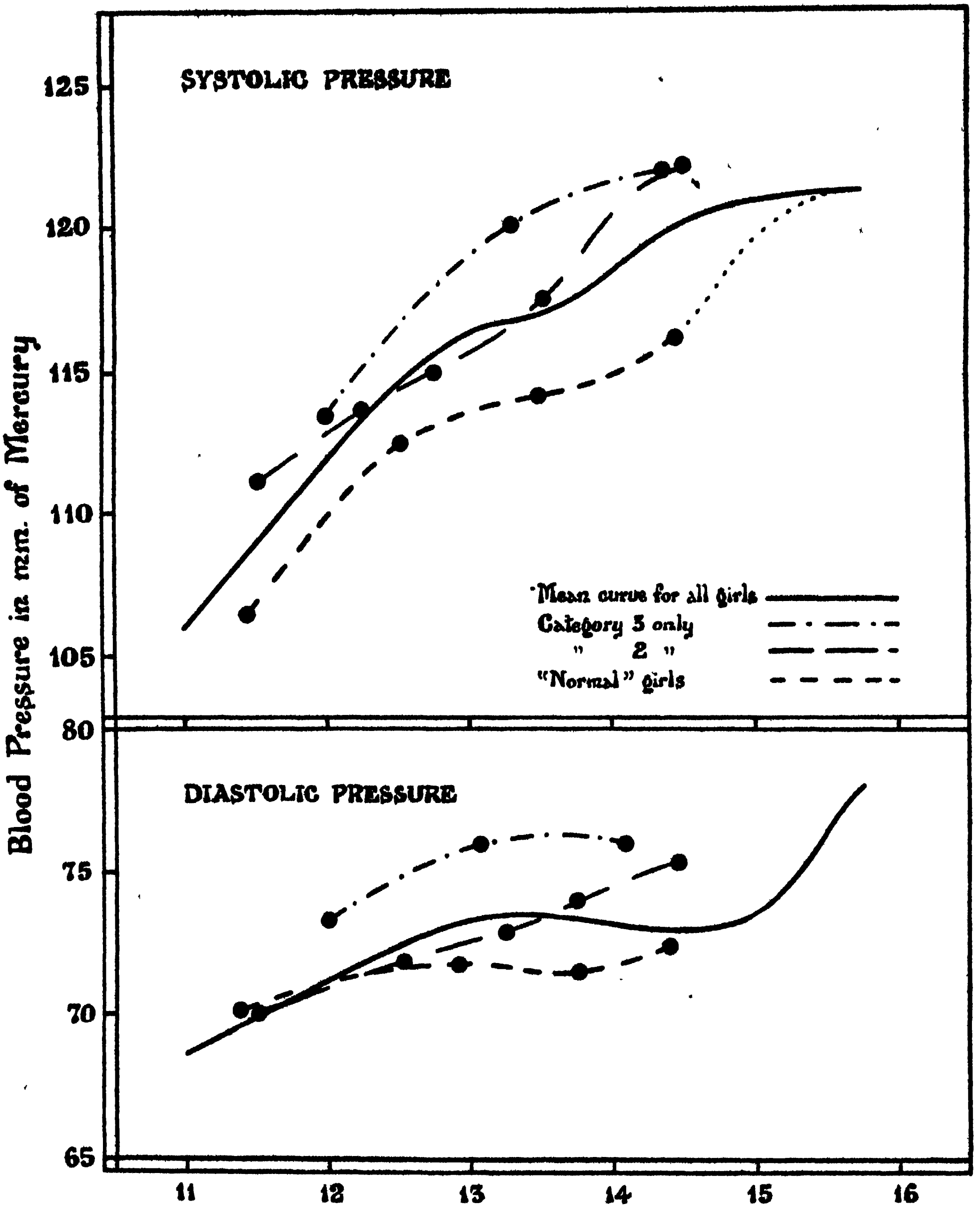


Fig. 14. Effect of age on blood pressure in goitrous and non-goitrous girls (Cheshire).

The relation of blood pressure to age has been worked out for boys in the memoir referred to above (8), but no extended series of blood pressure data in girls

has been hitherto published from the Galton Laboratory. In constructing the age curves shown in Figure 13 the data from the present study have been combined with blood pressure measurements carried out by an exactly similar technique with the same instrument on 326 women students of University College between the ages of 16 and 30. It is not proposed in the present paper to make any analysis of this data, and the details are not therefore given; the mean pressures at each age for the 537 school girls are shown in Table XXIII, and these and the students' data have been combined in producing the curves of Figure 13 which were readily drawn after combining a few neighbouring age-groups.

A comparison with the similar curves for males in Figures 5 and 6 of memoir (8), which have been reproduced as dotted lines in Figure 13, reveals interesting differences. The systolic pressure in girls is about the same as in boys at the 10th year but rises then more rapidly than in boys, almost reaching its adult level at 15, or some 3 years earlier than in boys; there is also in girls a pronounced pubescent dip about the 13th year. The diastolic pressure is some 3 mm. higher in girls than boys from 10 to 13, but a remarkable pubescent dip from 13 to 15 brings it below the boys' level; it then rises rapidly, reaching the adult level about 17, again about 3 years ahead of the boys in whom a similar rapid rise occurs from 17 to 20. In Figure 10 these curves are reduced to a scale unit equal to the pressure at age 11 in each case for comparison with the development of the thyroid gland on the same scale; this brings out a striking similarity between the curves for thyroid breadth and diastolic pressure, both showing a pronounced retardation about the period 12 to 14 followed by a rapid increase which is not however quite synchronous for the two factors. In Figure 14 the curves for normal girls have been separated from the composite data, and it then appears that in girls showing no visible enlargement the rapid rise in diastolic pressure begins at the end of the 14th year and is therefore synchronous with the rapid increase in normal thyroid size. In the lower part of Figure 10 the curve for pulse pressure has the same remarkable shape as was found in boys (memoir (8), Fig. 7), but the maximal peak is at 14—15 instead of 17—18 as in boys. The mean pulse pressures up to age 16 are shown in Table XXIII.

The mean pressures according to the smoothed curves from ages 10 to 20 in girls are given below in Table XXIX, and for comparison the corresponding values for the boys' curves referred to.

TABLE XXIX.

Age		*10*	*11*	*12*	*13*	*14*	*15*	*16*	*17*	*18*	*19*	*20*
Systolic	Girls ...	100·0	106·0	112·0	116·3	118·4	120·8	121·5	121·9	122·1	122·2	122·4
	Boys ...	*101*	*104*	*109*	*113*	*118*	*123*	*127*	*129*	*131*	*131*	*131*
Diastolic	Girls ...	66·0	68·6	71·3	73·3	73·2	73·6	78·9	81·1	82·3	82·8	82·2
	Boys ...	*63*	*65*	*68*	*70*	*73*	*75*	*76*	*77*	*79*	*82*	*84*
Pulse pressure	Girls ...	34·0	37·4	40·7	43·0	45·2	47·2	42·6	40·8	39·8	39·4	40·2
	Boys ...	*38*	*39*	*41*	*43*	*45*	*48*	*51*	*52*	*52*	*49*	*47*

The mean intervals between pulse beats at half-year periods are shown in Table XXIII. There is no very consistent change with age between 10 and 15 years though a slight slowing of the pulse is perhaps indicated up to the 13th year followed by a quickening from 13½ to 15. Continuation of the curve by means of the data for women students seems to indicate a slowing of the pulse from the 15th to 17th year where it attains the adult level. The complete curve reduced to a scale with the value at 11 years as unit is shown in Figure 10 and it is seen to be somewhat complementary to the pulse pressure curve. It is noteworthy that the rapid increase in average mean size of the thyroid from 13½ to 15 years is accompanied by a synchronous but less pronounced shortening of the mean pulse interval or quickening of pulse rate, which seems to be a temporary phenomenon only. There is no doubt that the changes manifested in all the curves in Figure 10 between 13½ and 15 years are directly associated with the establishment of menstruation.

In Figure 14 separate curves of systolic and diastolic pressure with age have been drawn in a somewhat rough manner through the points indicated, which are the mean pressures of the three categories of Cheshire girls after combining a few neighbouring half-year groups. The probable errors of individual points are of course considerable, but the manner in which the curves for the large and small goitre groups conform to the conclusions regarding growth at the end of the last section can scarcely be fortuitous.

The means and standard deviations after correction to age 12 were as follows:

TABLE XXX.

		Category 01	Categories 2+3	Complete sample
Systolic pressure, mm. of mercury	Mean ...	109·94 ± ·60	111·40 ± ·43	110·45 ± ·36
	σ ...	12·067	12·008	11·625
Diastolic pressure, mm. of mercury	Mean ...	71·05 ± ·39	72·58 ± ·36	71·27 ± ·25
	σ ...	7·343	8·850	7·569
Pulse pressure, mm. of mercury	Mean ...	40·76 ± ·60	43·11 ± ·42	41·11 ± ·36
	σ	11·131	10·239	11·045
Pulse interval, seconds	Mean ...	·668 ± ·005	·680 ± ·004	·670 ± ·003
	σ ...	·1065	·1144	·0951

The mean systolic pressures above differ from the sums of the mean diastolic and pulse pressures owing to the fact that in 12 °/ₒ of the girls the diastolic pressure could not be read with certainty, so the total systolic readings were in excess of those of diastolic or of pulse pressures. After correction to age 12 by the curves, the correlation coefficients with thyroid breadth were as shown in Table XXVI. After further correction for the normal effect of height by partial correlation as explained in Section (5) these become

TABLE XXXI.

Thyroid breadth with	Category 01	Categories 2+3	Complete sample
Systolic pressure ...	$-\cdot 0413 \pm \cdot 0500$	$\cdot 1527 \pm \cdot 0377$	$\cdot 0991 \pm \cdot 0303$
Diastolic „ ...	$\cdot 1059 \pm \cdot 0532$	$\cdot 1656 \pm \cdot 0398$	$\cdot 1310 \pm \cdot 0320$
Pulse „ ...	$\cdot 0317 \pm \cdot 0538$	$\cdot 0140 \pm \cdot 0409$	$\cdot 0684 \pm \cdot 0324$
Pulse interval ...	$\cdot 1139 \pm \cdot 0495$	$\cdot 0681 \pm \cdot 0384$	$\cdot 0546 \pm \cdot 0305$

Systolic pressure has a significant relation with thyroid size in the goitre group and the complete group, but not amongst normal girls alone. The regression line on thyroid breadth has been represented in Figure 15 by a cubic curve with the formula

$$p' = 112{\cdot}542 - 3{\cdot}2879B' + {\cdot}93206B'^2 - {\cdot}058180B'^3,$$

where p' = systolic pressure in mm. of mercury corrected to age 12, B' = thyroid breadth in cm. also corrected for age. The actual means and frequencies are given in Table XXIV.

Diastolic pressure has significant correlations with thyroid breadth in all groups after correction for height. The regression line shown in Figure 15 has been fitted with a cubic curve having the formula

$$p' = 55{\cdot}730 + 8{\cdot}1649B' - 1{\cdot}41097B'^2 + {\cdot}086324B'^3,$$

the actual means being shown in Table XXIV.

Pulse pressure has no significant relation with thyroid size either amongst the normal or goitrous girls.

The conclusion is that enlargement of the thyroid about the period of puberty is associated to a slight but significant extent with raised diastolic pressure whatever the degree of enlargement may be and after correction for age and height; when the enlargement becomes pronounced it becomes also associated with a slight rise of mean systolic pressure, the pulse pressure being therefore not affected.

Pulse interval has a smaller positive relation with thyroid size which is significant amongst normal girls but insignificant in the other groups; this indicates a slightly slower pulse in girls with larger thyroids. The mean points have been fitted with a cubic curve having the formula

$$\text{P.I.} = {\cdot}5433 + {\cdot}06495B' - {\cdot}009168B'^2 + {\cdot}0003743B'^3,$$

which is drawn in Figure 14. The points are somewhat irregular and would be almost as well fitted by the straight line

$$\text{P.I.} = {\cdot}6702 + {\cdot}00389\,(B' - 3{\cdot}533).$$

There is a suggestion from the mean points that thyroids enlarged to the extent of 6—8 cm. were associated with a more rapid pulse rate; but this certainly does not tend to occur with slight or moderate degrees of enlargement,

in fact there is a slight slowing of pulse rate as indicated by Figure 15 and by the significant correlation with pulse interval in the normal group. This, coupled with

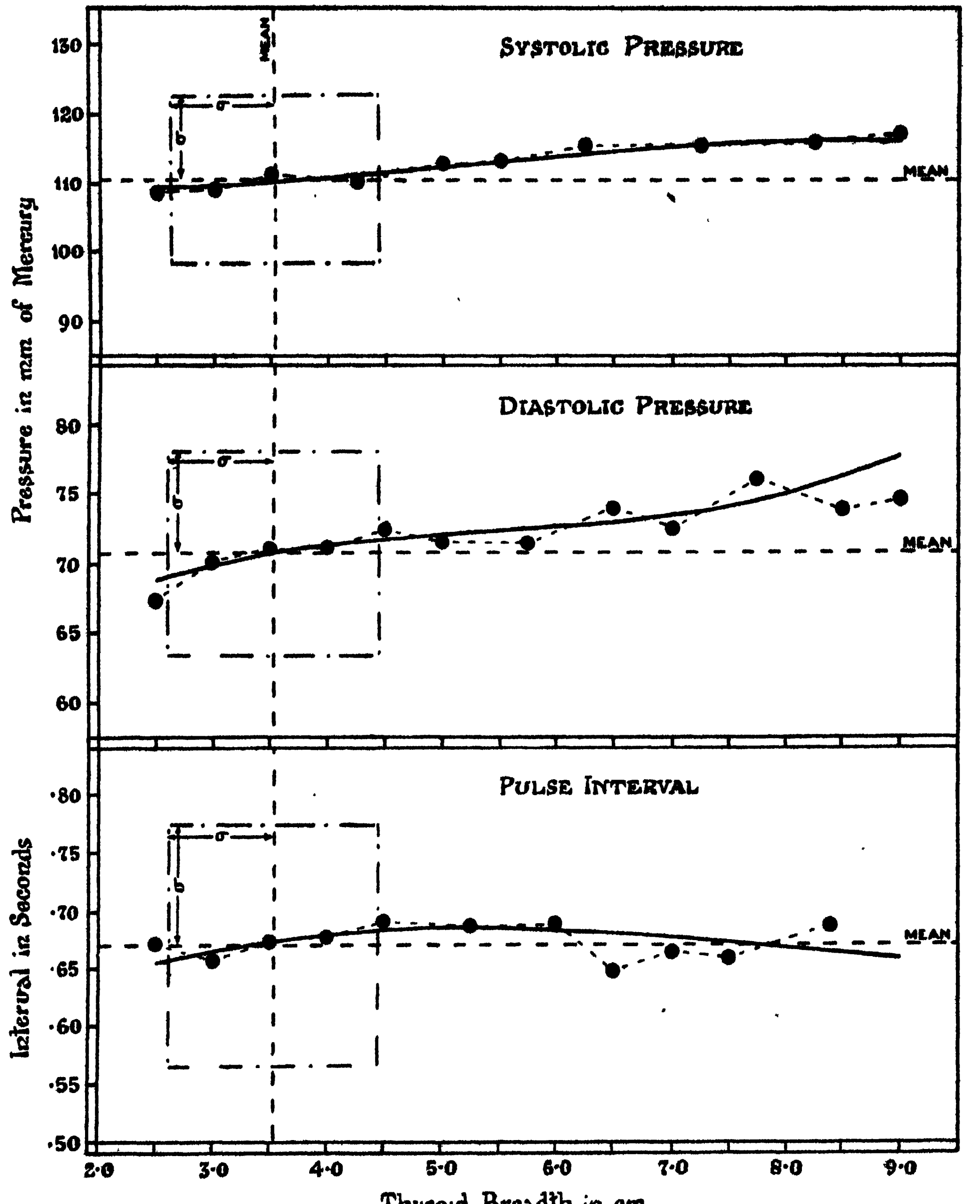


Fig. 15. Regression of blood pressure and pulse interval on thyroid breadth in girls: all corrected to age 12.

the absence of correlation with pulse pressure, seems to definitely rule out the presence of hyperthyroidism to any extent in the slight or moderate degrees of

enlargement met with about puberty, and to indicate rather the presence of factors tending to depress the pulse rate.

This again supports the view that the so-called "physiological" enlargements of the gland in girls are a manifestation of an effort by the gland, only partially successful, to overcome a deficiency of some substance stored by the gland and necessary to maintain growth—and that in this respect they are not essentially different in causation from the larger variety of goitre. That a large or small goitre may by the action of some nervous factor become complicated in some cases and lead to Graves' disease no doubt accounts for the presence of symptoms of hyperthyroidism in a certain proportion of girls with large goitres, but this is not evidence that the factors responsible for producing the goitre in the first instance were not of the same nature in all cases.

(9) RELATION TO COLOUR OF EYES AND HAIR.

The colour of the eyes and hair was recorded for every girl in order to determine whether these racial characters had any association with tendency to goitre. The *eye* scale used was Martin's and the 16 shades of diminishing brown pigment yielded the following frequencies, these girls being almost entirely drawn from rural schools in north Cheshire.

Eye shade number	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Frequency ...	—	3	9	41	30	28	34	29	13	44	45	13	51	66	52	24

For correlation purposes an arbitrary division was made into five groups, viz.: Very Dark = 1—4, Dark = 5—8, Medium = 9—12, Light = 13—14, Very Light = 15—16.

The mean thyroid breadths after age correction were

V. Dark	Dark	Medium	Light	V. Light
3·763 ± ·080	3·653 ± ·054	3·651 ± ·056	3·645 ± ·054	3·617 ± ·068.

These do not differ significantly. The correlation coefficient obtained by treating the groups as parts of a normal frequency distribution was with class index correction $r = \cdot0345 \pm \cdot0303$, where the + sign indicates more pigment. The corrected value of η is ·0754, $\eta^2 - \bar{\eta}^2 = \cdot003291$ and p.e. of $\bar{\eta}^2 = \cdot00186$. There is therefore no significant relation between thyroid size and colour of the eyes.

The *hair* colour was classified as Dark, Medium, Light, Very Light and Red. The mean thyroid breadths corrected for age and the actual numbers of girls in the groups were

	Dark	Medium	Light and V. Light	Red
Mean breadth	3·695 ± ·064	3·671 ± ·037	3·564 ± ·055	4·136 ± ·139
No. of girls	91	262	120	19

There are no significant differences between the first three groups, but the small group of red-haired girls gives a high value for the mean thyroid breadth

which differs significantly from that of the light-haired group, suggesting that red-haired girls are more liable to goitre: the number in this group however is small. The correlation coefficient obtained by the triserial method, combining the last two groups, is with class index correction $r = -\cdot 0140 \pm \cdot 0304$, the + sign indicating darkness; the corrected value of η is $\cdot 1068$, $\eta^2 - \bar{\eta}^2 = \cdot 001433$, p.e. of $\bar{\eta}^2 = \cdot 0016$. There is therefore no significant relation between darkness of hair and thyroid size.

(10) Relation to proficiency in school work.

This was assessed by the following method. The Head Teachers were asked to assign to each girl her rank, as based upon class lists or the Teacher's opinion, amongst all the girls in her class who were born in the same year, specifying the number of such girls. Where girls born in a given year were drawn from more than one class, it was assumed that the highest girl in the lower class followed on in order below the lowest girl in the higher class born in the same year. This assumption is admittedly open to objections since actually there would be an overlap, but in general the errors introduced by this assumption would be small. If the number of girls born in the year 1914 in the upper three classes of a school were n_1, n_2, n_3, and a girl born in 1914 was ranked 4th in the 3rd class amongst the n_3 girls of the same age, her proficiency would be measured by the fraction $\frac{n_1 + n_2 + 4}{n_1 + n_2 + n_3}$; if she was placed 5th in the 1st class amongst the n_1 girls, her proficiency would be given by $\frac{5}{n_1 + n_2 + n_3}$.

The fractions thus obtained were reduced to give the place in a class of 100 girls born in the same year. Thus if the girl's rank be x amongst n girls, her rank in 100 was taken to be $100 \times \frac{x - \frac{1}{2}}{n}$ to the nearest integer; this amounts to placing her at the centre of the xth interval of uniform size $\frac{100}{n}$.

The resulting distribution of these indices was roughly uniform over a linear scale of 1 to 99; the scale was then divided into 9 intervals symmetrically arranged about 50 and so graduated as to give the nearest approach to a normal frequency distribution.

The most suitable division of the scale was as follows and this was adopted for the correlation tables:

Proficiency group .	I	II	III	IV	V	VI	VII	VIII	IX
Rank in 100	1	2—5	6—16	17—37	38—62	63—83	84—94	95—98	99
Frequency observed ...	4	43	80	91	101	95	34	18	6

The distribution is slightly skew; taking the class interval of this scale as unit of proficiency and the centre of group V as origin, the means and standard deviations resulting were as follows:

Category 01. Mean $+ \cdot 3218 \pm \cdot 0860$, $\sigma = 1 \cdot 6806$.
Categories 2 + 3. „ $+ \cdot 4602 \pm \cdot 0675$, $\sigma = 1 \cdot 7005$.
Complete sample. „ $+ \cdot 3415 \pm \cdot 0572$, $\sigma = 1 \cdot 8246$.

The means do not differ significantly. The correlation coefficients between proficiency and thyroid breadth are shown in Table XXVI. There is a correlation with height after age correction, $r = \cdot 2081 \pm \cdot 0300$. After correction for height by partial correlation as in the previous sections, the coefficients with thyroid breadth become

Category 01. $r = \cdot 0413 \pm \cdot 0510$.
Categories 2 + 3. $r = \cdot 0641 \pm \cdot 0395$.
Complete sample. $r = \cdot 0766 \pm \cdot 0312$.

There is therefore no certainly significant correlation with proficiency in school work; the somewhat larger coefficient for the complete sample may be due to the effect of heterogeneity in this sample. The coefficients, however, seem to definitely disprove any association with backwardness.

The regression of proficiency in class on thyroid breadth is shown in Figure 16, the mean of all girls being here taken as the zero line; the actual deviations from this mean, and the original frequencies, are shown in Table XXXII.

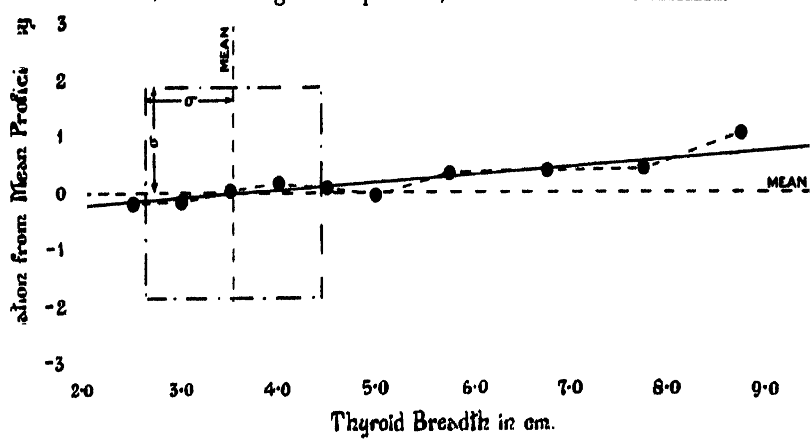


Fig. 16. Regression of proficiency in school work on thyroid breadth in girls: both corrected for age.

TABLE XXXII.

B' cm. =	2·5—	3—	3·5—	4—	4·5—	5—	5·5—	6—	6·5—	7—	7·5—	8—	8·5—	9—
Deviation from mean proficiency	−·191	−·141	+·040	+·177	+·074	−·060	+·536	+·059	−·499	+1·166	+·221	+·809	+1·809	−1·191
Number of girls	27	59	106	83	66	38	22	28	13	14	17	8	6	2

The equation of the regression line is $y = \cdot 1355 (B' - 3 \cdot 537)$.

(11) Conclusions.

1. The method of classifying the sizes of thyroid glands by broad categories without measurement is subject to large personal equations and is only to be recommended for a rapid and continuous survey by a single observer. Measurement of circumference or diameter of the neck is also of little value except as a means of registering the progress of a large goitre.

2. Measurement of the maximum breadth of the thyroid gland is recommended as the most reliable and useful index of the actual size of the gland; additional measurements of the vertical heights of the lobes, besides being difficult to carry out, add nothing to accuracy for biometric purposes.

3. Goitre prevalence increases with age to a maximum in boys about 13—14 and in girls about 17—18, and then declines. Where goitre is not endemic the thyroid gland changes very slightly in size from 10½ to 13½, this flattening of the growth curve just before puberty being unique amongst physical measurements so far investigated. In girls a rapid development of the gland occurs between 13½ and 15 which is doubtless associated with the onset of menstruation.

4. After correction for age and the normal variation with height, there remains in *boys* no evidence of any sensible relation between size of the thyroid gland and physical development, size of the head, rate of growth or strength of grip. In *girls*, however, there remains a significant positive association with height and weight, rates of growth, grip, systolic and diastolic pressures, but none with pulse pressure, pulse rate, colour of hair or eyes or proficiency in school work. The regressions of these physical characters on thyroid size for the most part become greater as the size of the goitre increases. The correspondence between growth curves of the thyroid and of diastolic pressure is noteworthy. Long-continued general development in excess of the average rate seems to be favourable to the appearance of large goitres whilst slow followed by temporary rapid growth is associated with small enlargements.

5. The complex relations to rate of physical development in girls brought to light in this and a previous paper (1) are capable of explanation on the theory of iodine deprivation for which they indirectly afford additional proof; the facts cannot be explained by an infective theory unless the infective agent acts by preventing proper absorption of iodine.

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SUPPLEMENTARY TABLES FOR DETERMINING CORRELATION FROM TETRACHORIC GROUPINGS.

(First Series.)

THE following table supplements and replaces the table published in Vol. VIII. pp. 385—395, of this Journal. It supplements it in that this table ranges from $r = \cdot 00$ to $r = 1\cdot 00$, and it replaces it in that the cell content is now given to six instead of four decimal places. The present table enables the value of the tetrachoric coefficient of correlation to be found for all positive values of r when h and k have the same sign, and for all negative values of r when h and k have different signs. It is hoped that the computer, Dr Alice Lee, may be able to extend her table of 1916* to the values of r between $\cdot 00$ and $-\cdot 80$ for h and k both of the same sign, which will complete the full range of correlation and enable us to determine any tetrachoric coefficient by simple interpolation and without the need in any case of solving a high order equation.

The computation of the present table has been rendered possible by aid from the Royal Society Government Grant Committee.

If we have a bivariate normal system:

$$z = \frac{N}{2\pi\,\sigma_x\sigma_y\sqrt{1-r^2}}\,e^{-\frac{1}{2}\frac{1}{1-r^2}\left(\frac{x^2}{\sigma_x^2} - \frac{2rxy}{\sigma_x\sigma_y} + \frac{y^2}{\sigma_y^2}\right)}$$

and

$$\frac{d}{N} = \frac{1}{2\pi\sqrt{1-r^2}}\int_h^\infty\int_k^\infty e^{-\frac{1}{2}\frac{1}{1-r^2}(x^2 - 2rxy + y^2)}\,dx\,dy,$$

then the present table gives the value of d/N in the "quadrant" from $h = x/\sigma_x$, $k = y/\sigma_y$, both up to infinity for values of r proceeding by $\cdot 05$ from $\cdot 00$ to $1\cdot 00$, and values of h and k from 0 to $2\cdot 6$ proceeding by $\cdot 1$. The table may be used for two purposes: first to find the tetrachoric coefficient of any fourfold table, and secondly to determine the contents of any cell (or of all the cells) in a bivariate table on the hypothesis that the distribution is normal; this is of course preliminary to testing whether the assumption of normality is legitimate, i.e. to see whether the observed distribution provides a reasonable value in the "goodness of fit" test. For the first purpose undoubtedly a table in which the arguments were $\frac{1}{2}(1-\alpha_1)$, $\frac{1}{2}(1-\alpha_2)$, i.e. the "tail areas" beyond h and k, respectively, would be most serviceable, in that these are directly given and we should not have

* *Biometrika*, Vol. XI. pp. 284—291.

to determine h and k to start with for our fourfold table. But when we need to determine cell contents from graduated scales, it seems more reasonable to proceed by h and k, which will be known as soon as the means and standard deviations have been found. h and k will then be affected by sampling errors in the means and standard deviations and not by irregularities in the individual marginal cell contents. Of course, theoretical marginal cell contents could be found from these values of h and k, and thus finally a table of $\frac{1}{2}(1-\alpha_1)$ and $\frac{1}{2}(1-\alpha_2)$ used, but the labour on the whole would be, we think, greater, although it would certainly be less in finding a tetrachoric correlation coefficient from the latter table.

First: To determine a tetrachoric coefficient of correlation we have primarily to determine the d/N, the $\frac{1}{2}(1-\alpha_1)$, and the $\frac{1}{2}(1-\alpha_2)$; from the two latter we determine h and k. Hence from a knowledge of d/N, h and k we have to find r. Now this clearly is a case of determining r from a table of triple argument, and thus the problem reduces to the choice of an appropriate method of interpolation. That method depends on the number of decimal places required in the answer. We may note the following points:

(*a*) Linear interpolation with regard to the argument r appears for the whole range of examples on which we have tested the table to be adequate.

(*b*) Hyperbolic interpolation* with regard to the arguments h and k is practically adequate for obtaining the value of r to three decimal places throughout the table. It is not adequate for obtaining r correctly to four decimal places.

(*c*) To obtain r correct to four decimal places we must use a central difference formula as far as the δ^2's. A forward difference formula going as far as the first Δ's was found to give poorer results than a hyperbolic interpolation.

(*d*) At the boundaries of the table, where a result to four-figure accuracy is required, it is needful to use a central difference finial or boundary formula.

It would undoubtedly have been a gain, if a single formula, i.e. the symmetrical hyperbolic interpolation, had sufficed, but this would have necessitated a table with the h and k intervals reduced to ·05, that is to say, a table occupying 64 and not 32 pages and so beyond our powers of publishing.

The following are the interpolation formulae needful for an effective use of the table. The diagrams explain the notation. Differences involving the first subscript are written δ^2, and those involving the second subscript δ'^2. As it was impossible on account of cost to publish the differences the reader must bear in mind that:

$$\delta^2 z_{r,s} = z_{r+1,s} + z_{r-1,s} - 2z_{r,s},$$
$$\delta'^2 z_{r,s} = z_{r,s+1} + z_{r,s-1} - 2z_{r,s},$$

and these are easily obtained from the table by aid of any calculating machine.

* The hyperbolic interpolation corresponds to fitting the hyperbolic paraboloid $z = a_0 + a_1 x + a_2 y + a_3 xy$ to the four nearest entries.

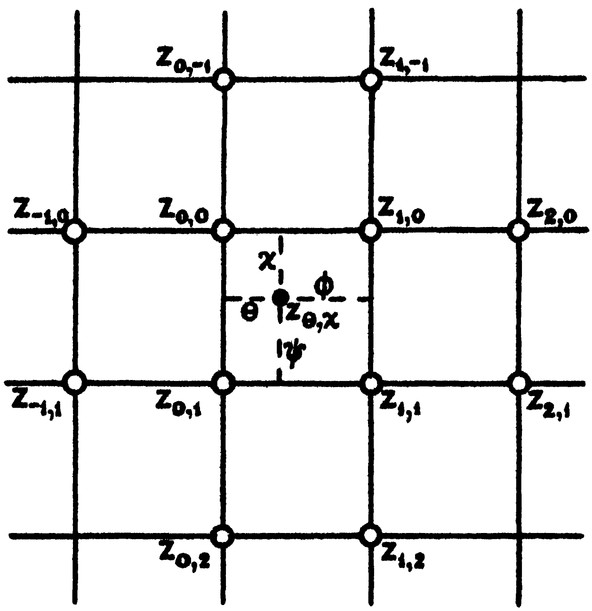
Formulae.

(α) *Hyperbolic Interpolating Formula.*

$$z_{\theta,\chi} = \phi\psi z_{0,0} + \phi\chi z_{0,1} + \theta\psi z_{1,0} + \theta\chi z_{1,1}.$$

(β) *Midpanel Central Difference Formula.* (Correct to third differences.)

$$\begin{aligned} z_{\theta,\chi} = {} & \phi\psi z_{0,0} + \phi\chi z_{0,1} + \theta\psi z_{1,0} + \theta\chi z_{1,1} \\ & - \tfrac{1}{6}\theta\phi\{(1+\phi)(\psi\delta^2 z_{0,0} + \chi\delta^2 z_{0,1}) + (1+\theta)(\psi\delta^2 z_{1,0} + \chi\delta^2 z_{1,1})\} \\ & - \tfrac{1}{6}\psi\chi\{(1+\psi)(\phi\delta'^2 z_{0,0} + \theta\delta'^2 z_{1,0}) + (1+\chi)(\phi\delta'^2 z_{0,1} + \theta\delta'^2 z_{1,1})\}. \end{aligned}$$



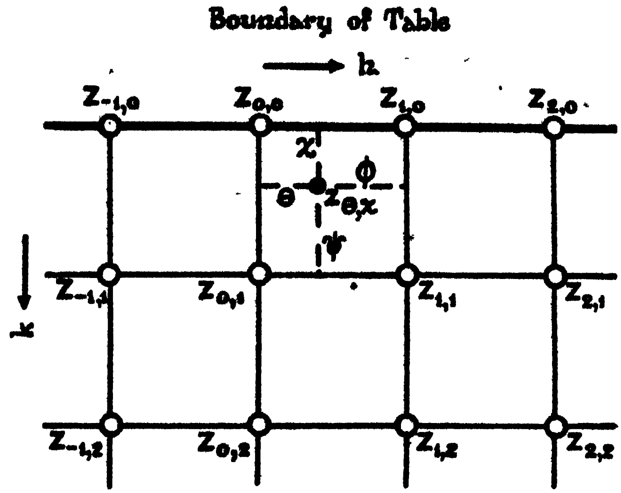
FORMULAE (α) & (β)
ORDINARY NON-FINIAL PANEL
(NEITHER h NOR k LIMITED)

Fig. 1.

(γ) *Midpanel Central Difference Singly Finial Formula**. (Correct to third differences.)

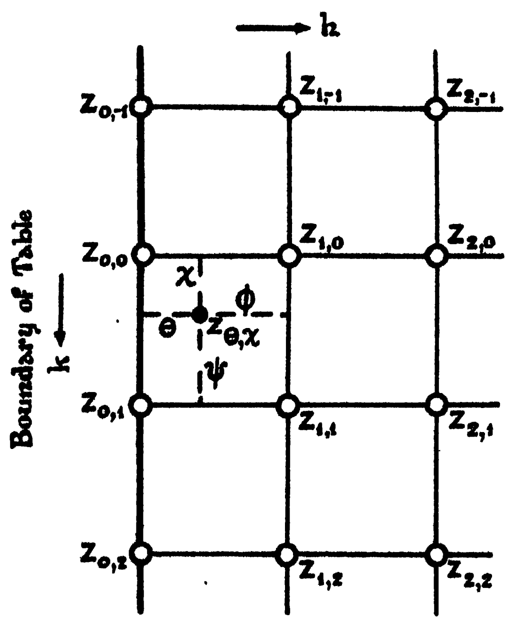
$$\begin{aligned} z_{\theta,\chi} = {} & \phi\psi z_{0,0} + \phi\chi z_{0,1} + \theta\psi z_{1,0} + \theta\chi z_{1,1} \\ & - \tfrac{1}{6}\theta\phi\{(1+\phi)(\psi\delta^2 z_{0,0} + \chi\delta^2 z_{0,1}) + (1+\theta)(\psi\delta^2 z_{1,0} + \chi\delta^2 z_{1,1})\} \\ & - \tfrac{1}{6}\psi\chi\{(4+\psi)(\phi\delta'^2 z_{0,1} + \theta\delta'^2 z_{1,1}) - (1+\psi)(\phi\delta'^2 z_{0,2} + \theta\delta'^2 z_{1,2})\}. \end{aligned}$$

* This formula applies when our value of h (corresponding to δ'^2 and the second subscript) lies between ·00 and ·10.



FORMULA (γ)
SINGLY FINIAL (k LIMITED)

Fig. 2.



FORMULA (γ) *bis*
SINGLY FINIAL (h LIMITED)

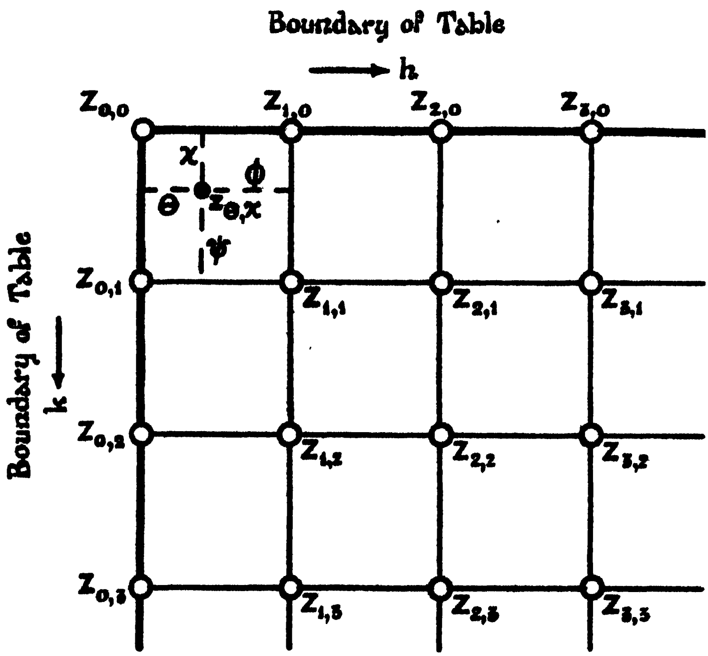
Fig. 3.

$(\gamma)^{bis}$ *Midpanel Central Difference Singly Finial Formula**. (Correct to third differences.)

$$\begin{aligned} z_{\theta,\chi} = {} & \phi\psi z_{0,0} + \phi\chi z_{0,1} + \theta\psi z_{1,0} + \theta\chi z_{1,1} \\ & - \tfrac{1}{6}\theta\phi\left\{(4+\phi)(\psi\delta^2 z_{1,0} + \chi\delta^2 z_{1,1}) - (1+\phi)(\psi\delta^2 z_{2,0} + \chi\delta^2 z_{2,1})\right\} \\ & - \tfrac{1}{6}\psi\chi\left\{(1+\psi)(\phi\delta'^2 z_{0,0} + \theta\delta'^2 z_{1,0}) + (1+\chi)(\phi\delta'^2 z_{0,1} + \theta\delta'^2 z_{1,1})\right\}. \end{aligned}$$

(δ) *Midpoint Central Difference Doubly Finial Formula*†. (Correct to third differences.)

$$\begin{aligned} z_{\theta,\chi} = {} & \phi\psi z_{0,0} + \phi\chi z_{0,1} + \theta\psi z_{1,0} + \theta\chi z_{1,1} \\ & - \tfrac{1}{6}\theta\phi\left\{(4+\phi)(\psi\delta^2 z_{1,0} + \chi\delta^2 z_{1,1}) - (1+\phi)(\psi\delta^2 z_{2,0} + \chi\delta^2 z_{2,1})\right\} \\ & - \tfrac{1}{6}\psi\chi\left\{(4+\psi)(\phi\delta'^2 z_{0,1} + \theta\delta'^2 z_{1,1}) - (1+\psi)(\phi\delta'^2 z_{0,2} + \theta\delta'^2 z_{1,2})\right\}. \end{aligned}$$



FORMULA (δ)

DOUBLY FINIAL PANEL (h AND k LIMITED)

Fig. 4.

In all cases a knowledge of h and k to five decimal places will be adequate. This can be found by linear interpolation from the table in *Biometrika*, Vol. VII. pp. 442—451, or in the book, *Tables for Statisticians*, Table XXIX, pp. 42—51, where the value of h to five decimal places is given for each value of $\frac{1}{2}(1-\alpha)$ from ·001 to ·500 by increments of ·001. We shall refer to this as Table XXIX.

* This formula applies when our value of h (corresponding to δ^2 and the first subscript) lies between ·00 and ·10.

† This formula applies when both h and k lie between ·00 and ·10.

It will be seen by our diagrams that θ and ϕ refer to variation of h, and χ and ψ to variation of k. Further the terms of (α) are common to all the formulae, and we propose to consider the degree of accuracy with which they suffice to give the correct value of r at different parts of the table.

Illustration I. Intelligence and Enlarged Glands. (Boys.)

Intelligence.

State of Glands.	Intelligent and Over	Slow Intelligent and Under	Totals
Normal	241	215	456
Enlarged	82	81	163
Totals	323	296	619

$$\tfrac{1}{2}(1-\alpha_h)=\tfrac{296}{619}=\cdot 478{,}191,\quad \tfrac{1}{2}(1-\alpha_k)=\tfrac{163}{619}=\cdot 263{,}328.$$

Hence by Table XXIX, using linear interpolation,

$$h=\cdot 05469,\quad k=\cdot 63312.$$

Further: $$d/N=\tfrac{81}{619}=\cdot 130{,}8562.$$

From these arguments we are to find r. Clearly h lies between $\cdot 0$ and $\cdot 1$ and k between $\cdot 6$ and $\cdot 7$. The value of d for these values of h and k might lie in the tables for $r=\cdot 00$, $r=\cdot 05$ or $r=\cdot 10$. The latter is very improbable because the point corresponding to our h and k lies in the triangle of which the z's are $\cdot 150\ldots$, $\cdot 139\ldots$ and $\cdot 133\ldots$, all greater than our $d/N=\cdot 130\ldots$. It is safest, however, to start interpolating from the mid-table $r=\cdot 05$.

We have $\theta=\cdot 5469$, $\phi=\cdot 4531$, $\chi=\cdot 3312$, $\psi=\cdot 6688$ and the scheme:

	$h=0\cdot 0$	$h=-0\cdot 1$
$k=0\cdot 6$	$\cdot 143{,}775\ (z_{0,0})$	$\cdot 132{,}829\ (z_{1,0})$
$k=0\cdot 7$	$\cdot 127{,}212\ (z_{0,1})$	$\cdot 117{,}555\ (z_{1,1})$

We then arrange our work thus, using formula (α):

$$d/N=\left.\begin{matrix}\cdot 6688\times\cdot 4531\\ \cdot 3030{,}3228\end{matrix}\right\}\left\{\begin{matrix}\cdot 143{,}775+\cdot 6688\times\cdot 5469\\ \cdot 137{,}127\quad \cdot 3657{,}6672\end{matrix}\right\}\left\{\begin{matrix}\cdot 132{,}829\\ \cdot 126{,}204\end{matrix}\right.$$

$$+\left.\begin{matrix}\cdot 3312\times\cdot 4531\\ \cdot 1500{,}6672\end{matrix}\right\}\left\{\begin{matrix}\cdot 127{,}212+\cdot 3312\times\cdot 5469\\ \cdot 120{,}984\quad \cdot 1811{,}3328\end{matrix}\right\}\left\{\begin{matrix}\cdot 117{,}555\\ \cdot 111{,}345\end{matrix}\right.$$

The lower number in the curled brackets facing left is the product of the θ, ϕ, χ, ψ pairs standing above them. Taking the upper numbers in the brackets facing right and multiplying them continuously on the machine we find

$$d/N=\cdot 132{,}5363.$$

This value is too high; we must therefore reduce the correlation by taking the table for $r=\cdot 00$. The corresponding values for the z's are those in the lower line of the curled brackets facing right, and repeating the continuous multiplication process

on the machine we have $d/N = \cdot 126{,}0388$. Interpolating linearly for $d/N = \cdot 130{,}8562$ we find

$$r = \cdot 00 + \cdot 05 \times \frac{48174}{64975} = \cdot 0371.$$

This result would be close enough for the great majority of practical investigations, but the true value of r is ·0379. It is of interest to determine how the table result may be brought into accord with the actual value of r. Our d/N is in a finial panel of the table; we cannot get the central difference for $z_{0,0}$ and $z_{0,1}$ as we are on the border of the table $h = 0{\cdot}0$. Accordingly, let us try the forward difference formula:

(ϵ) *Forward Difference Formula.* (Correct to second differences.)

$$\begin{aligned} z_{\theta,\chi} = z_{0,0} &+ \theta\,(z_{1,0} - z_{0,0}) + \chi\,(z_{0,1} - z_{0,0}) \\ &+ \tfrac{1}{2}\,[\theta\,(\theta - 1)\,(z_{2,0} - 2z_{1,0} + z_{0,0}) + 2\theta\chi\,(z_{1,1} - z_{0,1} - z_{1,0} + z_{0,0}) \\ &+ \chi\,(\chi - 1)\,(z_{0,2} - 2z_{0,1} + z_{0,0})]. \end{aligned}$$

Here $\theta = \cdot 5469$ and $\chi = \cdot 3312$ as before, and we have for the two tables $r = \cdot 00$ and $r = \cdot 05$:

$$\begin{aligned} z_{\theta,\chi} = \begin{Bmatrix}\cdot 137{,}127\\ \cdot 143{,}775\end{Bmatrix} &+ \cdot 5469 \begin{Bmatrix}-\cdot 010{,}923\\ -\cdot 010{,}946\end{Bmatrix} + \cdot 3312 \begin{Bmatrix}-\cdot 016{,}145\\ -\cdot 016{,}563\end{Bmatrix} \\ &- \cdot 1239{,}0019 \begin{Bmatrix}\cdot 000{,}108\\ \cdot 000{,}043\end{Bmatrix} - \cdot 1107{,}5328 \begin{Bmatrix}\cdot 001{,}091\\ \cdot 001{,}058\end{Bmatrix} \\ &+ \cdot 1811{,}3228 \begin{Bmatrix}\cdot 001{,}286\\ \cdot 001{,}289\end{Bmatrix} \\ &= \cdot 125{,}905, \text{ from upper line,} \\ &= \cdot 132{,}414, \text{ from lower line.} \end{aligned}$$

Interpolating linearly, $r = \cdot 00 + \cdot 05\,\dfrac{4951}{6509} = \cdot 0380$, which is in good agreement with the value ·0379 found by developing the tetrachoric equation and solving it as in the old method.

Let us now use the appropriate central difference finial formula $(\gamma)^{bis}$. We require the values of the following eight δ^2's and δ'^2's for the two tables of r*:

	$r = \cdot 00$	$r = \cdot 05$
$\delta^2 z_{1,0}$	·000,108	·000,043
$\delta^2 z_{1,1}$	·000,096	·000,034
$\delta^2 z_{2,0}$	·000,215	·000,151
$\delta^2 z_{2,1}$	·000,189	·000,129
$\delta'^2 z_{0,0}$	·000,997	·000,956
$\delta'^2 z_{1,0}$	·000,917	·000,876
$\delta'^2 z_{0,1}$	·001,091	·001,058
$\delta'^2 z_{1,1}$	·001,004	·000,971

* It is extremely easy to obtain by the machine the central differences of any table entry. To get δ^2 for any entry, add the entries to right and left of the given entry, place the entry itself on the machine and subtract it twice; the result on the slide is the δ^2. To get δ'^2 for any entry, add the entries above and below it and subtract twice the given entry. The discovery of either δ^2 or δ'^2 is thus a single continuous operation, and one of great simplicity and rapidity.

We need only compute the second and third lines of $(\gamma)^{bis}$, as we have already obtained the values of the first. Calling this remainder R, we have:

$$R = -\cdot 0413{,}0006\left[4\cdot 4531\left(\cdot 6688\begin{cases}\cdot 000{,}108\\ \cdot 000{,}043\end{cases} + \cdot 3312\begin{cases}\cdot 000{,}096\\ \cdot 000{,}034\end{cases}\right)\right.$$
$$\left. - 1\cdot 5469\left(\cdot 6688\begin{cases}\cdot 000{,}215\\ \cdot 000{,}151\end{cases} + \cdot 3312\begin{cases}\cdot 000{,}189\\ \cdot 000{,}129\end{cases}\right)\right]$$
$$-\cdot 0369{,}1776\left[1\cdot 6688\left(\cdot 4531\begin{cases}\cdot 000{,}997\\ \cdot 000{,}956\end{cases} + \cdot 5469\begin{cases}\cdot 000{,}917\\ \cdot 000{,}876\end{cases}\right)\right.$$
$$\left. + 1\cdot 3312\left(\cdot 4531\begin{cases}\cdot 001{,}091\\ \cdot 001{,}058\end{cases} + \cdot 5469\begin{cases}\cdot 001{,}004\\ \cdot 000{,}971\end{cases}\right)\right],$$

the upper line in the curled brackets referring to $r = \cdot 00$, and the lower to $r = \cdot 05$. These give:

$$R = \begin{cases}-\cdot 000{,}1159\\ -\cdot 000{,}1040\end{cases},$$

which must be taken away from the hyperbolic formula values of d/N, i.e. $\cdot 126{,}0388$ and $\cdot 132{,}5363$ respectively.

Thus we get

$$d/N = \cdot 125{,}9229 \quad \text{for} \quad r = 0\cdot 00,$$

and

$$\cdot 132{,}4323 \quad \text{for} \quad r = 0\cdot 05.$$

Hence, interpolating for $d/N = \cdot 130{,}8562$ we find $r = \cdot 0379$, the correct value. We accordingly conclude that the correct value in a finial panel will be obtained by the central difference formula $(\gamma)^{bis}$, although it fails to be given by a forward difference formula.

Illustration II. Intelligence and General Nutrition. (*Girls.*)

Intelligence.

Nutrition.	Intelligent and Above	Slow Intelligent and Under	Totals
Good	91	171	262
Medium to Bad	102	221	323
Totals	193	392	585

In this case $\frac{1}{2}(1-\alpha_h) = \cdot 329{,}914$, $\frac{1}{2}(1-\alpha_k) = \cdot 447{,}863$, leading by aid of Table XXIX to

$$h = \cdot 44015, \quad k = \cdot 13107,$$

while

$$d/N = \frac{91}{585} = \cdot 155{,}5556.$$

Clearly $\theta = \cdot 4015$, $\phi = \cdot 5985$, $\chi = \cdot 3107$, $\psi = \cdot 6893$.

Between the values $h = \cdot 4$ and $\cdot 5$ and $k = \cdot 1$ and $\cdot 2$, our value for d/N occurs in the r tables for values $\cdot 00$, $\cdot 05$, $\cdot 10$ and $\cdot 15$ but the h and k values are nearer to $\cdot 4$ and $\cdot 1$ than $\cdot 5$ and $\cdot 2$. This excludes $r = \cdot 15$, as clearly $d/N = \cdot 155\ldots$ lies

more than half-way from ·4 to ·5 and ·1 to ·2, i.e. the diagonal right top to left bottom corner is from ·163... to ·166.... Accordingly we start with interpolating into the $r = {}$·05 table. We have

$$d/N = z_{\theta,\chi} = \begin{matrix}\cdot6893\times\cdot5985\\ \cdot4125{,}4605\end{matrix}\left\{\begin{matrix}\cdot165{,}835\\ \cdot173{,}234\end{matrix} + \begin{matrix}\cdot6893\times\cdot4015\\ \cdot2767{,}5395\end{matrix}\left\{\begin{matrix}\cdot148{,}979\\ \cdot156{,}008\end{matrix}\right.\right.$$
$$+ \begin{matrix}\cdot3107\times\cdot5985\\ \cdot1859{,}5395\end{matrix}\left\{\begin{matrix}\cdot152{,}195\\ \cdot159{,}456\end{matrix} + \begin{matrix}\cdot3107\times\cdot4015\\ \cdot1247{,}4605\end{matrix}\left\{\begin{matrix}\cdot136{,}717\\ \cdot143{,}667\end{matrix}\right.\right.$$

The upper numbers in the right-facing curled brackets give $d/N = {}$·155,0218. This is not large enough, and we accordingly choose the $r = {}$·10 table for our second interpolation and this gives $d/N = {}$·162,2162. Finally, interpolating linearly for $d/N = {}$·155,5556, we have

$$r = \cdot05 + \cdot05 \times \frac{5338}{71944} = \cdot0537.$$

The true value of r is ·0542.

If we proceed by the forward difference formula, we find $r = {}$·0545, again a result slightly in excess of the true value. As our value of d/N does not fall into a finial panel, formula (β) applies, and we proceed to calculate the additional terms. We find:

	$r = \cdot05$	$r = \cdot10$
$\delta^2 z_{0,0}$	·000,614	·000,552
$\delta^2 z_{0,1}$	·000,756	·000,703
$\delta^2 z_{1,0}$	·000,558	·000,496
$\delta^2 z_{1,1}$	·000,687	·000,634
$\delta'^2 z_{0,0}$	·000,063	− ·000,010
$\delta'^2 z_{1,0}$	·000,053	− ·000,017
$\delta'^2 z_{0,1}$	·000,201	·000,130
$\delta'^2 z_{1,1}$	·000,175	·000,107

Substituting in the formula in a manner suitable for a nearly continuous operation we obtain—the upper figures in curled brackets referring to $r = {}$·05, and the lower to $r = {}$·10 interpolation:

$$R = \begin{matrix}-\tfrac{1}{8}\cdot4015\times\cdot5985\\ -\cdot0400{,}4963\end{matrix}\left[1\cdot5985\left(\cdot6893\begin{Bmatrix}\cdot000{,}614\\ \cdot000{,}552\end{Bmatrix} + \cdot3107\begin{Bmatrix}\cdot000{,}558\\ \cdot000{,}496\end{Bmatrix}\right) + 1\cdot4015\left(\cdot6893\begin{Bmatrix}\cdot000{,}756\\ \cdot000{,}703\end{Bmatrix} + \cdot3107\begin{Bmatrix}\cdot000{,}687\\ \cdot000{,}634\end{Bmatrix}\right)\right]$$
$$\begin{matrix}-\tfrac{1}{8}\cdot3107\times\cdot6893\\ -\cdot0356{,}9425\end{matrix}\left[1\cdot6893\left(\cdot5985\begin{Bmatrix}\cdot000{,}063\\ -\cdot000{,}010\end{Bmatrix} + \cdot4015\begin{Bmatrix}\cdot000{,}053\\ -\cdot000{,}017\end{Bmatrix}\right) + 1\cdot3107\left(\cdot5985\begin{Bmatrix}\cdot000{,}201\\ \cdot000{,}130\end{Bmatrix} + \cdot4015\begin{Bmatrix}\cdot000{,}175\\ \cdot000{,}107\end{Bmatrix}\right)\right]$$
$$= -\cdot0400{,}4963\left[\begin{Bmatrix}\cdot000{,}9537\\ \cdot000{,}8546\end{Bmatrix} + \begin{Bmatrix}\cdot001{,}0295\\ \cdot000{,}9552\end{Bmatrix}\right]$$
$$-\cdot0356{,}9425\left[\begin{Bmatrix}\cdot000{,}0996\\ -\cdot000{,}0216\end{Bmatrix} + \begin{Bmatrix}\cdot000{,}2498\\ \cdot000{,}1583\end{Bmatrix}\right]$$
$$= -\begin{Bmatrix}\cdot000{,}0794\\ \cdot000{,}0072\end{Bmatrix} - \begin{Bmatrix}\cdot000{,}0125\\ \cdot000{,}0049\end{Bmatrix} = -\begin{Bmatrix}\cdot000{,}0919\\ \cdot000{,}0121\end{Bmatrix}.$$

Accordingly:

$$\text{For } r = \cdot 05: \ d/N = \cdot 155{,}0218 - \cdot 000{,}0919 = \cdot 154{,}9299,$$
$$\text{,, } r = \cdot 10: \ d/N = \cdot 162{,}2162 - \cdot 000{,}0121 = \cdot 162{,}2041,$$

and finally for $d/N = \cdot 155{,}5556$,

$$r = \cdot 05 + \cdot 05 \times \frac{6257}{72742} = \cdot 0543.$$

The result is thus in excellent agreement with the result found by forming the high order tetrachoric equation and solving it. We conclude that formula (α) will suffice for three and formula (β) or formula (γ) for four-figure accuracy. We shall give further illustrations with greater brevity as the method of arranging the work will now be clear to the reader.

Illustration III. Mother's Habits and Baby's Health.

Mother's Habits.

Baby Heal[th]	Good	Not Good	Totals
Good ...	625	218	843
Not Good	206	136	342
Totals	831	354	1185

Here $\quad \frac{1}{2}(1-\alpha_h) = \cdot 298\cdot 734, \quad \frac{1}{2}(1-\alpha_k) = \cdot 288{,}608,$

and thus $\quad h = \cdot 52805, \quad k = \cdot 55746,$

while $\quad d/N = \cdot 114{,}7679,$

$$\theta = \cdot 2805, \quad \phi = \cdot 7195, \quad \chi = \cdot 5746, \quad \psi = \cdot 4254.$$

Between $h = \cdot 50$ and $\cdot 60$, and $k = \cdot 50$ and $\cdot 60$, the above value of d/N occurs in tables of r for $\cdot 20$, $\cdot 25$ and $\cdot 30$, but not in those for $r = \cdot 15$ and $\cdot 35$. We therefore start by interpolating into $\cdot 25$. Using formula (α)

$$d/N = \cdot 3060{,}7530 \begin{Bmatrix} \cdot 127{,}375 \\ \cdot 120{,}715 \end{Bmatrix} + \cdot 4134{,}2470 \begin{Bmatrix} .115{,}238 \\ \cdot 108{,}878 \end{Bmatrix}$$
$$+ \cdot 1193{,}2470 \begin{Bmatrix} \cdot 115{,}238 \\ \cdot 108{,}878 \end{Bmatrix} + \cdot 1611{,}7530 \begin{Bmatrix} \cdot 104{,}390 \\ \cdot 098{,}392 \end{Bmatrix}.$$

The first or upper series of numbers gives us $d/N = \cdot 117{,}2044$, or too high a value. Hence the second system is inserted for $r = \cdot 20$. This gives $d/N = \cdot 110{,}7964$. Thus, by linear interpolation,

$$r = \cdot 20 + \cdot 05 \times \frac{39715}{64080} = \cdot 2310.$$

The true value is $\cdot 2317$.

We now examine this result by aid of formula (β). The δ^2's and δ'^2's are as follows:

	$r = \cdot 20$	$r = \cdot 25$
$\delta^2 z_{0,0} = \delta'^2 z_{0,0}$	$\cdot 000{,}337$	$\cdot 000{,}278$
$\delta^2 z_{1,0} = \delta'^2 z_{0,1}$	$\cdot 000{,}284$	$\cdot 000{,}395$
$\delta^2 z_{0,1} = \delta'^2 z_{1,0}$	$\cdot 000{,}445$	$\cdot 000{,}227$
$\delta^2 z_{1,1} = \delta'^2 z_{1,1}$	$\cdot 000{,}383$	$\cdot 000{,}333$

Hence

$$R = -\cdot 0336{,}3662_{(\delta)}\left[1\cdot 7195\left(\cdot 4254\left\{\begin{matrix}\cdot 000{,}337\\ \cdot 000{,}278\end{matrix}\right\} + \cdot 5746\left\{\begin{matrix}\cdot 000{,}445\\ \cdot 000{,}227\end{matrix}\right\}\right)\right.$$
$$\left. + 1\cdot 2805\left(\cdot 4254\left\{\begin{matrix}\cdot 000{,}284\\ \cdot 000{,}395\end{matrix}\right\} + \cdot 5746\left\{\begin{matrix}\cdot 000{,}383\\ \cdot 000{,}333\end{matrix}\right\}\right)\right]$$
$$-\cdot 0407{,}3914\left[1\cdot 4254\left(\cdot 7195\left\{\begin{matrix}\cdot 000{,}337\\ \cdot 000{,}278\end{matrix}\right\} + \cdot 2805\left\{\begin{matrix}\cdot 000{,}445\\ \cdot 000{,}227\end{matrix}\right\}\right)\right.$$
$$\left. + 1\cdot 5746\left(\cdot 7195\left\{\begin{matrix}\cdot 000{,}284\\ \cdot 000{,}395\end{matrix}\right\} + \cdot 2805\left\{\begin{matrix}\cdot 000{,}383\\ \cdot 000{,}333\end{matrix}\right\}\right)\right]$$
$$= -\cdot 000{,}0791 \text{ for } r = \cdot 20, \text{ and } = -\cdot 000{,}0694 \text{ for } r = \cdot 25.$$

Accordingly we have

$$d/N = \cdot 110{,}7173 \text{ for } r = \cdot 20 \text{ and } = \cdot 117{,}1350 \text{ for } r = \cdot 25.$$

Interpolating linearly for $d/N = \cdot 114{,}7679$, we find

$$r = \cdot 20 + \cdot 05 \times \frac{40506}{64177} = \cdot 2316.$$

The result from the tetrachoric equation is ·2317.

If we use the forward difference formula (ϵ) we find $r = \cdot 2316$, also a quite good result.

Illustration IV. Ventilation of Home and Health of Mother.

Health of Mother.

Ventilation.	Good	Indifferent and Bad	Totals
Good and Fair	1080	352	1432
Poor	133	131	264
Totals	1213	483	1696

Here $\frac{1}{2}(1-\alpha_h) = \cdot 284{,}788, \quad \frac{1}{2}(1-\alpha_k) = \cdot 155{,}660,$

while $d/N = \cdot 077{,}2406.$

We find $h = \cdot 56868, \quad k = 1\cdot 01245,$

$$\theta = \cdot 6868, \quad \phi = \cdot 3132, \quad \chi = \cdot 1245, \quad \psi = \cdot 8755.$$

The required value of d/N will be found in the section $h = \cdot 5$ to $\cdot 6$ and $k = 1\cdot 0$ to $1\cdot 1$ in the tables for $r = \cdot 35$, ·40 and ·45. We therefore first determine d/N for ·40.

$z_{\theta,\chi}$ from (α) is given by

$$z_{\theta,\chi} = \cdot 2742{,}0660\left\{\begin{matrix}\cdot 086{,}679\\ \cdot 081{,}519\end{matrix}\right\} + \cdot 6012{,}9340\left\{\begin{matrix}\cdot 079{,}913\\ \cdot 074{,}857\end{matrix}\right\}$$
$$+ \cdot 0389{,}9340\left\{\begin{matrix}\cdot 075{,}987\\ \cdot 071{,}320\end{matrix}\right\} + \cdot 0855{,}0660\left\{\begin{matrix}\cdot 070{,}212\\ \cdot 065{,}619\end{matrix}\right\}.$$

$d/N = \cdot 080{,}7857$ for $r = \cdot 40$, or upper figures. This is too high; we therefore take the value for ·35, the lower figures. These give

$$d/N = \cdot 075{,}7559.$$

Hence by linear interpolation

$$r = \cdot 35 + \cdot 05 \times \frac{14847}{50298} = \cdot 3648.$$

The exact value is $r = \cdot 3653$.

The answer is correct to three figures. To get a better approximation we must proceed to second differences.

Let us try forward differences in this case using formula (ϵ). We have

$$\begin{aligned} z_{\theta,\chi} &= \begin{Bmatrix} \cdot 081{,}519 \\ \cdot 086{,}670 \end{Bmatrix} + \cdot 6868 \begin{Bmatrix} -\cdot 006{,}662 \\ -\cdot 006{,}757 \end{Bmatrix} + \cdot 1245 \begin{Bmatrix} -\cdot 010{,}199 \\ -\cdot 010{,}683 \end{Bmatrix} \\ &\quad - \cdot 1075{,}5288 \begin{Bmatrix} \cdot 000{,}051 \\ -\cdot 000{,}013 \end{Bmatrix} + \cdot 0855{,}0660 \begin{Bmatrix} \cdot 000{,}961 \\ \cdot 000{,}982 \end{Bmatrix} \\ &\quad - \cdot 0544{,}9988 \begin{Bmatrix} \cdot 000{,}758 \\ \cdot 000{,}747 \end{Bmatrix} \\ &= \cdot 075{,}709, \text{ for } r = \cdot 35, \\ &= \cdot 086{,}670, \text{ for } r = \cdot 40. \end{aligned}$$

Hence, by linear interpolation for $d/N = \cdot 077{,}241$, we have

$$r = \cdot 35 + \cdot 05 \times \frac{1532}{5035} = \cdot 3652.$$

The result could scarcely agree better with the actual value $r = \cdot 3653$ found from the tetrachoric equation had we proceeded by the central difference instead of a forward difference formula. For a table of this type the latter formula gives very fair results and can be applied to any panel whatever*.

Illustration V. Cleanliness of Home and Mother's Health.

Health of Mother.

Cleanliness of Home.

	Good	Indifferent and Bad	Totals
Clean	939	234	1173
"Fair" and Dirty	274	249	523
Totals	1213	483	1696

Here $\frac{1}{2}(1-\alpha_h) = \cdot 284{,}788$, $\frac{1}{2}(1-\alpha_k) = \cdot 308{,}373$,

$$d/N = \cdot 146{,}8160,$$

Hence $h = \cdot 56868$, $k = \cdot 50047$.

We have $\theta = \cdot 6868$, $\phi = \cdot 3132$, $\chi = \cdot 0047$, $\psi = \cdot 9953$.

* It is clearly less exact than the central difference formula in that it neglects third differences, but these do not seem to reach practical importance in any of the cases tested in this introduction.

The value of d/N occurs in the tables for h and k between ·5 and ·6 for $r = ·40$, ·45, ·50 and ·55. The last clearly need not be considered as k is only slightly over ·5 and a value such as ·146... could not occur in this region. We may also discard the first ($r = ·40$), for in the close neighbourhood of $k = ·50$ the value $d/N = ·146...$ could only occur when h is slightly greater than ·50. Working with the ·45 and ·50 table we have

$$z_{\theta,\chi} \quad ·3117{,}2796 \begin{Bmatrix} ·155{,}684 \\ ·162{,}320 \end{Bmatrix} + ·6835{,}7204 \begin{Bmatrix} ·142{,}361 \\ ·149{,}694 \end{Bmatrix}$$

$$+ ·0014{,}7204 \begin{Bmatrix} ·142{,}361 \\ ·149{,}694 \end{Bmatrix} + ·0032{,}2796 \begin{Bmatrix} ·130{,}483 \\ ·137{,}570 \end{Bmatrix}$$

$$= \begin{Bmatrix} ·146{,}4758 \\ ·153{,}9025 \end{Bmatrix}.$$

Interpolating linearly we have

$$r = ·45 + ·05 \times \frac{3402}{74267} = ·4523.$$

The value found from the full tetrachoric equation is ·4524.

Or, the agreement is complete without passing to second order differences.

Illustration VI. Eye Colour in Brother and Sister.

Brother.

Sister.	Light Eyes	Dark Eyes	Totals
Light Eyes	616	219	835
Dark Eyes	293	372	665
Totals	909	591	1500

Here $\quad \frac{1}{2}(1 - \alpha_h) = ·394{,}000, \quad \frac{1}{2}(1 - \alpha_k) = ·443{,}333,$

$$d/N = ·248{,}0000,$$

Hence $\quad h = ·26891, \quad k = ·14253.$

This value of d/N can lie in the ·40, ·45, or ·50 tables for r, we therefore start by interpolating into that for $r = ·45$. We have

$$\theta = ·6891, \quad \phi = ·3109, \quad \chi = ·4253, \quad \psi = ·5747.$$

Thus $$z_{\theta,\chi} = ·1786{,}7423 \begin{Bmatrix} ·266{,}295 \\ ·275{,}161 \end{Bmatrix} + ·3960{,}2577 \begin{Bmatrix} ·246{,}755 \\ ·255{,}392 \end{Bmatrix}$$

$$+ ·1322{,}2577 \begin{Bmatrix} ·248{,}905 \\ ·257{,}709 \end{Bmatrix} + ·2930{,}7423 \begin{Bmatrix} ·231{,}089 \\ ·239{,}718 \end{Bmatrix}.$$

Hence $$z_{\theta,\chi} = \begin{Bmatrix} ·245{,}9366 \\ ·254{,}6369 \end{Bmatrix},$$

the upper number showing that we need $r = ·50$, and not ·40.

Thus by linear interpolation

$$r = \cdot 45 + \cdot 05 \times \frac{20634}{87003} = \cdot 4619.$$

The actual value of r is ·4614.

We will next inquire how much closer we get to this result by using a higher difference formula. The following δ^2's and δ'^2's are needed for (β):

	$r = \cdot 45$	$r = \cdot 50$
$\delta^2 z_{0,0}$	− ·000,407	− ·000,508
$\delta^2 z_{1,0}$	− ·000,176	− ·000,278
$\delta^2 z_{0,1}$	− ·000,426	− ·000,539
$\delta^2 z_{1,1}$	− ·000,225	− ·000,330
$\delta'^2 z_{0,0}$	− ·000,614	− ·000,727
$\delta'^2 z_{1,0}$	− ·000,611	− ·000,722
$\delta'^2 z_{0,1}$	− ·000,426	− ·000,539
$\delta'^2 z_{1,1}$	− ·000,444	− ·000,556

Hence

$$\begin{aligned} R = - \cdot 0357{,}0686_{(5)} & \left[1\cdot 3109 \left(\cdot 5747 \begin{Bmatrix} - \cdot 000{,}407 \\ - \cdot 000{,}508 \end{Bmatrix} + \cdot 4253 \begin{Bmatrix} - \cdot 000{,}426 \\ - \cdot 000{,}539 \end{Bmatrix}\right)\right. \\ & \left. + 1\cdot 6891 \left(\cdot 5747 \begin{Bmatrix} - \cdot 000{,}176 \\ - \cdot 000{,}278 \end{Bmatrix} + \cdot 4253 \begin{Bmatrix} - \cdot 000{,}225 \\ - \cdot 000{,}330 \end{Bmatrix}\right)\right] \\ - \cdot 0407{,}3665 & \left[1\cdot 5747 \left(\cdot 3109 \begin{Bmatrix} - \cdot 000{,}614 \\ - \cdot 000{,}727 \end{Bmatrix} + \cdot 6891 \begin{Bmatrix} - \cdot 000{,}611 \\ - \cdot 000{,}722 \end{Bmatrix}\right)\right. \\ & \left. + 1\cdot 4253 \left(\cdot 3109 \begin{Bmatrix} - \cdot 000{,}426 \\ - \cdot 000{,}539 \end{Bmatrix} + \cdot 6891 \begin{Bmatrix} - \cdot 000{,}444 \\ - \cdot 000{,}556 \end{Bmatrix}\right)\right]. \end{aligned}$$

Thus $R = \cdot 000{,}0960$, for $r = \cdot 45$,

$\quad = \cdot 000{,}1209$, for $r = \cdot 50$.

These give $d/N = \cdot 246{,}0326$, for $r = \cdot 45$,

$\quad d/N = \cdot 254{,}7578$, for $r = \cdot 50$.

Whence, interpolating for $d/N = \cdot 248{,}0000$, we have

$$r = \cdot 45 + \cdot 05 \times \frac{19674}{87252} = \cdot 4613,$$

only differing by a unit in the last figure from the actual value.

Illustration VII. Intelligence and Conscientiousness. (Girls.)

Intelligence.

Conscientiousness.	Slow Intelligent and below	Intelligent and above	Totals
Dull and Moderate	250	67	317
Keen	76	96	172
Totals	326	163	489

Here $\frac{1}{2}(1-\alpha_h) = \cdot 333{,}333, \quad \frac{1}{2}(1-\alpha_k) = \cdot 351{,}738,$

and $d/N = \cdot 196{,}3190.$

Hence $h = \cdot 43072, \quad k = \cdot 38063.$

We have $\theta = \cdot 3072, \quad \phi = \cdot 6928, \quad \chi = \cdot 8063, \quad \psi = \cdot 1937.$

Thus
$$z_{\theta,\chi} = \cdot 1341{,}9536 \begin{Bmatrix} \cdot 206{,}888 \\ \cdot 215{,}474 \end{Bmatrix} + \cdot 0595{,}0464 \begin{Bmatrix} \cdot 190{,}114 \\ \cdot 198{,}360 \end{Bmatrix}$$
$$+ \cdot 5586{,}0464 \begin{Bmatrix} \cdot 191{,}979 \\ \cdot 200{,}401 \end{Bmatrix} + \cdot 2476{,}9536 \begin{Bmatrix} \cdot 176{,}847 \\ \cdot 184{,}994 \end{Bmatrix}.$$

The observed value of d/N occurs in the $h = \cdot 4$ to $\cdot 5$ and $k = \cdot 3$ to $\cdot 4$ in the r tables for ·45, ·50 and ·55; we therefore try $r = \cdot 50$ first, i.e. the upper numbers in the curled brackets, and find $d/N = \cdot 190{,}1206$ too small; we then compute for the higher number $r = \cdot 55$, i.e. the lower figures. These give us $d/N = \cdot 198{,}4860$. Interpolating linearly for $d/N = \cdot 196{,}3190$, we have

$$r = \cdot 50 + \cdot 05 \times \frac{61934}{83654} = \cdot 5370.$$

The value found from the full tetrachoric equation is $r = \cdot 5370$, in exact agreement. Thus the hyperbolic formula alone appears adequate in this case.

Illustration VIII. Habits of Mother and Cleanliness of Home. (Bradford.)

Habits of Mother.

Cleanliness of Home.

	Good	Indifferent and Bad	Totals
Clean... ...	1009	164	1173
Fair and Dirty	144	379	523
Totals	1153	543	1696

Here $\frac{1}{2}(1-\alpha_h) = \cdot 320{,}165, \quad \frac{1}{2}(1-\alpha_k) = \cdot 308{,}373,$

and $d/N = \cdot 223{,}4670.$

Hence $h = \cdot 46724, \quad k = \cdot 50047.$

We have $\theta = \cdot 6724, \quad \phi = \cdot 3276, \quad \chi = \cdot 0047, \quad \psi = \cdot 9953.$

The value of d/N occurs only in the (h, k) range for $r = \cdot 80$ and $r = \cdot 85$. It is only necessary therefore to interpolate for these tables. We have

$$z_{\theta,\chi} = \cdot 3260{,}6028 \begin{Bmatrix} \cdot 233{,}252 \\ \cdot 245{,}704 \end{Bmatrix} + \cdot 6692{,}3972 \begin{Bmatrix} \cdot 218{,}560 \\ \cdot 230{,}869 \end{Bmatrix}$$
$$+ \cdot 0015{,}3972 \begin{Bmatrix} \cdot 215{,}207 \\ \cdot 226{,}826 \end{Bmatrix} + \cdot 0031{,}6028 \begin{Bmatrix} \cdot 202{,}791 \\ \cdot 214{,}580 \end{Bmatrix}$$
$$= \begin{Bmatrix} \cdot 223{,}2955 \\ \cdot 235{,}6484 \end{Bmatrix}.$$

Hence, interpolating for the observed value of d/N, we find

$$r = \cdot 80 + \cdot 05 \times \frac{1715}{123{,}529} = \cdot 8007.$$

Using 18 tetrachoric functions and adjusting for the remainder we find $r = \cdot 80013$. The labour of determining an equation of this order is very great, and we cannot even then be absolutely sure to a unit in the last figure of r. We accordingly proceeded to a more close interpolation from the tables. We need the following δ^2's and δ'^2's:

$$\begin{array}{lrr}
 & r = \cdot 80 & r = \cdot 85 \\
\delta^2 z_{0,0} & -\cdot 001{,}305 & -\cdot 001{,}695 \\
\delta^2 z_{1,0} & -\cdot 001{,}077 & -\cdot 001{,}454 \\
\delta^2 z_{0,1} & -\cdot 001{,}279 & -\cdot 001{,}633 \\
\delta^2 z_{1,1} & -\cdot 001{,}116 & -\cdot 001{,}486 \\
\delta'^2 z_{0,0} & -\cdot 000{,}989 & -\cdot 001{,}343 \\
\delta'^2 z_{1,0} & -\cdot 001{,}077 & -\cdot 001{,}454 \\
\delta'^2 z_{0,1} & -\cdot 000{,}648 & -\cdot 000{,}930 \\
\delta'^2 z_{1,1} & -\cdot 000{,}795 & -\cdot 001{,}125
\end{array}$$

Using formula (β) we have for the remainder

$$\begin{aligned}
R = {} & \left[\begin{matrix} -\tfrac{1}{6}(\cdot 6724 \times \cdot 3276) \\ -\cdot 0367{,}1304 \end{matrix}\right] \left[1\cdot 3276 \left(\cdot 9953 \left\{\begin{matrix} -\cdot 001{,}305 \\ -\cdot 001{,}695 \end{matrix}\right\} + \cdot 0047 \left\{\begin{matrix} -\cdot 001{,}279 \\ -\cdot 001{,}633 \end{matrix}\right\} \right) \right. \\
& \qquad + 1\cdot 6724 \left(\cdot 9953 \left\{\begin{matrix} -\cdot 001{,}077 \\ -\cdot 001{,}454 \end{matrix}\right\} + \cdot 0047 \left\{\begin{matrix} -\cdot 001{,}116 \\ -\cdot 001{,}486 \end{matrix}\right. \right. \\
& \left[\begin{matrix} -\tfrac{1}{6}(\cdot 0047 \times \cdot 9953) \\ -\cdot 0007{,}7965 \end{matrix}\right] \left[1\cdot 9953 \left(\cdot 3276 \left\{\begin{matrix} -\cdot 000{,}989 \\ -\cdot 001{,}343 \end{matrix}\right\} + \cdot 6724 \left\{\begin{matrix} -\cdot 001{,}077 \\ -\cdot 001{,}454 \end{matrix}\right\} \right) \right. \\
& \qquad \left. \cdot 0047 \left(\cdot 3276 \left\{\begin{matrix} -\cdot 000{,}648 \\ -\cdot 000{,}930 \end{matrix}\right\} + \cdot 6724 \left\{\begin{matrix} -\cdot 000{,}795 \\ -\cdot 001{,}125 \end{matrix}\right\} \right) \right],
\end{aligned}$$

or

$$\begin{aligned}
R &= +\cdot 000{,}1320, \text{ for } r = \cdot 80, \\
&= +\cdot 000{,}1749, \text{ for } r = \cdot 85.
\end{aligned}$$

Hence

$$\begin{aligned}
d/N &= \cdot 223{,}4275, \text{ for } r = \cdot 80, \\
&= \cdot 235{,}9233, \text{ for } r = \cdot 85.
\end{aligned}$$

Then interpolating linearly for $d/N = \cdot 223{,}4670$, we have

$$r = \cdot 80 + \cdot 05 \times \frac{395}{124958} = \cdot 80016,$$

which is adequately close to the result $\cdot 80013$ of the equation method.

The forward difference formula (ϵ) gives a slightly worse result, namely $r = \cdot 80021$.

Illustration IX. Binocular Vision and Vision of Better Eye. (Boys.)

Vision of Better Eye.

Binocular Vision.	6/9 and better	Below 6/9	Totals
6/6 and better	296	9	305
Below 6/6 ...	35	99	134
Totals	331	108	439

When the correlation is over ·90 it is hard to test the accuracy of the result by the complete tetrachoric equation; the terms converge so slowly that it is practically impossible to compute enough of them. On the other hand, if we take material of which the correlation is known, say by the product-moment meth d, it is again not easy to obtain adequate numbers in the cells off the "correlation diagonal." The above table is one for the correlation of binocular vision and the vision of the better eye. The correlation coefficient worked by the product-moment process on the full table was ·9485. The above was the best fourfold table we could make, i.e. the others gave two or three units only in one or other cell.

We have $\frac{1}{2}(1-\alpha_h) = \cdot 246{,}0136, \quad \frac{1}{2}(1-\alpha_k) = \cdot 305{,}2391,$

and $$d/N = \cdot 225{,}5125.$$

Hence we have $h = \cdot 68709, \quad k = \cdot 50939.$

Further $\theta = \cdot 8709, \quad \phi = \cdot 1291, \quad \chi = \cdot 0939, \quad \psi = \cdot 9061.$

The value required of d/N occurs within the given (h, k) range only for $r = \cdot 95$ and $r = \cdot 90$. Accordingly we have

$$z_{\theta,\chi} = \cdot 1169{,}7751 \begin{Bmatrix} \cdot 245{,}878 \\ \cdot 228{,}353 \end{Bmatrix} + \cdot 7891{,}2249 \begin{Bmatrix} \cdot 225{,}035 \\ \cdot 209{,}730 \end{Bmatrix}$$
$$+ \cdot 0121{,}2249 \begin{Bmatrix} \cdot 232{,}102 \\ \cdot 214{,}481 \end{Bmatrix} + \cdot 0817{,}7751 \begin{Bmatrix} \cdot 215{,}259 \\ \cdot 198{,}793 \end{Bmatrix}$$

or
$$d/N = \cdot 226{,}759, \text{ for } r = \cdot 95,$$
$$= \cdot 211{,}072, \text{ for } r = \cdot 90.$$

Interpolating linearly for $d/N = \cdot 225{,}513$, we have

$$r = \cdot 90 + \cdot 05 \times \frac{14446}{15687} = \cdot 9460.$$

This is as good a result as could possibly be anticipated for the agreement of a product-moment and a tetrachoric coefficient, when we bear in mind that the accordance depends on the existence of nine individuals only in the right-hand upper cell.

To sum up we see that the new table throughout the range of r gives very satisfactory results with far less labour than the equation method, if we use second differences for h and k and linear interpolation for r. There is not much difference in accuracy between applying a central difference and a forward difference second order interpolation. For many purposes where two to three decimals are adequate in the value of r the simple hyperbolic formula (α) is sufficient.

We will now proceed to illustrate the application of the present table to determine the cell contents of a contingency table of which the frequency distribution is supposed to be normal, and it is desired to test the accuracy of the hypothesis.

Unfortunately until the second series of tables is completed, namely those for r negative and h and k of the same sign, it is not possible to work out all the cell contents of a contingency table by aid of the present short method. It is necessary to work out a certain number of them still by aid of the tetrachoric expansion. The reader who has this to do will rapidly appreciate how much the present table alone saves him.

Illustration X. To test whether the following contingency table for Stature in Father and Son may reasonably be considered normal.

Stature of Father.

Stature of Son.	Short	Shortish	Medium	Tallish	Tall	Totals
Short	203*	91	26†	9†	6†	335
Shortish	95	75	66	22	26	284
Medium	30†	36	37*	14*	20*	137
Tallish	18†	27	26*	11*	23*	105
Tall	12†	35	25*	13*	54*	139
Totals	358	264	180	69	129	1000

This table has been reduced to broad categories from a much larger table where the categories were all quantitatively given. The ten cell contents in which an asterisk is placed were found with great ease by means of formula (α). The contents of six further cells—those in which a dagger is placed—were found far less easily by aid of tetrachoric expansions. The contents of the remaining nine cells could then be found from the marginal totals and a knowledge of the sixteen cell contents already found.

The actual correlation coefficient of this table as found by the product-moment method on the original data was $r = \cdot 5189$. The result would have been practically the same had the value obtained by mean square contingency corrected for class indices, namely $r = \cdot 5179$, been used. The following table gives the computed frequencies:

Stature of Father.

Stature of Son.	Short	Shortish	Medium	Tallish	Tall	Totals
Short	196·4	83·4	36·2	9·4	9·6	335
Shortish	99·3	86·5	54·6	18·5	25·1	284
Medium	32·2	39·9	31·4	12·5	21·0	137
Tallish	17·8	22·4	27·6	12·3	24·9	105
Tall	12·3	31·8	30·2	16·3	48·4	139
Totals	358	264	180	69	129	1000

The method of determining the cell contents from the table is easily indicated. Take for example the cell for "Tall" Sons and "Tall" Fathers. We have

$$\tfrac{1}{2}(1-\alpha_h) = \cdot 129, \quad \tfrac{1}{2}(1-\alpha_k) = \cdot 139,$$

whence we have $h = 1 \cdot 13113$ and $k = 1 \cdot 12639$. Thus

$$\theta = \cdot 3113, \quad \phi = \cdot 6887, \quad \chi = \cdot 2639, \text{ and } \psi = \cdot 7361,$$

Here we find for

$$r = \cdot 50,\ d/N = \cdot 046{,}831\,;\quad r = \cdot 55,\ d/N = \cdot 050880,$$

or by interpolation for $r = \cdot 5189$ we have

$$d/N = \cdot 04836, \text{ and } d = 48\cdot 4.$$

Next we take "Tall" Fathers with "Tall" and "Tallish" Sons and find in the same way $d = 73\cdot 3$; the difference of this and the former d gives 24·9 as the cell frequency of "Tall" Fathers and "Tallish" Sons. Thus gradually the individual cell frequency is built up from combinations of cell frequencies.

In the cases where we still had to determine cell frequencies from tetrachoric series, two sets of ten tetrachoric functions had to be determined from the tables of these functions; then the product of these functions, pair and pair was taken, and these products again had to be multiplied by the respective powers of r from 1 to 10. The series thus formed, i.e.

$$\frac{d}{N} = \tau_0\tau_0' + r\tau_1\tau_1' + r^2\tau_2\tau_2' + \ldots + r^{10}\tau_{10}\tau_{10}',$$

gives the required cell content. The process is a very laborious one, and we can only hope to avoid it completely when the second series of these tables is computed; it will then be exceedingly easy to determine the cell contents of any contingency table on the hypothesis that it is a normal distribution.

It seemed worth while applying the goodness of fit test. Here, as we have applied the normal surface *assuming the marginal frequencies to be the same for the surface and the sample*, we have introduced ten restrictions instead of the single usual one, accordingly we must look up P for $n = 25 - 9 = 16$. The $\chi^2 = 15\cdot 566$ and we have $P = \cdot 412$, a very reasonable fit. Thus the distribution of Stature in Fathers and Sons may be described legitimately by a normal surface.

k	r = ·00									k
	h = 0·0	h = 0·1	h = 0·2	h = 0·3	h = 0·4	h = 0·5	h = 0·6	h = 0·7	h = 0·8	
0·0	·250000	·230086	·210370	·191044	·172289	·154269	·137127	·120982	·105928	0·0
0·1	·230086	·211758	·193613	·175827	·158565	·141980	·126204	·111345	·097490	0·1
0·2	·210370	·193613	·177022	·160760	·144978	·129814	·115389	·101804	·089136	0·2
0·3	·191044	·175827	·160760	·145992	·131659	·117889	·104789	·092452	·080948	0·3
0·4	·172289	·158565	·144978	·131659	·118734	·106315	·094502	·083375	·073001	0·4
0·5	·154269	·141980	·129814	·117889	·106315	·095195	·084617	·074655	·065365	0·5
0·6	·137127	·126204	·115389	·104789	·094502	·084617	·075215	·066359	·058102	0·6
0·7	·120982	·111345	·101804	·092452	·083375	·074655	·066359	·058546	·051261	0·7
0·8	·105928	·097490	·089136	·080948	·073001	·065365	·058102	·051261	·044883	0·8
0·9	·092030	·084699	·077442	·070327	·063423	·056789	·050479	·044536	·038994	0·9
1·0	·079328	·073009	·066753	·060620	·054669	·048951	·043512	·038389	·033612	1·0
1·1	·067833	·062430	·057080	·051836	·046748	·041858	·037207	·032826	·028742	1·1
1·2	·057535	·052952	·048414	·043967	·039650	·035503	·031558	·027843	·024378	1·2
1·3	·048400	·044545	·040728	·036986	·033355	·029867	·026548	·023422	·020508	1·3
1·4	·040378	·037162	·033978	·030856	·027827	·024916	·022148	·019540	·017109	1·4
1·5	·033404	·030743	·028108	·025526	·023020	·020613	·018322	·016165	·014153	1·5
1·6	·027400	·025217	·023056	·020938	·018883	·016908	·015029	·013259	·011610	1·6
1·7	·022283	·020508	·018750	·017028	·015356	·013750	·012222	·010783	·009441	1·7
1·8	·017965	·016534	·015117	·013729	·012381	·011086	·009854	·008694	·007612	1·8
1·9	·014358	·013215	·012082	·010972	·009895	·008860	·007876	·006948	·006084	1·9
2·0	·011375	·010469	·009572	·008693	·007839	·007019	·006239	·005505	·004820	2·0
2·1	·008932	·008221	·007516	·006826	·006156	·005512	·004899	·004323	·003785	2·1
2·2	·006952	·006398	·005850	·005312	·004791	·004290	·003813	·003364	·002946	2·2
2·3	·005362	·004935	·004512	·004098	·003695	·003309	·002941	·002595	·002272	2·3
2·4	·004099	·003772	·003449	·003132	·002825	·002529	·002248	·001984	·001737	2·4
2·5	·003105	·002858	·002613	·002373	·002140	·001916	·001703	·001503	·001316	2·5
2·6	·002331	·002145	·001961	·001781	·001606	·001438	·001278	·001128	·000987	2·6

k	r = ·00									k
	h = 0·9	h = 1·0	h = 1·1	h = 1·2	h = 1·3	h = 1·4	h = 1·5	h = 1·6	h = 1·7	
0·0	·092030	·079328	·067833	·057535	·048400	·040378	·033404	·027400	·022283	0·0
0·1	·084699	·073009	·062430	·052952	·044545	·037162	·030743	·025217	·020508	0·1
0·2	·077442	·066753	·057080	·048414	·040728	·033978	·028108	·023056	·018750	0·2
0·3	·070327	·060620	·051836	·043967	·036986	·030856	·025526	·020938	·017028	0·3
0·4	·063423	·054669	·046748	·039650	·033355	·027827	·023020	·018883	·015356	0·4
0·5	·056789	·048951	·041858	·035503	·029867	·024916	·020613	·016908	·013750	0·5
0·6	·050479	·043512	·037207	·031558	·026548	·022148	·018322	·015029	·012222	0·6
0·7	·044536	·038389	·032826	·027843	·023422	·019540	·016165	·013259	·010783	0·7
0·8	·038994	·033612	·028742	·024378	·020508	·017109	·014153	·011610	·009441	0·8
0·9	·033878	·029202	·024971	·021180	·017817	·014864	·012297	·010086	·008203	0·9
1·0	·029202	·025171	·021524	·018256	·015358	·012812	·010599	·008694	·007071	1·0
1·1	·024971	·021524	·018405	·015611	·013133	·010956	·009063	·007434	·006046	1·1
1·2	·021180	·018256	·015611	·013241	·011139	·009293	·007688	·006306	·005128	1·2
1·3	·017817	·015358	·013133	·011139	·009370	·007817	·006467	·005305	·004314	1·3
1·4	·014864	·012812	·010956	·009293	·007817	·006522	·005395	·004425	·003599	1·4
1·5	·012297	·010599	·009063	·007688	·006467	·005395	·004463	·003661	·002977	1·5
1·6	·010086	·008694	·007434	·006306	·005305	·004425	·003661	·003003	·002442	1·6
1·7	·008203	·007071	·006046	·005128	·004314	·003599	·002977	·002442	·001986	1·7
1·8	·006613	·005700	·004875	·004134	·003478	·002902	·002400	·001969	·001601	1·8
1·9	·005286	·004556	·003896	·003304	·002780	·002319	·001918	·001574	·001280	1·9
2·0	·004187	·003609	·003086	·002618	·002202	·001837	·001520	·001247	·001014	2·0
2·1	·003288	·002834	·002424	·002056	·001729	·001443	·001194	·000979	·000796	2·1
2·2	·002559	·002206	·001886	·001600	·001346	·001123	·000929	·000762	·000620	2·2
2·3	·001974	·001701	·001455	·001234	·001038	·000866	·000716	·000588	·000478	2·3
2·4	·001509	·001301	·001112	·000943	·000794	·000662	·000548	·000449	·000365	2·4
2·5	·001143	·000985	·000842	·000715	·000601	·000502	·000415	·000340	·000277	2·5
2·6	·000858	·000740	·000632	·000536	·000451	·000376	·000311	·000255	·000208	2·6

k	r=·00									k
	h=*1·8*	*h*=*1·9*	*h*=*2·0*	*h*=*2·1*	*h*=*2·2*	*h*=*2·3*	*h*=*2·4*	*h*=*2·5*	*h*=*2·6*	
0·0	·017965	·014358	·011375	·008932	·006952	·005362	·004099	·003105	·002331	*0·0*
0·1	·016534	·013215	·010469	·008221	·006398	·004935	·003772	·002858	·002145	*0·1*
0·2	·015117	·012082	·009572	·007516	·005850	·004512	·003449	·002613	·001961	*0·2*
0·3	·013729	·010972	·008693	006826	·005312	·004098	·003132	·002373	·001781	*0·3*
0·4	·012381	·009895	·007839	·006156	·004791	·003695	·002825	·002140	·001606	*0·4*
0·5	·011086	·008860	·007019	·005512	·004290	·003309	·002529	·001916	·001438	*0·5*
0·6	·009854	·007876	·006239	·004899	·003813	·002941	·002248	·001703	·001278	*0·6*
0·7	·008694	·006948	·005505	·004323	·003364	·002595	·001984	·001503	·001128	*0·7*
0·8	·007612	·006084	·004820	·003785	·002946	·002272	·001737	·001316	·000987	*0·8*
0·9	·006613	·005286	·004187	·003288	·002559	·001974	·001509	·001143	·000858	*0·9*
1·0	·005700	·004556	·003609	·002834	·002206	·001701	·001301	·000985	·000740	*1·0*
1·1	·004875	·003896	·003086	·002424	·001886	·001455	·001112	·000842	·000632	*1·1*
1·2	·004134	·003304	·002618	·002056	·001600	·001234	·000943	·000715	·000536	*1·2*
1·3	·003478	·002780	·002202	·001729	·001346	·001038	·000794	·000601	·000451	*1·3*
1·4	·002902	·002319	·001837	·001443	·001123	·000866	·000662	·000502	·000376	*1·4*
1·5	·002400	·001918	·001520	·001194	·000929	·000716	·000548	·000415	·000311	*1·5*
1·6	·001969	·001574	·001247	·000979	·000762	·000588	·000449	·000340	·000255	*1·6*
1·7	·001601	·001280	·001014	·000796	·000620	·000478	·000365	·000277	·000208	*1·7*
1·8	·001291	·001032	·000817	·000642	·000500	·000385	·000295	·000223	·000168	*1·8*
1·9	·001032	·000825	·000653	·000513	·000399	·000308	·000235	·000178	·000134	*1·9*
2·0	·000817	·000653	·000518	·000406	·000316	·000244	·000186	·000141	·000106	*2·0*
2·1	·000642	·000513	·000406	·000319	·000248	·000192	·000146	·000111	·000083	*2·1*
2·2	·000500	·000399	·000316	·000248	·000193	·000149	·000114	·000086	·000065	*2·2*
2·3	·000385	·000308	·000244	·000192	·000149	·000115	·000088	·000067	·000050	*2·3*
2·4	·000295	·000235	·000186	·000146	·000114	·000088	·000067	·000051	·000038	*2·4*
2·5	·000223	·000178	·000141	·000111	·000086	·000067	·000051	·000039	·000029	*2·5*
2·6	·000168	·000134	·000106	·000083	·000065	·000050	·000038	·000029	·000022	*2·6*

k	r=·05									k
	h=*0·0*	*h*=*0·1*	*h*=*0·2*	*h*=*0·3*	*h*=*0·4*	*h*=*0·5*	*h*=*0·6*	*h*=*0·7*	*h*=*0·8*	
0·0	·257961	·238007	·218173	·198655	·179638	·161294	·143775	·127212	·111707	*0·0*
0·1	·238007	·219642	·201381	·183405	·165885	·148979	·132829	·117555	·103252	*0·1*
0·2	·218173	·201381	·184679	·168231	·152195	·136717	·121926	·107932	·094824	*0·2*
0·3	·198655	·183405	·168231	·153283	·138706	·124630	·111174	·098438	·086506	*0·3*
0·4	·179638	·165885	·152195	·138706	·125545	·112832	·100676	·089166	·078378	*0·4*
0·5	·161294	·148979	·136717	·124630	·112832	·101433	·090528	·080201	·070517	*0·5*
0·6	·143775	·132829	·121926	·111174	·100676	·090528	·080818	·071617	·062987	*0·6*
0·7	·127212	·117555	·107932	·098438	·089166	·080201	·071617	·063482	·055848	*0·7*
0·8	·111707	·103252	·094824	·086506	·078378	·070517	·062987	·055848	·049146	*0·8*
0·9	·097338	·089993	·082668	·075436	·068367	·061527	·054973	·048756	·042918	*0·9*
1·0	·084154	·077823	·071507	·065269	·059169	·053264	·047604	·042233	·037187	*1·0*
1·1	·072178	·066765	·061363	·056025	·050803	·045746	·040896	·036293	·031967	*1·1*
1·2	·061408	·056817	·052233	·047703	·043269	·038972	·034851	·030938	·027258	*1·2*
1·3	·051818	·047956	·044099	·040285	·036551	·032932	·029458	·026158	·023054	*1·3*
1·4	·043364	·040143	·036925	·033740	·030622	·027597	·024694	·021935	·019338	*1·4*
1·5	·035986	·033322	·030659	·028023	·025440	·022934	·020528	·018240	·016085	*1·5*
1·6	·029611	·027426	·025241	·023078	·020957	·018898	·016921	·015039	·013267	*1·6*
1·7	·024157	·022381	·020604	·018843	·017116	·015440	·013828	·012295	·010850	*1·7*
1·8	·019539	·018107	·016673	·015253	·013860	·012506	·011204	·009965	·008796	*1·8*
1·9	·015666	·014522	·013376	·012240	·011125	·010042	·008999	·008006	·007070	*1·9*
2·0	·012451	·011545	·010637	·009736	·008852	·007993	·007165	·006377	·005633	*2·0*
2·1	·009808	·009097	·008384	·007677	·006982	·006306	·005655	·005034	·004448	*2·1*
2·2	·007658	·007105	·006550	·005999	·005458	·004931	·004423	·003939	·003482	*2·2*
2·3	·005926	·005499	·005071	·004646	·004228	·003821	·003429	·003054	·002701	*2·3*
2·4	·004545	·004219	·003891	·003566	·003246	·002935	·002634	·002348	·002076	*2·4*
2·5	·003454	·003207	·002959	·002713	·002470	·002234	·002006	·001788	·001582	*2·5*
2·6	·002601	·002416	·002230	·002044	·001862	·001685	·001513	·001349	·001194	*2·6*

k	$r=\cdot05$									k
	$h=0\cdot9$	$h=1\cdot0$	$h=1\cdot1$	$h=1\cdot2$	$h=1\cdot3$	$h=1\cdot4$	$h=1\cdot5$	$h=1\cdot6$	$h=1\cdot7$	
0·0	·097338	·084154	·072178	·061408	·051818	·043364	·035986	·029611	·024157	0·0
0·1	·089993	·077823	·066765	·056817	·047956	·040143	·033322	·027426	·022381	0·1
0·2	·082668	·071507	·061363	·052233	·044099	·036925	·030659	·025241	·020604	0·2
0·3	·075436	·065269	·056025	·047703	·040285	·033740	·028023	·023078	·018843	0·3
0·4	·068367	·059169	·050803	·043269	·036551	·030622	·025440	·020957	·017116	0·4
0·5	·061527	·053264	·045746	·038972	·032932	·027597	·022934	·018898	·015440	0·5
0·6	·054973	·047604	·040896	·034851	·029458	·024694	·020528	·016921	·013828	0·6
0·7	·048756	·042233	·036293	·030938	·026158	·021935	·018240	·015039	·012295	0·7
0·8	·042918	·037187	·031967	·027258	·023054	·019338	·016085	·013267	·010850	0·8
0·9	·037490	·032494	·027941	·023833	·020164	·016919	·014078	·011615	·009502	0·9
1·0	·032494	·028172	·024232	·020676	·017499	·014687	·012225	·010090	·008257	1·0
1·1	·027941	·024232	·020850	·017796	·015066	·012650	·010533	·008696	·007119	1·1
1·2	·023833	·020676	·017796	·015195	·012868	·010808	·009002	·007435	·006088	1·2
1·3	·020164	·017499	·015066	·012868	·010901	·009159	·007631	·006305	·005165	1·3
1·4	·016919	·014687	·012650	·010808	·009159	·007698	·006416	·005303	·004345	1·4
1·5	·014078	·012225	·010533	·009002	·007631	·006416	·005350	·004423	·003626	1·5
1·6	·011615	·010090	·008696	·007435	·006305	·005303	·004423	·003658	·003000	1·6
1·7	·009502	·008257	·007119	·006088	·005165	·004345	·003626	·003000	·002461	1·7
1·8	·007706	·006699	·005777	·004943	·004195	·003530	·002947	·002439	·002002	1·8
1·9	·006196	·005388	·004648	·003978	·003377	·002843	·002374	·001966	·001614	1·9
2·0	·004938	·004295	·003707	·003174	·002695	·002270	·001896	·001571	·001290	2·0
2·1	·003901	·003394	·002930	·002510	·002132	·001797	·001501	·001244	·001022	2·1
2·2	·003054	·002659	·002296	·001967	·001672	·001409	·001178	·000976	·000803	2·2
2·3	·002370	·002064	·001783	·001528	·001299	·001096	·000916	·000760	·000625	2·3
2·4	·001823	·001588	·001372	·001177	·001001	·000844	·000706	·000586	·000482	2·4
2·5	·001389	·001211	·001047	·000898	·000764	·000644	·000539	·000448	·000368	2·5
2·6	·001049	·000915	·000791	·000679	·000578	·000488	·000408	·000339	·000279	2·6

k	$r=\cdot05$									k
	$h=1\cdot8$	$h=1\cdot9$	$h=2\cdot0$	$h=2\cdot1$	$h=2\cdot2$	$h=2\cdot3$	$h=2\cdot4$	$h=2\cdot5$	$h=2\cdot6$	
0·0	·019539	·015666	·012451	·009808	·007658	·005926	·004545	·003454	·002601	0·0
0·1	·018107	·014522	·011545	·009097	·007105	·005499	·004219	·003207	·002416	0·1
0·2	·016673	·013376	·010637	·008384	·006550	·005071	·003891	·002959	·002230	0·2
0·3	·015253	·012240	·009736	·007677	·005999	·004646	·003566	·002713	·002044	0·3
0·4	·013860	·011125	·008852	·006982	·005458	·004228	·003246	·002470	·001862	0·4
0·5	·012506	·010042	·007993	·006306	·004931	·003821	·002935	·002234	·001685	0·5
0·6	·011204	·008999	·007165	·005655	·004423	·003429	·002634	·002006	·001513	0·6
0·7	·009965	·008006	·006377	·005034	·003939	·003054	·002348	·001788	·001349	0·7
0·8	·008796	·007070	·005633	·004448	·003482	·002701	·002076	·001582	·001194	0·8
0·9	·007706	·006196	·004938	·003901	·003054	·002370	·001823	·001389	·001049	0·9
1·0	·006699	·005388	·004295	·003394	·002659	·002064	·001588	·001211	·000915	1·0
1·1	·005777	·004648	·003707	·002930	·002296	·001783	·001372	·001047	·000791	1·1
1·2	·004943	·003978	·003174	·002510	·001967	·001528	·001177	·000898	·000679	1·2
1·3	·004195	·003377	·002695	·002132	·001672	·001299	·001001	·000764	·000578	1·3
1·4	·003530	·002843	·002270	·001797	·001409	·001096	·000844	·000644	·000488	1·4
1·5	·002947	·002374	·001896	·001501	·001178	·000916	·000706	·000539	·000408	1·5
1·6	·002439	·001966	·001571	·001244	·000976	·000760	·000586	·000448	·000339	1·6
1·7	·002002	·001614	·001290	·001022	·000803	·000625	·000482	·000368	·000279	1·7
1·8	·001629	·001314	·001050	·000832	·000654	·000509	·000393	·000300	·000228	1·8
1·9	·001314	·001060	·000848	·000672	·000528	·000411	·000318	·000243	·000184	1·9
2·0	·001050	·000848	·000678	·000538	·000423	·000330	·000255	·000195	·000148	2·0
2·1	·000832	·000672	·000538	·000427	·000336	·000262	·000202	·000155	·000117	2·1
2·2	·000654	·000528	·000423	·000336	·000264	·000206	·000159	·000122	·000093	2·2
2·3	·000509	·000411	·000330	·000262	·000206	·000161	·000124	·000095	·000072	2·3
2·4	·000393	·000318	·000255	·000202	·000159	·000124	·000096	·000074	·000056	2·4
2·5	·000300	·000243	·000195	·000155	·000122	·000095	·000074	·000057	·000043	2·5
2·6	·000228	·000184	·000148	·000117	·000093	·000072	·000056	·000043	·000033	2·6

k	$r=\cdot10$									k
	h = *0·0*	*h* = *0·1*	*h* = *0·2*	*h* = *0·3*	*h* = *0·4*	*h* = *0·5*	*h* = *0·6*	*h* = *0·7*	*h* = *0·8*	
0·0	·265942	·245948	·225996	·206283	·187002	·168332	·150435	·133450	·117492	*0·0*
0·1	·245948	·227549	·209176	·191012	·173234	·156008	·139485	·123794	·109042	*0·1*
0·2	·225996	·209176	·192368	·175741	·159456	·143667	·128512	·114110	·100562	*0·2*
0·3	·206283	·191012	·175741	·160623	·145808	·131433	·117625	·104496	·092135	*0·3*
0·4	·187002	·173234	·159456	·145808	·132422	·119425	·106933	·095045	·083846	*0·4*
0·5	·168332	·156008	·143667	·131433	·119425	·107758	·096536	·085849	·075774	*0·5*
0·6	·150435	·139485	·128512	·117625	·106933	·096536	·086527	·076990	·067992	*0·6*
0·7	·133450	·123794	·114110	·104496	·095045	·085849	·076990	·068541	·060564	*0·7*
0·8	·117492	·109042	·100562	·092135	·083846	·075774	·067992	·060564	·053546	*0·8*
0·9	·102649	·095313	·087944	·080616	·073402	·066371	·059588	·053109	·046982	*0·9*
1·0	·088981	·082662	·076310	·069988	·063759	·057684	·051818	·046210	·040903	*1·0*
1·1	·076521	·071122	·065690	·060280	·054945	·049738	·044705	·039890	·035330	*1·1*
1·2	·065276	·060701	·056094	·051502	·046970	·042542	·038260	·034159	·030273	*1·2*
1·3	·055229	·051384	·047509	·043643	·039825	·036091	·032477	·029014	·025729	*1·3*
1·4	·046342	·043138	·039906	·036678	·033488	·030366	·027342	·024441	·021687	*1·4*
1·5	·038560	·035912	·033239	·030568	·027925	·025337	·022827	·020417	·018128	*1·5*
1·6	·031813	·029641	·027452	·025260	·023089	·020962	·018896	·016913	·015026	*1·6*
1·7	·026023	·024261	·022480	·020696	·018928	·017194	·015510	·013890	·012348	*1·7*
1·8	·021103	·019685	·018249	·016811	·015384	·013983	·012621	·011310	·010061	*1·8*
1·9	·016965	·015833	·014686	·013536	·012394	·011272	·010180	·009129	·008126	*1·9*
2·0	·013518	·012623	·011715	·010804	·009898	·009008	·008140	·007304	·006506	*2·0*
2·1	·010677	·009975	·009263	·008547	·007836	·007135	·006452	·005793	·005163	*2·1*
2·2	·008358	·007813	·007259	·006702	·006147	·005601	·005068	·004553	·004061	*2·2*
2·3	·006484	·006064	·005638	·005208	·004780	·004358	·003946	·003547	·003166	*2·3*
2·4	·004985	·004665	·004339	·004011	·003683	·003360	·003044	·002738	·002446	*2·4*
2·5	·003798	·003556	·003309	·003061	·002813	·002567	·002327	·002095	·001872	*2·5*
2·6	·002867	·002686	·002501	·002315	·002128	·001944	·001763	·001588	·001420	*2·6*

k	$r=\cdot10$									k
	h = *0·9*	*h* = *1·0*	*h* = *1·1*	*h* = *1·2*	*h* = *1·3*	*h* = *1·4*	*h* = *1·5*	*h* = *1·6*	*h* = *1·7*	
·0	·102649	·088981	·076521	·065276	·055229	·046342	·038560	·031813	·026023	*0·0*
·1	·095313	·082662	·071122	·060701	·051384	·043138	·035912	·029644	·024261	*0·1*
·2	·087944	·076310	·065690	·056094	·047509	·039906	·033239	·027452	·022480	*0·2*
·3	·080616	·069988	·060280	·051502	·043643	·036678	·030568	·025260	·020696	*0·3*
·4	·073402	·063759	·054945	·046970	·039825	·033488	·027925	·023089	·018928	*0·4*
·5	·066371	·057684	·049738	·042542	·036091	·030366	·025337	·020962	·017194	*0·5*
·6	·059588	·051818	·044705	·038260	·032477	·027342	·022827	·018896	·015510	*0·6*
·7	·053109	·046210	·039890	·034159	·029014	·024441	·020417	·016913	·013890	*0·7*
·8	·046982	·040903	·035330	·030273	·025729	·021687	·018128	·015026	·012348	*0·8*
·9	·041247	·035931	·031055	·026626	·022643	·019098	·015974	·013249	·010895	*0·9*
·0	·035931	·031320	·027086	·023238	·019775	·016690	·013969	·011593	·009540	*1·0*
·1	·031055	·027086	·023440	·020122	·017134	·014470	·012119	·010065	·008288	*1·1*
·2	·026626	·023238	·020122	·017285	·014728	·012447	·010432	·008669	·007143	*1·2*
·3	·022643	·019775	·017134	·014728	·012558	·010620	·008906	·007407	·006107	*1·3*
·4	·019098	·016690	·014470	·012447	·010620	·008987	·007542	·006276	·005179	*1·4*
·5	·015974	·013969	·012119	·010432	·008906	·007542	·006334	·005275	·004356	*1·5*
·6	·013249	·011593	·010065	·008669	·007407	·006276	·005275	·004396	·003632	*1·6*
·7	·010895	·009540	·008288	·007143	·006107	·005179	·004356	·003632	·003004	*1·7*
·8	·008883	·007783	·006766	·005836	·004993	·004237	·003566	·002976	·002463	*1·8*
·9	·007179	·006294	·005476	·004726	·004046	·003436	·002894	·002417	·002002	*1·9*
·0	·005752	·005046	·004393	·003794	·003251	·002763	·002328	·001946	·001613	*2·0*
·1	·004568	·004010	·003493	·003019	·002589	·002202	·001857	·001553	·001288	*2·1*
·2	·003595	·003158	·002753	·002381	·002043	·001739	·001468	·001228	·001020	*2·2*
·3	·002804	·002465	·002151	·001862	·001598	·001361	·001150	·000963	·000800	*2·3*
·4	·002168	·001907	·001665	·001442	·001239	·001056	·000893	·000748	·000622	*2·4*
·5	·001661	·001462	·001277	·001107	·000952	·000812	·000687	·000576	·000479	*2·5*
·6	·001261	·001111	·000971	·000842	·000725	·000619	·000524	·000440	·000366	*2·6*

k	r = ·10									k
	h = *1·8*	*h* = *1·9*	*h* = *2·0*	*h* = *2·1*	*h* = *2·2*	*h* = *2·3*	*h* = *2·4*	*h* = *2·5*	*h* = *2·6*	
0·0	·021103	·016965	·013518	·010677	·008358	·006484	·004985	·003798	·002867	*0·0*
0·1	·019685	·015833	·012623	·009975	·007813	·006064	·004665	·003556	·002686	*0·1*
0·2	·018249	·014686	·011715	·009263	·007259	·005638	·004339	·003309	·002501	*0·2*
0·3	·016811	·013536	·010804	·008547	·006702	·005208	·004011	·003061	·002315	*0·3*
0·4	·015384	·012394	·009898	·007836	·006147	·004780	·003683	·002813	·002128	*0·4*
0·5	·013983	·011272	·009008	·007135	·005601	·004358	·003360	·002567	·001944	*0·5*
0·6	·012621	·010180	·008140	·006452	·005068	·003946	·003044	·002327	·001763	*0·6*
0·7	·011310	·009129	·007304	·005793	·004553	·003547	·002738	·002095	·001588	*0·7*
0·8	·010061	·008126	·006506	·005163	·004061	·003166	·002446	·001872	·001420	*0·8*
0·9	·008883	·007179	·005752	·004568	·003595	·002804	·002168	·001661	·001261	*0·9*
1·0	·007783	·006294	·005046	·004010	·003158	·002465	·001907	·001462	·001111	*1·0*
1·1	·006766	·005476	·004393	·003493	·002753	·002151	·001665	·001277	·000971	*1·1*
1·2	·005836	·004726	·003794	·003019	·002381	·001862	·001442	·001107	·000842	*1·2*
1·3	·004993	·004046	·003251	·002589	·002043	·001598	·001239	·000952	·000725	*1·3*
1·4	·004237	·003436	·002763	·002202	·001739	·001361	·001056	·000812	·000619	*1·4*
1·5	·003566	·002894	·002328	·001857	·001468	·001150	·000893	·000687	·000524	*1·5*
1·6	·002976	·002417	·001946	·001553	·001228	·000963	·000748	·000576	·000440	*1·6*
1·7	·002463	·002002	·001613	·001288	·001020	·000800	·000622	·000479	·000366	*1·7*
1·8	·002020	·001643	·001325	·001059	·000839	·000659	·000513	·000395	·000302	*1·8*
1·9	·001643	·001338	·001080	·000864	·000685	·000538	·000419	·000323	·000247	*1·9*
2·0	·001325	·001080	·000872	·000698	·000554	·000435	·000339	·000262	·000201	*2·0*
2·1	·001059	·000864	·000698	·000559	·000444	·000349	·000273	·000211	·000161	*2·1*
2·2	·000839	·000685	·000554	·000444	·000353	·000278	·000217	·000168	·000129	*2·2*
2·3	·000659	·000538	·000435	·000349	·000278	·000219	·000171	·000132	·000102	*2·3*
2·4	·000513	·000419	·000339	·000273	·000217	·000171	·000134	·000104	·000080	*2·4*
2·5	·000395	·000323	·000262	·000211	·000168	·000132	·000104	·000080	·000062	*2·5*
2·6	·000302	·000247	·000201	·000161	·000129	·000102	·000080	·000062	·000047	*2·6*

k	r = ·15									k
	h = *0·0*	*h* = *0·1*	*h* = *0·2*	*h* = *0·3*	*h* = *0·4*	*h* = *0·5*	*h* = *0·6*	*h* = *0·7*	*h* = *0·8*	
0·0	·273964	·253929	·233856	·213946	·194397	·175397	·157115	·139703	·123287	*0·0*
0·1	·253929	·235500	·217016	·198665	·180629	·163082	·146183	·130072	·114867	*0·1*
0·2	·233856	·217016	·200109	·183307	·166777	·150679	·135159	·120348	·106357	*0·2*
0·3	·213946	·198665	·183307	·168028	·152981	·138312	·124155	·110631	·097843	*0·3*
0·4	·194397	·180629	·166777	·152981	·139380	·126106	·113283	·101019	·089411	*0·4*
0·5	·175397	·163082	·150679	·138312	·126106	·114181	·102648	·091607	·081144	*0·5*
0·6	·157115	·146183	·135159	·124155	·113283	·102648	·092352	·082484	·073122	*0·6*
0·7	·139703	·130072	·120348	·110631	·101019	·091607	·082484	·073730	·065416	*0·7*
0·8	·123287	·114867	·106357	·097843	·089411	·081144	·073122	·065416	·058088	*0·8*
0·9	·107964	·100662	·093273	·085871	·078532	·071328	·064330	·057598	·051190	*0·9*
1·0	·093807	·087525	·081161	·074778	·068441	·062214	·056156	·050323	·044763	*1·0*
1·1	·080859	·075499	·070061	·064602	·059175	·053835	·048634	·043620	·038835	*1·1*
1·2	·069136	·064600	·059993	·055361	·050752	·046211	·041782	·037508	·033423	*1·2*
1·3	·058628	·054822	·050952	·047055	·043173	·039343	·035604	·031989	·028532	*1·3*
1·4	·049305	·046139	·042915	·039665	·036422	·033219	·030088	·027058	·024156	*1·4*
1·5	·041117	·038505	·035842	·033155	·030470	·027814	·025215	·022696	·020280	*1·5*
1·6	·033998	·031862	·029682	·027479	·025274	·023092	·020952	·018876	·016883	*1·6*
1·7	·027870	·026139	·024370	·022579	·020785	·019007	·017262	·015566	·013935	*1·7*
1·8	·022650	·021259	·019835	·018393	·016946	·015510	·014098	·012725	·011403	*1·8*
1·9	·018246	·017139	·016004	·014852	·013696	·012546	·011414	·010312	·009249	*1·9*
2·0	·014569	·013696	·012799	·011888	·010971	·010059	·009160	·008283	·007437	*2·0*
2·1	·011530	·010847	·010145	·009431	·008711	·007994	·007286	·006595	·005926	*2·1*
2·2	·009044	·008515	·007970	·007415	·006855	·006296	·005744	·005204	·004681	*2·2*
2·3	·007030	·006623	·006205	·005777	·005346	·004914	·004487	·004069	·003664	*2·3*
2·4	·005415	·005106	·004787	·004461	·004131	·003801	·003474	·003153	·002842	*2·4*
2·5	·004133	·003900	·003659	·003413	·003163	·002913	·002665	·002421	·002184	*2·5*
2·6	·003126	·002952	·002772	·002587	·002400	·002213	·002026	·001842	·001664	*2·6*

k	$r = \cdot 15$									k
	$h=0{\cdot}9$	$h=1{\cdot}0$	$h=1{\cdot}1$	$h=1{\cdot}2$	$h=1{\cdot}3$	$h=1{\cdot}4$	$h=1{\cdot}5$	$h=1{\cdot}6$	$h=1{\cdot}7$	
0·0	·107964	·093807	·080859	·069136	·058628	·049305	·041117	·033998	·027870	0·0
0·1	·100662	·087525	·075499	·064600	·054822	·046139	·038505	·031862	·026139	0·1
0·2	·093273	·081161	·070061	·059993	·050952	·042915	·035842	·029682	·024370	0·2
0·3	·085871	·074778	·064602	·055361	·047055	·039665	·033155	·027479	·022579	0·3
0·4	·078532	·068441	·059175	·050752	·043173	·036422	·030470	·025273	·020785	0·4
0·5	·071328	·062214	·053835	·046211	·039343	·033219	·027814	·023092	·019007	0·5
0·6	·064330	·056156	·048634	·041782	·035604	·030088	·025215	·020952	·017262	0·6
0·7	·057598	·050323	·043620	·037508	·031989	·027058	·022696	·018876	·015566	0·7
0·8	·051190	·044763	·038835	·033423	·028532	·024156	·020280	·016883	·013935	0·8
0·9	·045151	·039517	·034315	·029560	·025257	·021403	·017987	·014988	·012382	0·9
1·0	·039517	·034618	·030088	·025943	·022188	·018820	·015831	·013204	·010920	1·0
1·1	·034315	·030088	·026176	·022591	·019339	·016420	·013825	·011543	·009555	1·1
1·2	·029560	·025943	·022591	·019515	·016723	·014212	·011979	·010011	·008296	1·2
1·3	·025257	·022188	·019339	·016723	·014344	·012203	·010295	·008613	·007144	1·3
1·4	·021403	·018820	·016420	·014212	·012203	·010392	·008776	·007350	·006103	1·4
1·5	·017987	·015831	·013825	·011979	·010295	·008776	·007420	·006220	·005170	1·5
1·6	·014988	·013204	·011543	·010011	·008613	·007350	·006220	·005220	·004344	1·6
1·7	·012382	·010920	·009555	·008296	·007144	·006103	·005170	·004344	·003618	1·7
1·8	·010142	·008953	·007842	·006815	·005875	·005024	·004261	·003584	·002988	1·8
1·9	·008235	·007276	·006380	·005550	·004790	·004100	·003481	·002931	·002447	1·9
2·0	·006627	·005862	·005145	·004481	·003871	·003317	·002819	·002376	·001986	2·0
2·1	·005287	·004681	·004113	·003585	·003101	·002660	·002263	·001910	·001598	2·1
2·2	·004180	·003704	·003258	·002843	·002462	·002114	·001801	·001521	·001274	2·2
2·3	·003275	·002905	·002558	·002235	·001937	·001665	·001420	·001201	·001007	2·3
2·4	·002543	·002258	·001990	·001740	·001510	·001300	·001109	·000939	·000789	2·4
2·5	·001956	·001739	·001534	·001343	·001167	·001005	·000859	·000728	·000612	2·5
2·6	·001492	·001327	·001172	·001027	·000893	·000770	·000659	·000559	·000471	2·6

k	$r = \cdot 15$									k
	$h=1{\cdot}8$	$h=1{\cdot}9$	$h=2{\cdot}0$	$h=2{\cdot}1$	$h=2{\cdot}2$	$h=2{\cdot}3$	$h=2{\cdot}4$	$h=2{\cdot}5$	$h=2{\cdot}6$	
0·0	·022650	·018246	·014569	·011530	·009044	·007030	·005415	·004133	·003126	0·0
0·1	·021259	·017139	·013696	·010827	·008545	·006623	·005106	·003900	·002952	0·1
0·2	·019835	·016004	·012799	·010145	·007970	·006205	·004787	·003659	·002772	0·2
0·3	·018393	·014852	·011888	·009431	·007415	·005777	·004461	·003413	·002587	0·3
0·4	·016946	·013696	·010971	·008711	·006855	·005346	·004131	·003163	·002400	0·4
0·5	·015510	·012546	·010059	·007994	·006296	·004914	·003801	·002913	·002213	0·5
0·6	·014098	·011414	·009160	·007286	·005744	·004487	·003474	·002665	·002026	0·6
0·7	·012725	·010312	·008283	·006595	·005204	·004069	·003153	·002421	·001842	0·7
0·8	·011403	·009249	·007437	·005926	·004681	·003664	·002842	·002184	·001664	0·8
0·9	·010142	·008235	·006627	·005287	·004180	·003275	·002543	·001956	·001492	0·9
1·0	·008953	·007276	·005862	·004681	·003704	·002905	·002258	·001739	·001327	1·0
1·1	·007842	·006380	·005145	·004113	·003258	·002558	·001990	·001534	·001172	1·1
1·2	·006815	·005550	·004481	·003585	·002843	·002235	·001740	·001343	·001027	1·2
1·3	·005875	·004790	·003871	·003101	·002462	·001937	·001510	·001167	·000893	1·3
1·4	·005024	·004100	·003317	·002660	·002114	·001665	·001300	·001005	·000770	1·4
1·5	·004261	·003481	·002819	·002263	·001801	·001420	·001109	·000859	·000659	1·5
1·6	·003584	·002931	·002376	·001910	·001521	·001201	·000939	·000728	·000559	1·6
1·7	·002988	·002447	·001986	·001598	·001274	·001007	·000789	·000612	·000471	1·7
1·8	·002471	·002025	·001646	·001325	·001058	·000837	·000656	·000510	·000393	1·8
1·9	002025	·001662	·001352	·001090	·000871	·000690	·000542	·000421	·000325	1·9
2·0	·001646	·001352	·001101	·000889	·000711	·000564	·000443	·000345	·000266	2·0
2·1	·001325	·001090	·000889	·000718	·000575	·000457	·000359	·000280	·000217	2·1
2·2	·001058	·000871	·000711	·000575	·000461	·000367	·000289	·000226	·000174	2·2
2·3	·000837	·000690	·000564	·000457	·000367	·000292	·000230	·000180	·000139	2·3
2·4	·000656	·000542	·000443	·000359	·000289	·000230	·000182	·000142	·000110	2·4
2·5	·000510	·000421	·000345	·000280	·000226	·000180	·000142	·000111	·000086	2·5
2·6	·000393	·000325	·000266	·000217	·000174	·000139	·000110	·000086	·000067	2·6

k	r = ·20									k
	h = 0·0	*h* = 0·1	*h* = 0·2	*h* = 0·3	*h* = 0·4	*h* = 0·5	*h* = 0·6	*h* = 0·7	*h* = 0·8	
0·0	·282047	·261971	·241774	·221662	·201840	·182502	·163829	·145981	·129097	0·0
0·1	·261971	·243515	·224922	·206384	·188087	·170215	·152934	·136397	·120731	0·1
0·2	·241774	·224922	·207922	·190949	·174174	·157765	·141879	·126655	·112215	0·2
0·3	·221662	·206384	·190949	·175516	·160242	·145281	·130776	·116856	·103635	0·3
0·4	·201840	·188087	·174174	·160242	·146434	·132889	·119738	·107100	·095079	0·4
0·5	·182502	·170215	·157765	·145281	·132889	·120715	·108878	·097486	·086634	0·5
0·6	·163829	·152934	·141879	·130776	·119738	·108878	·098302	·088109	·078384	0·6
0·7	·145981	·136397	·126655	·116856	·107100	·097486	·088109	·079057	·070408	0·7
0·8	·129097	·120731	·112215	·103635	·095079	·086634	·078384	·070408	·062776	0·8
0·9	·113286	·106044	·098659	·091206	·083762	·076403	·069203	·062230	·055547	0·9
1·0	·098633	·092413	·086062	·079641	·073217	·066856	·060622	·054576	·048771	1·0
1·1	·085189	·079893	·074475	·068989	·063491	·058038	·052686	·047485	·042484	1·1
1·2	·072981	·068509	·063927	·059279	·054613	·049978	·045420	·040984	·036711	1·2
1·3	·062008	·058265	·054422	·050518	·046592	·042684	·038836	·035084	·031464	1·3
1·4	·052246	·049139	·045945	·042693	·039418	·036153	·032931	·029784	·026744	1·4
1·5	·043649	·041094	·038461	·035777	·033068	·030363	·027689	·025073	·022541	1·5
1·6	·036156	·034072	·031922	·029725	·027505	·025283	·023084	·020928	·018838	1·6
1·7	·029690	·028007	·026266	·024484	·022680	·020872	·019079	·017318	·015608	1·7
1·8	·024169	·022820	·021424	·019992	·018539	·017081	·015632	·014206	·012819	1·8
1·9	·019501	·018431	·017321	·016181	·015022	·013856	·012695	·011552	·010437	1·9
2·0	·015596	·014755	·013880	·012980	·012063	·011140	·010219	·009310	·008422	2·0
2·1	·012361	·011706	·011023	·010319	·009601	·008877	·008152	·007436	·006735	2·1
2·2	·009709	·009204	·008676	·008130	·007573	·007009	·006445	·005886	·005338	2·2
2·3	·007558	·007171	·006766	·006348	·005919	·005485	·005050	·004617	·004193	2·3
2·4	·005829	·005536	·005229	·004911	·004584	·004253	·003920	·003589	·003263	2·4
2·5	·004455	·004235	·004004	·003764	·003518	·003267	·003015	·002764	·002516	2·5
2·6	·003373	·003210	·003038	·002859	·002675	·002487	·002298	·002109	·001922	2·6

k	r = ·20									k
	h = 0·9	*h* = 1·0	*h* = 1·1	*h* = 1·2	*h* = 1·3	*h* = 1·4	*h* = 1·5	*h* = 1·6	*h* = 1·7	
0·0	·113286	·098633	·085189	·072981	·062008	·052246	·043649	·036156	·029690	0·0
0·1	·106044	·092413	·079893	·068509	·058265	·049139	·041094	·034072	·028007	0·1
0·2	·098659	·086062	·074475	·063927	·054422	·045945	·038461	·031922	·026266	0·2
0·3	·091206	·079641	·068989	·059279	·050518	·042693	·035777	·029725	·024484	0·3
0·4	·083762	·073217	·063491	·054613	·046592	·039418	·033068	·027505	·022680	0·4
0·5	·076403	·066856	·058038	·049978	·042684	·036153	·030363	·025283	·020872	0·5
0·6	·069203	·060622	·052686	·045420	·038836	·032931	·027689	·023084	·019079	0·6
0·7	·062230	·054576	·047485	·040984	·035084	·029784	·025073	·020928	·017318	0·7
0·8	·055547	·048771	·042484	·036711	·031464	·026744	·022541	·018838	·015608	0·8
0·9	·049208	·043256	·037725	·032638	·028007	·023835	·020114	·016831	·013962	0·9
1·0	·043256	·038069	·033241	·028794	·024740	·021081	·017813	·014924	·012397	1·0
1·1	·037725	·033241	·029062	·025205	·021684	·018500	·015653	·013132	·010922	1·1
1·2	·032638	·028794	·025205	·021889	·018855	·016108	·013646	·011463	·009548	1·2
1·3	·028007	·024740	·021684	·018855	·016262	·013911	·011801	·009927	·008279	1·3
1·4	·023835	·021081	·018500	·016108	·013911	·011916	·010123	·008527	·007121	1·4
1·5	·020114	·017813	·015653	·013646	·011801	·010123	·008611	·007263	·006074	1·5
1·6	·016831	·014924	·013132	·011463	·009927	·008527	·007263	·006135	·005138	1·6
1·7	·013962	·012397	·010922	·009548	·008279	·007121	·006074	·005138	·004309	1·7
1·8	·011483	·010208	·009006	·007883	·006845	·005896	·005036	·004266	·003583	1·8
1·9	·009361	·008333	·007361	·006452	·005610	·004839	·004140	·003512	·002954	1·9
2·0	·007563	·006742	·005963	·005234	·004558	·003937	·003372	·002865	·002413	2·0
2·1	·006056	·005406	·004788	·004209	·003670	·003174	·002723	·002317	·001955	2·1
2·2	·004806	·004296	·003810	·003353	·002928	·002536	·002179	·001857	·001569	2·2
2·3	·003780	·003383	·003004	·002648	·002316	·002009	·001728	·001475	·001248	2·3
2·4	·002946	·002639	·002348	·002072	·001814	·001576	·001358	·001161	·000984	2·4
2·5	·002274	·002041	·001818	·001606	·001409	·001225	·001058	·000905	·000768	2·5
2·6	·001740	·001563	·001394	·001234	·001084	·000944	·000816	·000699	·000594	2·6

k	r = ·20									k
	h = 1·8	h = 1·9	h = 2·0	h = 2·1	h = 2·2	h = 2·3	h = 2·4	h = 2·5	h = 2·6	
0·0	·024169	·019501	·015596	·012361	·009709	·007558	·005829	·004455	·003373	0·0
0·1	·022820	·018431	·014755	·011706	·009204	·007171	·005536	·004235	·003210	0·1
0·2	·021424	·017321	·013880	·011023	·008676	·006766	·005229	·004004	·003038	0·2
0·3	·019992	·016181	·012980	·010319	·008130	·006348	·004911	·003764	·002859	0·3
0·4	·018539	·015022	·012063	·009601	·007573	·005919	·004584	·003518	·002675	0·4
0·5	·017081	·013856	·011140	·008877	·007009	·005485	·004253	·003267	·002487	0·5
0·6	·015632	·012695	·010219	·008152	·006445	·005050	·003920	·003015	·002298	0·6
0·7	·014206	·011552	·009310	·007436	·005886	·004617	·003589	·002764	·002109	0·7
0·8	·012819	·010437	·008422	·006735	·005338	·004193	·003263	·002516	·001922	0·8
0·9	·011483	·009361	·007563	·006056	·004806	·003780	·002946	·002274	·001740	0·9
1·0	·010208	·008333	·006742	·005406	·004296	·003383	·002639	·002041	·001563	1·0
1·1	·009006	·007361	·005963	·004788	·003810	·003004	·002348	·001818	·001394	1·1
1·2	·007883	·006452	·005234	·004209	·003353	·002648	·002072	·001606	·001234	1·2
1·3	·006845	·005610	·004558	·003670	·002928	·002316	·001814	·001409	·001084	1·3
1·4	·005896	·004839	·003937	·003174	·002536	·002009	·001576	·001225	·000944	1·4
1·5	·005036	·004140	·003372	·002723	·002179	·001728	·001358	·001058	·000816	1·5
1·6	·004266	·003512	·002865	·002317	·001857	·001475	·001161	·000905	·000699	1·6
1·7	·003583	·002954	·002413	·001955	·001569	·001248	·000984	·000768	·000594	1·7
1·8	·002984	·002463	·002016	·001635	·001314	·001047	·000826	·000646	·000501	1·8
1·9	·002463	·002037	·001669	·001356	·001091	·000871	·000688	·000539	·000419	1·9
2·0	·002016	·001669	·001370	·001114	·000898	·000718	·000569	·000446	·000347	2·0
2·1	·001635	·001356	·001114	·000908	·000733	·000587	·000465	·000366	·000285	2·1
2·2	·001314	·001091	·000898	·000733	·000593	·000475	·000378	·000297	·000232	2·2
2·3	·001047	·000871	·000718	·000587	·000475	·000382	·000304	·000239	·000187	2·3
2·4	·000826	·000688	·000569	·000465	·000378	·000304	·000242	·000191	·000149	2·4
2·5	·000646	·000539	·000446	·000366	·000297	·000239	·000191	·000151	·000118	2·5
2·6	·000501	·000419	·000347	·000285	·000232	·000187	·000149	·000118	·000093	2·6

k	r = ·25									k
	h = 0·0	h = 0·1	h = 0·2	h = 0·3	h = 0·4	h = 0·5	h = 0·6	h = 0·7	h = 0·8	
0·0	·290215	·270096	·249772	·229452	·209348	·189662	·170586	·152291	·134927	0·0
0·1	·270096	·251617	·232916	·214188	·195627	·177422	·159752	·142778	·126642	0·1
0·2	·249772	·232916	·215828	·198685	·181666	·164945	·148686	·133042	·118145	0·2
0·3	·229452	·214188	·198685	·183105	·167609	·152357	·137501	·123181	·109521	0·3
0·4	·209348	·195627	·181666	·167609	·153602	·139790	·126313	·113298	·100861	0·4
0·5	·189662	·177422	·164945	·152357	·139790	·127375	·115238	·103496	·092254	0·5
0·6	·170586	·159752	·148686	·137501	·126313	·115238	·104390	·093875	·083789	0·6
0·7	·152291	·142778	·133042	·123181	·113298	·103496	·093875	·084532	·075552	0·7
0·8	·134927	·126642	·118145	·109521	·100861	·092254	·083789	·075552	·067620	0·8
0·9	·118617	·111461	·104106	·096626	·089099	·081603	·074216	·067012	·060061	0·9
1·0	·103456	·097326	·091014	·084580	·078091	·071617	·065223	·058975	·052933	1·0
1·1	·089507	·084302	·078930	·073442	·067897	·062351	·056863	·051489	·046282	1·1
1·2	·076806	·072424	·067892	·063252	·058553	·053844	·049174	·044590	·040140	1·2
1·3	·065362	·061705	·057914	·054026	·050078	·046114	·042173	·038298	·034527	1·3
1·4	·055155	·052131	·048988	·045758	·042471	·039163	·035867	·032619	·029452	1·4
1·5	·046146	·043667	·041086	·038426	·035714	·032977	·030246	·027547	·024910	1·5
1·6	·038276	·036263	·034161	·031991	·029773	·027530	·025286	·023065	·020888	1·6
1·7	·031473	·029852	·028157	·026402	·024605	·022783	·020957	·019144	·017364	1·7
1·8	·025651	·024359	·023004	·021599	·020155	·018689	·017216	·015750	·014308	1·8
1·9	·020721	·019700	·018628	·017512	·016364	·015195	·014017	·012843	·011685	1·9
2·0	·016589	·015790	·014949	·014072	·013167	·012244	·011311	·010379	·009458	2·0
2·1	·013161	·012542	·011889	·011205	·010499	·009776	·009045	·008312	·007586	2·1
2·2	·010347	·009872	·009369	·008842	·008296	·007735	·007167	·006596	·006030	2·2
2·3	·008061	·007699	·007316	·006913	·006495	·006065	·005627	·005187	·004749	2·3
2·4	·006222	·005950	·005660	·005355	·005038	·004711	·004377	·004041	·003705	2·4
2·5	·004758	·004555	·004339	·004110	·003872	·003625	·003374	·003119	·002864	2·5
2·6	·003605	·003455	·003295	·003125	·002948	·002764	·002576	·002385	·002194	2·6

k	*r* = ·25									*k*
	h = *0·9*	*h* = *1·0*	*h* = *1·1*	*h* = *1·2*	*h* = *1·3*	*h* = *1·4*	*h* = *1·5*	*h* = *1·6*	*h* = *1·7*	
0·0	·118617	·103456	·089507	·076806	·065362	·055155	·046146	·038276	·031473	*0·0*
0·1	·111461	·097326	·084302	·072424	·061705	·052131	·043667	·036263	·029852	*0·1*
0·2	·104106	·091014	·078930	·067892	·057914	·048988	·041086	·034161	·028157	*0·2*
0·3	·096626	·084580	·073442	·063252	·054026	·045758	·038426	·031991	·026402	*0·3*
0·4	·089099	·078091	·067897	·058553	·050078	·042471	·035714	·029773	·024605	*0·4*
0·5	·081603	·071617	·062351	·053844	·046114	·039163	·032977	·027530	·022783	*0·5*
0·6	·074216	·065223	·056863	·049174	·042173	·035867	·030246	·025286	·020957	*0·6*
0·7	·067012	·058975	·051489	·044590	·038298	·032619	·027547	·023065	·019144	*0·7*
0·8	·060061	·052933	·046282	·040140	·034527	·029452	·024910	·020888	·017364	*0·8*
0·9	·053424	·047153	·041289	·035863	·030896	·026395	·022359	·018778	·015635	*0·9*
1·0	·047153	·041680	·036552	·031797	·027435	·023475	·019917	·016755	·013972	*1·0*
1·1	·041289	·036552	·032104	·027972	·024173	·020716	·017605	·014834	·012390	*1·1*
1·2	·035863	·031797	·027972	·024410	·021128	·018137	·015438	·013030	·010902	*1·2*
1·3	·030896	·027435	·024173	·021128	·018318	·015750	·013429	·011353	·009515	*1·3*
1·4	·026395	·023475	·020716	·018137	·015750	·013565	·011586	·009812	·008237	*1·4*
1·5	·022359	·019917	·017605	·015438	·013429	·011586	·009912	·008409	·007072	*1·5*
1·6	·018778	·016755	·014834	·013030	·011353	·009812	·008409	·007146	·006020	*1·6*
1·7	·015635	·013972	·012390	·010902	·009515	·008237	·007072	·006020	·005081	*1·7*
1·8	·012903	·011550	·010259	·009042	·007905	·006856	·005896	·005028	·004252	*1·8*
1·9	·010555	·009463	·008420	·007433	·006510	·005656	·004873	·004163	·003526	*1·9*
2·0	·008557	·007684	·006848	·006056	·005313	·004624	·003991	·003416	·002899	*2·0*
2·1	·006874	·006183	·005520	·004890	·004298	·003747	·003240	·002778	·002362	*2·1*
2·2	·005472	·004931	·004409	·003912	·003445	·003008	·002606	·002239	·001907	*2·2*
2·3	·004317	·003896	·003489	·003102	·002736	·002393	·002077	·001788	·001526	*2·3*
2·4	·003374	·003050	·002736	·002436	·002153	·001887	·001640	·001414	·001209	*2·4*
2·5	·002612	·002365	·002126	·001896	·001678	·001473	·001283	·001109	·000950	*2·5*
2·6	·002004	·001817	·001636	·001462	·001296	·001140	·000995	·000861	·000739	*2·6*

k	*r* = ·25									*k*
	h = *1·8*	*h* = *1·9*	*h* = *2·0*	*h* = *2·1*	*h* = *2·2*	*h* = *2·3*	*h* = *2·4*	*h* = *2·5*	*h* = *2·6*	
0·0	·025651	·020721	·016589	·013161	·010347	·008061	·006222	·004758	·003605	*0·0*
0·1	·024359	·019700	·015790	·012542	·009872	·007699	·005950	·004555	·003455	*0·1*
0·2	·023004	·018628	·014949	·011889	·009369	·007316	·005660	·004339	·003295	*0·2*
0·3	·021599	·017512	·014072	·011205	·008842	·006913	·005355	·004110	·003125	*0·3*
0·4	·020155	·016364	·013167	·010499	·008296	·006495	·005038	·003872	·002948	*0·4*
0·5	·018689	·015195	·012244	·009776	·007735	·006065	·004711	·003625	·002764	*0·5*
0·6	·017216	·014017	·011311	·009045	·007167	·005627	·004377	·003374	·002576	*0·6*
0·7	·015750	·012843	·010379	·008312	·006596	·005187	·004041	·003119	·002385	*0·7*
0·8	·014308	·011685	·009458	·007586	·006030	·004749	·003705	·002864	·002194	*0·8*
0·9	·012903	·010555	·008557	·006874	·005472	·004317	·003374	·002612	·002004	*0·9*
1·0	·011550	·009463	·007684	·006183	·004931	·003896	·003050	·002365	·001817	*1·0*
1·1	·010259	·008420	·006848	·005520	·004409	·003489	·002736	·002126	·001636	*1·1*
1·2	·009042	·007433	·006056	·004890	·003912	·003102	·002436	·001896	·001462	*1·2*
1·3	·007905	·006510	·005313	·004298	·003445	·002736	·002153	·001678	·001296	*1·3*
1·4	·006856	·005656	·004624	·003747	·003008	·002393	·001887	·001473	·001140	*1·4*
1·5	·005896	·004873	·003991	·003240	·002606	·002077	·001640	·001283	·000995	*1·5*
1·6	·005028	·004163	·003416	·002778	·002239	·001788	·001414	·001109	·000861	*1·6*
1·7	·004252	·003526	·002899	·002362	·001907	·001526	·001209	·000950	·000739	*1·7*
1·8	·003564	·002962	·002439	·001991	·001610	·001291	·001025	·000807	·000629	*1·8*
1·9	·002962	·002466	·002034	·001664	·001348	·001083	·000862	·000679	·000531	*1·9*
2·0	·002439	·002034	·001682	·001378	·001119	·000900	·000718	·000567	·000444	*2·0*
2·1	·001991	·001664	·001378	·001131	·000920	·000742	·000593	·000469	·000368	*2·1*
2·2	·001610	·001348	·001119	·000920	·000750	·000606	·000485	·000385	·000302	*2·2*
2·3	·001291	·001083	·000900	·000742	·000606	·000490	·000393	·000313	·000246	*2·3*
2·4	·001025	·000862	·000718	·000593	·000485	·000393	·000316	·000252	·000199	*2·4*
2·5	·000807	·000679	·000567	·000469	·000385	·000313	·000252	·000201	·000159	*2·5*
2·6	·000629	·000531	·000444	·000368	·000302	·000246	·000199	·000159	·000126	*2·6*

k	$r = \cdot30$									k
	$h = 0\cdot0$	$h = 0\cdot1$	$h = 0\cdot2$	$h = 0\cdot3$	$h = 0\cdot4$	$h = 0\cdot5$	$h = 0\cdot6$	$h = 0\cdot7$	$h = 0\cdot8$	
0·0	·298493	·278330	·257873	·237337	·216939	·196893	·177399	·158641	·140782	0·0
0·1	·278330	·259830	·241021	·222100	·203268	·184722	·166651	·149228	·132606	0·1
0·2	·257873	·241021	·223850	·206540	·189274	·172235	·155596	·139521	·124154	0·2
0·3	·237337	·222100	·206540	·190819	·175103	·159558	·144347	·129620	·115511	0·3
0·4	·216939	·203268	·189274	·175103	·160904	·146828	·133023	·119628	·106767	0·4
0·5	·196893	·184722	·172235	·159558	·146828	·134179	·121745	·109652	·098015	0·5
0·6	·177399	·166651	·155596	·144347	·133023	·121745	·110632	·099798	·089349	0·6
0·7	·158641	·149228	·139521	·129620	·119628	·109652	·099798	·090168	·080859	0·7
0·8	·140782	·132606	·124154	·115511	·106767	·098015	·089349	·080859	·072631	0·8
0·9	·123958	·116917	·109620	·102139	·094551	·086936	·079378	·071954	·064742	0·9
1·0	·108275	·102264	·096019	·089599	·083070	·076501	·069964	·063528	·057259	1·0
1·1	·093808	·088722	·083424	·077962	·072394	·066777	·061172	·055639	·050236	1·1
1·2	·080604	·076339	·071884	·067279	·062571	·057810	·053047	·048332	·043715	1·2
1·3	·068679	·065135	·061422	·057574	·053630	·049630	·045618	·041635	·037725	1·3
1·4	·058021	·055103	·052037	·048852	·045577	·042247	·038897	·035564	·032282	1·4
1·5	·048596	·046215	·043708	·041094	·038400	·035653	·032882	·030117	·027387	1·5
1·6	·040348	·038424	·036392	·034268	·032072	·029827	·027556	·025283	·023034	1·6
1·7	·033206	·031665	·030034	·028324	·026551	·024733	·022889	·021039	·019202	1·7
1·8	·027085	·025864	·024566	·023203	·021785	·020327	·018844	·017351	·015865	1·8
1·9	·021895	·020936	·019914	·018837	·017714	·016556	·015374	·014181	·012991	1·9
2·0	·017540	·016794	·015997	·015154	·014273	·013362	·012429	·011485	·010541	2·0
2·1	·013923	·013349	·012733	·012080	·011396	·010686	·009957	·009217	·008474	2·1
2·2	·010951	·010513	·010042	·009541	·009015	·008467	·007903	·007329	·006750	2·2
2·3	·008534	·008203	·007847	·007466	·007065	·006646	·006214	·005773	·005327	2·3
2·4	·006589	·006342	·006074	·005788	·005486	·005169	·004841	·004505	·004165	2·4
2·5	·005040	·004857	·004658	·004445	·004219	·003982	·003736	·003482	·003225	2·5
2·6	·003819	·003685	·003539	·003382	·003215	·003039	·002855	·002667	·002474	2·6

k	$r = \cdot30$									k
	$h = 0\cdot9$	$h = 1\cdot0$	$h = 1\cdot1$	$h = 1\cdot2$	$h = 1\cdot3$	$h = 1\cdot4$	$h = 1\cdot5$	$h = 1\cdot6$	$h = 1\cdot7$	
0·0	·123958	·108275	·093808	·080604	·068679	·058021	·048596	·040348	·033206	0·0
0·1	·116917	·102264	·088722	·076339	·065135	·055103	·046215	·038424	·031665	0·1
0·2	·109620	·096019	·083424	·071884	·061422	·052037	·043708	·036392	·030034	0·2
0·3	·102139	·089599	·077962	·067279	·057574	·048852	·041094	·034268	·028324	0·3
0·4	·094551	·083070	·072394	·062571	·053630	·045577	·038400	·032072	·026551	0·4
0·5	·086936	·076501	·066777	·057810	·049630	·042247	·035653	·029827	·024733	0·5
0·6	·079378	·069964	·061172	·053047	·045618	·038897	·032882	·027556	·022889	0·6
0·7	·071954	·063528	·055639	·048332	·041635	·035564	·030117	·025283	·021039	0·7
0·8	·064742	·057259	·050236	·043715	·037725	·032282	·027387	·023034	·019202	0·8
0·9	·057810	·051217	·045015	·039243	·033928	·029086	·024722	·020831	·017398	0·9
1·0	·051217	·045458	·040026	·034958	·030279	·026007	·022147	·018697	·015646	1·0
1·1	·045015	·040026	·035308	·030895	·026811	·023073	·019686	·016652	·013962	1·1
1·2	·039243	·034958	·030895	·027085	·023551	·020306	·017360	·014714	·012362	1·2
1·3	·033928	·030279	·026811	·023551	·020517	·017726	·015185	·012896	·010857	1·3
1·4	·029086	·026007	·023073	·020306	·017726	·015345	·013172	·011210	·009457	1·4
1·5	·024722	·022147	·019686	·017360	·015185	·013172	·011330	·009662	·008168	1·5
1·6	·020831	·018697	·016652	·014714	·012896	·011210	·009662	·008257	·006996	1·6
1·7	·017398	·015646	·013962	·012362	·010857	·009457	·008168	·006996	·005940	1·7
1·8	·014402	·012977	·011603	·010294	·009059	·007908	·006845	·005875	·004999	1·8
1·9	·011815	·010666	·009556	·008495	·007492	·006553	·005685	·004890	·004170	1·9
2·0	·009605	·008688	·007799	·006948	·006140	·005382	·004679	·004033	·003447	2·0
2·1	·007737	·007012	·006307	·005630	·004986	·004380	·003816	·003296	·002824	2·1
2·2	·006174	·005607	·005054	·004520	·004011	·003531	·003083	·002670	·002292	2·2
2·3	·004882	·004442	·004012	·003595	·003197	·002821	·002468	·002142	·001843	2·3
2·4	·003824	·003486	·003155	·002833	·002525	·002232	·001958	·001703	·001468	2·4
2·5	·002967	·002710	·002457	·002211	·001975	·001750	·001538	·001341	·001159	2·5
2·6	·002280	·002087	·001896	·001710	·001530	·001359	·001197	·001046	·000906	2·6

k	$r = \cdot 30$									k
	$h=1\cdot 8$	$h=1\cdot 9$	$h=2\cdot 0$	$h=2\cdot 1$	$h=2\cdot 2$	$h=2\cdot 3$	$h=2\cdot 4$	$h=2\cdot 5$	$h=2\cdot 6$	
0·0	·027085	·021895	·017540	·013923	·010951	·008534	·006589	·005040	·003819	0·0
0·1	·025864	·020936	·016794	·013349	·010513	·008203	·006342	·004857	·003685	0·1
0·2	·024566	·019914	·015997	·012733	·010042	·007847	·006074	·004658	·003539	0·2
0·3	·023203	·018837	·015154	·012080	·009541	·007466	·005788	·004445	·003382	0·3
0·4	·021785	·017714	·014273	·011396	·009015	·007065	·005486	·004219	·003215	0·4
0·5	·020327	·016556	·013362	·010686	·008467	·006646	·005169	·003982	·003039	0·5
0·6	·018844	·015374	·012429	·009957	·007903	·006214	·004841	·003736	·002855	0·6
0·7	·017351	·014181	·011485	·009217	·007329	·005773	·004505	·003482	·002667	0·7
0·8	·015865	·012991	·010541	·008474	·006750	·005327	·004165	·003225	·002474	0·8
0·9	·014402	·011815	·009605	·007737	·006174	·004882	·003824	·002967	·002280	0·9
1·0	·012977	·010666	·008688	·007012	·005607	·004442	·003486	·002710	·002087	1·0
1·1	·011603	·009556	·007799	·006307	·005054	·004012	·003155	·002457	·001896	1·1
1·2	·010294	·008495	·006948	·005630	·004520	·003595	·002833	·002211	·001710	1·2
1·3	·009059	·007492	·006140	·004986	·004011	·003197	·002525	·001975	·001530	1·3
1·4	·007908	·006553	·005382	·004380	·003531	·002821	·002232	·001750	·001359	1·4
1·5	·006845	·005685	·004679	·003816	·003083	·002468	·001958	·001538	·001197	1·5
1·6	·005875	·004890	·004033	·003296	·002670	·002142	·001703	·001341	·001046	1·6
1·7	·004999	·004170	·003447	·002824	·002292	·001843	·001468	·001159	·000906	1·7
1·8	·004216	·003525	·002920	·002398	·001951	·001572	·001255	·000993	·000778	1·8
1·9	·003525	·002953	·002453	·002018	·001646	·001329	·001064	·000844	·000663	1·9
2·0	·002920	·002453	·002041	·001684	·001376	·001114	·000894	·000710	·000559	2·0
2·1	·002398	·002018	·001684	·001392	·001140	·000925	·000744	·000593	·000468	2·1
2·2	·001951	·001646	·001376	·001140	·000936	·000762	·000614	·000490	·000388	2·2
2·3	·001572	·001329	·001114	·000925	·000762	·000621	·000502	·000402	·000319	2·3
2·4	·001255	·001064	·000894	·000744	·000614	·000502	·000407	·000326	·000259	2·4
2·5	·000993	·000844	·000710	·000593	·000490	·000402	·000326	·000262	·000209	2·5
2·6	·000778	·000663	·000559	·000468	·000388	·000319	·000259	·000209	·000167	2·6

k	$r = \cdot 35$									k
	$h=0\cdot 0$	$h=0\cdot 1$	$h=0\cdot 2$	$h=0\cdot 3$	$h=0\cdot 4$	$h=0\cdot 5$	$h=0\cdot 6$	$h=0\cdot 7$	$h=0\cdot 8$	
0·0	·306909	·286699	·266102	·245340	·224635	·204210	·184280	·165040	·146666	0·0
0·1	·286699	·268182	·249265	·230146	·211033	·192133	·173644	·155754	·138630	0·1
0·2	·266102	·249265	·232016	·214539	·197022	·179656	·162625	·146105	·130253	0·2
0·3	·245340	·230146	·214539	·198682	·182747	·166907	·151333	·136187	·121617	0·3
0·4	·224635	·211033	·197022	·182747	·168362	·154025	·139890	·126106	·112811	0·4
0·5	·204210	·192133	·179656	·166907	·154025	·141148	·128418	·115971	·103932	0·5
0·6	·184280	·173644	·162625	·151333	·139890	·128418	·117045	·105893	·095077	0·6
0·7	·165040	·155754	·146105	·136187	·126106	·115971	·105893	·095983	·086343	0·7
0·8	·146666	·138630	·130253	·121617	·112811	·103932	·095077	·086343	·077823	0·8
0·9	·129308	·122415	·115206	·107752	·100127	·092416	·084702	·077070	·069602	0·9
1·0	·113085	·107225	·101078	·094700	·088158	·081519	·074857	·068246	·061758	1·0
1·1	·098086	·093150	·087955	·082549	·076984	·071320	·065619	·059943	·054355	1·1
1·2	·084366	·080247	·075897	·071356	·066667	·061879	·057043	·052214	·047444	1·2
1·3	·071950	·068545	·064937	·061158	·057244	·053233	·049169	·045098	·041063	1·3
1·4	·060834	·058046	·055082	·051967	·048729	·045401	·042018	·038617	·035237	1·4
1·5	·050987	·048727	·046315	·043772	·041120	·038385	·035595	·032781	·029974	1·5
1·6	·042359	·040543	·038600	·036544	·034392	·032166	·029887	·027581	·025273	1·6
1·7	·034878	·033434	·031884	·030237	·028509	·026713	·024870	·022998	·021119	1·7
1·8	·028460	·027323	·026099	·024793	·023418	·021985	·020508	·019004	·017488	1·8
1·9	·023013	·022127	·021169	·020145	·019061	·017929	·016757	·015560	·014349	1·9
2·0	·018438	·017755	·017013	·016217	·015372	·014486	·013566	·012622	·011665	2·0
2·1	·014638	·014116	·013547	·012935	·012283	·011596	·010881	·010144	·009395	2·1
2·2	·011513	·011118	·010687	·010221	·009722	·009195	·008645	·008076	·007495	2·2
2·3	·008971	·008676	·008352	·008000	·007623	·007223	·006804	·006369	·005923	2·3
2·4	·006925	·006706	·006465	·006203	·005921	·005621	·005304	·004975	·004637	2·4
2·5	·005295	·005135	·004958	·004764	·004555	·004332	·004096	·003849	·003595	2·5
2·6	·004011	·003895	·003766	·003624	·003471	·003307	·003132	·002950	·002760	2·6

k	r = ·35									k
	h = 0·9	h = 1·0	h = 1·1	h = 1·2	h = 1·3	h = 1·4	h = 1·5	h = 1·6	h = 1·7	
0·0	·129308	·113085	·098086	·084366	·071950	·060834	·050987	·042359	·034878	0·0
0·1	·122415	·107225	·093150	·080247	·068545	·058046	·048727	·040543	·033434	0·1
0·2	·115206	·101078	·087955	·075897	·064937	·055082	·046315	·038600	·031884	0·2
0·3	·107752	·094700	·082549	·071356	·061158	·051967	·043772	·036544	·030237	0·3
0·4	·100127	·088158	·076984	·066667	·057244	·048729	·041120	·034392	·028509	0·4
0·5	·092416	·081519	·071320	·061879	·053233	·045401	·038385	·032166	·026713	0·5
0·6	·084702	·074857	·065619	·057043	·049169	·042018	·035595	·029887	·024870	0·6
0·7	·077070	·068246	·059943	·052214	·045098	·038617	·032781	·027581	·022998	0·7
0·8	·069602	·061758	·054355	·047444	·041063	·035237	·029974	·025273	·021119	0·8
0·9	·062376	·055460	·048914	·042785	·037110	·031913	·027206	·022989	·019252	0·9
1·0	·055460	·049414	·043675	·038285	·033279	·028682	·024506	·020754	·017419	1·0
1·1	·048914	·043675	·038685	·033986	·029608	·025576	·021902	·018591	·015640	1·1
1·2	·042785	·038285	·033986	·029924	·026129	·022622	·019417	·016521	·013931	1·2
1·3	·037110	·033279	·029608	·026129	·022868	·019845	·017074	·014561	·012308	1·3
1·4	·031913	·028682	·025576	·022622	·019845	·017262	·014887	·012727	·010785	1·4
1·5	·027206	·024506	·021902	·019417	·017074	·014887	·012870	·011030	·009370	1·5
1·6	·022989	·020754	·018591	·016521	·014561	·012727	·011030	·009477	·008070	1·6
1·7	·019252	·017419	·015640	·013931	·012308	·010785	·009370	·008070	·006890	1·7
1·8	·015977	·014489	·013039	·011642	·010311	·009056	·007888	·006811	·005830	1·8
1·9	·013138	·011941	·010771	·009640	·008559	·007536	·006580	·005697	·004889	1·9
2·0	·010704	·009751	·008816	·007909	·007039	·006214	·005439	·004721	·004062	2·0
2·1	·008640	·007889	·007149	·006429	·005735	·005076	·004454	·003876	·003344	2·1
2·2	·006908	·006322	·005742	·005176	·004629	·004107	·003614	·003153	·002727	2·2
2·3	·005471	·005018	·004569	·004128	·003701	·003292	·002904	·002540	·002203	2·3
2·4	·004292	·003945	·003601	·003261	·002931	·002613	·002311	·002027	·001763	2·4
2·5	·003335	·003072	·002810	·002551	·002299	·002055	·001822	·001602	·001397	2·5
2·6	·002566	·002369	·002172	·001977	·001785	·001600	·001422	·001254	·001096	2·6

k	r = ·35									k
	h = 1·8	h = 1·9	h = 2·0	h = 2·1	h = 2·2	h = 2·3	h = 2·4	h = 2·5	h = 2·6	
0·0	·028460	·023013	·018438	·014638	·011513	·008971	·006925	·005295	·004011	0·0
0·1	·027323	·022127	·017755	·014116	·011118	·008676	·006706	·005135	·003895	0·1
0·2	·026099	·021169	·017013	·013547	·010687	·008352	·006465	·004958	·003766	0·2
0·3	·024793	·020145	·016217	·012935	·010221	·008000	·006203	·004764	·003624	0·3
0·4	·023418	·019061	·015372	·012283	·009722	·007623	·005921	·004555	·003471	0·4
0·5	·021985	·017929	·014486	·011596	·009195	·007223	·005621	·004332	·003307	0·5
0·6	·020508	·016757	·013566	·010881	·008645	·006804	·005304	·004096	·003132	0·6
0·7	·019004	·015560	·012622	·010144	·008076	·006369	·004975	·003849	·002950	0·7
0·8	·017488	·014349	·011665	·009395	·007495	·005923	·004637	·003595	·002760	0·8
0·9	·015977	·013138	·010704	·008640	·006908	·005471	·004292	·003335	·002566	0·9
1·0	·014489	·011941	·009751	·007889	·006322	·005018	·003945	·003072	·002369	1·0
1·1	·013039	·010771	·008816	·007149	·005742	·004569	·003601	·002810	·002172	1·1
1·2	·011642	·009640	·007909	·006429	·005176	·004128	·003261	·002551	·001977	1·2
1·3	·010311	·008559	·007039	·005735	·004629	·003701	·002931	·002299	·001785	1·3
1·4	·009056	·007536	·006214	·005076	·004107	·003292	·002613	·002055	·001600	1·4
1·5	·007888	·006580	·005439	·004454	·003614	·002904	·002311	·001822	·001422	1·5
1·6	·006811	·005697	·004721	·003876	·003153	·002540	·002027	·001602	·001254	1·6
1·7	·005830	·004889	·004062	·003344	·002727	·002203	·001763	·001397	·001096	1·7
1·8	·004946	·004159	·003464	·002860	·002338	·001894	·001519	·001207	·000950	1·8
1·9	·004159	·003505	·002928	·002423	·001987	·001614	·001298	·001035	·000816	1·9
2·0	·003464	·002928	·002453	·002035	·001673	·001363	·001100	·000879	·000695	2·0
2·1	·002860	·002423	·002035	·001694	·001396	·001140	·000923	·000739	·000587	2·1
2·2	·002338	·001987	·001673	·001396	·001155	·000946	·000767	·000616	·000491	2·2
2·3	·001894	·001614	·001363	·001140	·000946	·000777	·000632	·000509	·000406	2·3
2·4	·001519	·001298	·001100	·000923	·000767	·000632	·000516	·000417	·000334	2·4
2·5	·001207	·001035	·000879	·000739	·000616	·000509	·000417	·000338	·000271	2·5
2·6	·000950	·000816	·000695	·000587	·000491	·000406	·000334	·000271	·000218	2·6

k	r = ·40 h = 0·0	h = 0·1	h = 0·2	h = 0·3	h = 0·4	h = 0·5	h = 0·6	h = 0·7	h = 0·8	k
0·0	·315495	·295235	·274491	·253487	·232457	·211633	·191242	·171495	·152582	0·0
0·1	·295235	·276706	·257677	·238353	·218947	·199675	·180750	·162371	·144720	0·1
0·2	·274491	·257677	·240356	·222710	·204936	·187232	·169794	·152809	·136451	0·2
0·3	·253487	·238353	·222710	·206724	·190569	·174428	·158481	·142902	·127851	0·3
0·4	·232457	·218947	·204936	·190569	·176005	·161405	·146934	·132752	·119009	0·4
0·5	·211633	·199675	·187232	·174428	·161405	·148306	·135281	·122474	·110023	0·5
0·6	·191242	·180750	·169794	·158481	·146934	·135281	·123653	·112182	·100994	0·6
0·7	·171495	·162371	·152809	·142902	·132752	·122474	·112182	·101994	·092023	0·7
0·8	·152582	·144720	·136451	·127851	·119009	·110023	·100994	·092023	·083213	0·8
0·9	·134666	·127955	·120869	·113473	·105840	·098055	·090203	·082375	·074659	0·9
1·0	·117883	·112208	·106193	·099890	·093363	·086679	·079913	·073143	·066446	1·0
1·1	·102331	·097578	·092521	·087202	·081671	·075987	·070212	·064411	·058651	1·1
1·2	·088079	·084137	·079926	·075480	·070839	·066051	·061167	·056243	·051335	1·2
1·3	·075160	·071923	·068451	·064771	·060914	·056920	·052829	·048689	·044547	1·3
1·4	·063577	·060945	·058111	·055095	·051922	·048622	·045229	·041782	·038320	1·4
1·5	·053305	·051187	·048897	·046449	·043864	·041165	·038379	·035537	·032671	1·5
1·6	·044294	·042606	·040774	·038809	·036724	·034538	·032273	·029953	·027605	1·6
1·7	·036474	·035144	·033693	·032131	·030466	·028715	·026892	·025017	·023111	1·7
1·8	·029762	·028724	·027588	·026358	·025043	·023653	·022201	·020702	·019171	1·8
1·9	·024063	·023261	·022380	·021423	·020395	·019303	·018158	·016971	·015754	1·9
2·0	·019276	·018663	·017987	·017249	·016454	·015606	·014712	·013782	·012825	2·0
2·1	·015297	·014834	·014321	·013758	·013149	·012497	·011807	·011086	·010341	2·1
2·2	·012026	·011680	·011295	·010870	·010409	·009913	·009386	·008833	·008258	2·2
2·3	·009366	·009110	·008823	·008507	·008161	·007788	·007389	·006969	·006531	2·3
2·4	·007225	·007038	·006827	·006594	·006337	·006059	·005761	·005446	·005116	2·4
2·5	·005521	·005386	·005233	·005062	·004874	·004669	·004449	·004214	·003968	2·5
2·6	·004179	·004082	·003972	·003848	·003712	·003563	·003401	·003229	·003047	2·6

k	r = ·40 h = 0·9	h = 1·0	h = 1·1	h = 1·2	h = 1·3	h = 1·4	h = 1·5	h = 1·6	h = 1·7	k
0·0	·134666	·117883	·102331	·088079	·075160	·063577	·053305	·044294	·036474	0·0
0·1	·127955	·112208	·097578	·084137	·071923	·060945	·051187	·042606	·035144	0·1
0·2	·120869	·106193	·092521	·079926	·068451	·058111	·048897	·040774	·033693	0·2
0·3	·113473	·099890	·087202	·075480	·064771	·055095	·046449	·038809	·032131	0·3
0·4	·105840	·093363	·081671	·070839	·060914	·051922	·043864	·036724	·030466	0·4
0·5	·098055	·086679	·075987	·066051	·056920	·048622	·041165	·034538	·028715	0·5
0·6	·090203	·079913	·070212	·061167	·052829	·045229	·038379	·032273	·026892	0·6
0·7	·082375	·073143	·064411	·056243	·048689	·041782	·035537	·029953	·025017	0·7
0·8	·074659	·066446	·058651	·051335	·044547	·038320	·032671	·027605	·023111	0·8
0·9	·067140	·059896	·052997	·046500	·040451	·034882	·029814	·025254	·021196	0·9
1·0	·059896	·053563	·047510	·041789	·036445	·031508	·026999	·022929	·019294	1·0
1·1	·052997	·047510	·042246	·037254	·032573	·028234	·024258	·020655	·017427	1·1
1·2	·046500	·041789	·037254	·032937	·028874	·025094	·021618	·018457	·015615	1·2
1·3	·040451	·036445	·032573	·028874	·025380	·022117	·019105	·016356	·013876	1·3
1·4	·034882	·031508	·028234	·025094	·022117	·019326	·016741	·014373	·012228	1·4
1·5	·029814	·026999	·024258	·021618	·019105	·016741	·014542	·012520	·010683	1·5
1·6	·025254	·022929	·020655	·018457	·016356	·014373	·012520	·010811	·009251	1·6
1·7	·021196	·019294	·017427	·015615	·013876	·012228	·010683	·009251	·007940	1·7
1·8	·017627	·016086	·014568	·013089	·011664	·010308	·009032	·007844	·006752	1·8
1·9	·014521	·013287	·012065	·010870	·009714	·008609	·007565	·006590	·005690	1·9
2·0	·011851	·010871	·009898	·008942	·008013	·007122	·006277	·005484	·004750	2·0
2·1	·009579	·008810	·008043	·007286	·006548	·005836	·005159	·004521	·003927	2·1
2·2	·007669	·007071	·006473	·005879	·005299	·004737	·004199	·003691	·003216	2·2
2·3	·006080	·005621	·005158	·004699	·004246	·003807	·003385	·002984	·002608	2·3
2·4	·004774	·004424	·004071	·003718	·003370	·003030	·002702	·002389	·002095	2·4
2·5	·003711	·003448	·003181	·002914	·002648	·002388	·002136	·001894	·001666	2·5
2·6	·002857	·002661	·002462	·002260	·002060	·001863	·001671	·001487	·001311	2·6

k	$r = \cdot 40$									k
	$h = 1\cdot 8$	$h = 1\cdot 9$	$h = 2\cdot 0$	$h = 2\cdot 1$	$h = 2\cdot 2$	$h = 2\cdot 3$	$h = 2\cdot 4$	$h = 2\cdot 5$	$h = 2\cdot 6$	
0·0	·029762	·024063	·019276	·015297	·012026	·009366	·007225	·005521	·004179	0·0
0·1	·028724	·023261	·018663	·014834	·011680	·009110	·007038	·005386	·004082	0·1
0·2	·027588	·022380	·017987	·014321	·011295	·008823	·006827	·005233	·003972	0·2
0·3	·026358	·021423	·017249	·013758	·010870	·008507	·006594	·005062	·003848	0·3
0·4	·025043	·020395	·016454	·013149	·010409	·008161	·006337	·004874	·003712	0·4
0·5	·023653	·019303	·015606	·012497	·009913	·007788	·006059	·004669	·003563	0·5
0·6	·022201	·018158	·014712	·011807	·009386	·007389	·005761	·004449	·003401	0·6
0·7	·020702	·016971	·013782	·011086	·008833	·006969	·005446	·004214	·003229	0·7
0·8	·019171	·015754	·012825	·010341	·008258	·006531	·005116	·003968	·003047	0·8
0·9	·017627	·014521	·011851	·009579	·007669	·006080	·004774	·003711	·002857	0·9
1·0	·016086	·013287	·010871	·008810	·007071	·005621	·004424	·003448	·002661	1·0
1·1	·014568	·012065	·009898	·008043	·006473	·005158	·004071	·003181	·002462	1·1
1·2	·013089	·010870	·008942	·007286	·005879	·004699	·003718	·002914	·002260	1·2
1·3	·011664	·009714	·008013	·006548	·005299	·004246	·003370	·002648	·002060	1·3
1·4	·010308	·008609	·007122	·005836	·004737	·003807	·003030	·002388	·001863	1·4
1·5	·009032	·007565	·006277	·005159	·004199	·003385	·002702	·002136	·001671	1·5
1·6	·007844	·006590	·005484	·004521	·003691	·002984	·002389	·001894	·001487	1·6
1·7	·006752	·005690	·004750	·003927	·003216	·002608	·002095	·001666	·001311	1·7
1·8	·005760	·004868	·004076	·003381	·002777	·002260	·001820	·001452	·001147	1·8
1·9	·004868	·004127	·003467	·002884	·002377	·001940	·001568	·001255	·000994	1·9
2·0	·004076	·003467	·002921	·002438	·002015	·001650	·001338	·001074	·000854	2·0
2·1	·003381	·002884	·002438	·002041	·001693	·001391	·001131	·000911	·000727	2·1
2·2	·002777	·002377	·002015	·001693	·001409	·001161	·000948	·000766	·000613	2·2
2·3	·002260	·001940	·001650	·001391	·001161	·000960	·000786	·000637	·000512	2·3
2·4	·001820	·001568	·001338	·001131	·000948	·000786	·000646	·000525	·000423	2·4
2·5	·001452	·001255	·001074	·000911	·000766	·000637	·000525	·000429	·000347	2·5
2·6	·001147	·000994	·000854	·000727	·000613	·000512	·000423	·000347	·000281	2·6

k	$r = \cdot 45$									k
	$h = 0\cdot 0$	$h = 0\cdot 1$	$h = 0\cdot 2$	$h = 0\cdot 3$	$h = 0\cdot 4$	$h = 0\cdot 5$	$h = 0\cdot 6$	$h = 0\cdot 7$	$h = 0\cdot 8$	
0·0	·324288	·303974	·283071	·261810	·240432	·219182	·198300	·178014	·158530	0·0
0·1	·303974	·285438	·266295	·246755	·227039	·207375	·187987	·169091	·150884	0·1
0·2	·283071	·266295	·248905	·231089	·213048	·194990	·177124	·159651	·142759	0·2
0·3	·261810	·246755	·231089	·214979	·198603	·182152	·165817	·149783	·134229	0·3
0·4	·240432	·227039	·213048	·198603	·183864	·169000	·154185	·139590	·125380	0·4
0·5	·219182	·207375	·194990	·182152	·169000	·155684	·142361	·129186	·116309	0·5
0·6	·198300	·187987	·177124	·165817	·154185	·142361	·130483	·118691	·107120	0·6
0·7	·178014	·169091	·159651	·149783	·139590	·129186	·118691	·108229	·097923	0·7
0·8	·158530	·150884	·142759	·134229	·125380	·116309	·107120	·097923	·088825	0·8
0·9	·140029	·133540	·126615	·119312	·111704	·103870	·095901	·087889	·079932	0·9
1·0	·122658	·117207	·111362	·105172	·098693	·091993	·085148	·078236	·071341	1·0
1·1	·106532	·101999	·097116	·091921	·086459	·080785	·074962	·069056	·063139	1·1
1·2	·091729	·087998	·083961	·079645	·075087	·070330	·065426	·060429	·055401	1·2
1·3	·078294	·075255	·071951	·068403	·064637	·060690	·056601	·052416	·048186	1·3
1·4	·066235	·063786	·061110	·058223	·055145	·051903	·048529	·045060	·041536	1·4
1·5	·055531	·053579	·051435	·049111	·046621	·043986	·041230	·038384	·035480	1·5
1·6	·046136	·044596	·042897	·041046	·039053	·036934	·034708	·032397	·030027	1·6
1·7	·037981	·036779	·035447	·033989	·032411	·030725	·028945	·027088	·025176	1·7
1·8	·030979	·030052	·029019	·027882	·026646	·025320	·023912	·022436	·020909	1·8
1·9	·025034	·024326	·023534	·022658	·021701	·020667	·019566	·018406	·017200	1·9
2·0	·020041	·019507	·018906	·018238	·017505	·016709	·015857	·014955	·014012	2·0
2·1	·015893	·015494	·015044	·014540	·013985	·013379	·012727	·012033	·011304	2·1
2·2	·012485	·012191	·011856	·011481	·011065	·010609	·010115	·009588	·009031	2·2
2·3	·009714	·009500	·009255	·008978	·008670	·008330	·007961	·007564	·007143	2·3
2·4	·007487	·007332	·007155	·006953	·006727	·006477	·006204	·005909	·005594	2·4
2·5	·005715	·005605	·005478	·005332	·005169	·004987	·004787	·004570	·004338	2·5
2·6	·004321	·004243	·004153	·004050	·003933	·003802	·003657	·003500	·003330	2·6

k	r = ·45									k
	h = *0·9*	*h* = *1·0*	*h* = *1·1*	*h* = *1·2*	*h* = *1·3*	*h* = *1·4*	*h* = *1·5*	*h* = *1·6*	*h* = *1·7*	
0·0	·140029	·122658	·106532	·091729	·078294	·066235	·055531	·046136	·037981	*0·0*
0·1	·133540	·117207	·101999	·087998	·075255	·063786	·053579	·044596	·036779	*0·1*
0·2	·126615	·111362	·097116	·083961	·071951	·061110	·051435	·042897	·035447	*0·2*
0·3	·119312	·105172	·091921	·079645	·068403	·058223	·049111	·041046	·033989	*0·3*
0·4	·111704	·098693	·086459	·075087	·064637	·055145	·046621	·039053	·032411	*0·4*
0·5	·103870	·091993	·080785	·070330	·060690	·051903	·043986	·036934	·030725	*0·5*
0·6	·095901	·085148	·074962	·065426	·056601	·048529	·041230	·034708	·028945	*0·6*
0·7	·087889	·078236	·069056	·060429	·052416	·045060	·038384	·032397	·027088	*0·7*
0·8	·079932	·071341	·063139	·055401	·048186	·041536	·035480	·030027	·025176	*0·8*
0·9	·072120	·064543	·057280	·050400	·043959	·038000	·032551	·027627	·023229	*0·9*
1·0	·064543	·057922	·051549	·045486	·039787	·034494	·029634	·025225	·021272	*1·0*
1·1	·057280	·051549	·046008	·040715	·035719	·031059	·026763	·022850	·019327	*1·1*
1·2	·050400	·045486	·040715	·036138	·031798	·027734	·023971	·020529	·017418	*1·2*
1·3	·043959	·039787	·035719	·031798	·028066	·024554	·021289	·018290	·015568	*1·3*
1·4	·038000	·034494	·031059	·027734	·024554	·021549	·018743	·016154	·013795	*1·4*
1·5	·032551	·029634	·026763	·023971	·021289	·018743	·016355	·014142	·012116	*1·5*
1·6	·027627	·025225	·022850	·020529	·018290	·016154	·014142	·012269	·010547	*1·6*
1·7	·023229	·021272	·019327	·017418	·015568	·013795	·012116	·010547	·009097	*1·7*
1·8	·019347	·017769	·016193	·014639	·013126	·011669	·010283	·008981	·007773	*1·8*
1·9	·015960	·014700	·013438	·012186	·010961	·009776	·008644	·007576	·006579	*1·9*
2·0	·013038	·012045	·011043	·010046	·009065	·008112	·007197	·006329	·005515	*2·0*
2·1	·010548	·009772	·008987	·008201	·007424	·006665	·005933	·005235	·004578	*2·1*
2·2	·008450	·007851	·007241	·006628	·006019	·005422	·004843	·004288	·003763	*2·2*
2·3	·006702	·006245	·005777	·005304	·004832	·004367	·003913	·003477	·003062	*2·3*
2·4	·005263	·004917	·004562	·004202	·003840	·003482	·003131	·002791	·002467	*2·4*
2·5	·004091	·003833	·003567	·003295	·003021	·002748	·002479	·002218	·001967	*2·5*
2·6	·003149	·002958	·002761	·002558	·002352	·002147	·001943	·001744	·001553	*2·6*

k	r = ·45									k
	h = *1·8*	*h* = *1·9*	*h* = *2·0*	*h* = *2·1*	*h* = *2·2*	*h* = *2·3*	*h* = *2·4*	*h* = *2·5*	*h* = *2·6*	
0·0	·030979	·025034	·020041	·015893	·012485	·009714	·007487	·005715	·004321	*0·0*
0·1	·030052	·024326	·019507	·015494	·012191	·009500	·007332	·005605	·004243	*0·1*
0·2	·029019	·023534	·018906	·015044	·011856	·009255	·007155	·005478	·004153	*0·2*
0·3	·027882	·022658	·018238	·014540	·011481	·008978	·006953	·005332	·004050	*0·3*
0·4	·026646	·021701	·017505	·013985	·011065	·008670	·006727	·005169	·003933	*0·4*
0·5	·025320	·020667	·016709	·013379	·010609	·008330	·006477	·004987	·003802	*0·5*
0·6	·023912	·019566	·015857	·012727	·010115	·007961	·006204	·004787	·003657	*0·6*
0·7	·022436	·018406	·014955	·012033	·009588	·007564	·005909	·004570	·003500	*0·7*
0·8	·020909	·017200	·014012	·011304	·009031	·007143	·005594	·004338	·003330	*0·8*
0·9	·019347	·015960	·013038	·010548	·008450	·006702	·005263	·004091	·003149	*0·9*
1·0	·017769	·014700	·012045	·009772	·007851	·006245	·004917	·003833	·002958	*1·0*
1·1	·016193	·013438	·011043	·008987	·007241	·005777	·004562	·003567	·002761	*1·1*
1·2	·014639	·012186	·010046	·008201	·006628	·005304	·004202	·003295	·002558	*1·2*
1·3	·013126	·010961	·009065	·007424	·006019	·004832	·003840	·003021	·002352	*1·3*
1·4	·011669	·009776	·008112	·006665	·005422	·004367	·003482	·002748	·002147	*1·4*
1·5	·010283	·008644	·007197	·005933	·004843	·003913	·003131	·002479	·001943	*1·5*
1·6	·008981	·007576	·006329	·005235	·004288	·003477	·002791	·002218	·001744	*1·6*
1·7	·007773	·006579	·005515	·004578	·003763	·003062	·002467	·001967	·001553	*1·7*
1·8	·006665	·005661	·004763	·003968	·003273	·002673	·002161	·001729	·001370	*1·8*
1·9	·005661	·004826	·004074	·003406	·002820	·002312	·001876	·001507	·001198	*1·9*
2·0	·004763	·004074	·003452	·002897	·002407	·001980	·001613	·001300	·001038	*2·0*
2·1	·003968	·003406	·002897	·002440	·002035	·001680	·001374	·001112	·000890	*2·1*
2·2	·003273	·002820	·002407	·002035	·001703	·001412	·001159	·000941	·000757	*2·2*
2·3	·002673	·002312	·001980	·001680	·001412	·001175	·000968	·000789	·000637	*2·3*
2·4	·002161	·001876	·001613	·001374	·001159	·000968	·000800	·000655	·000531	*2·4*
2·5	·001729	·001507	·001300	·001112	·000941	·000789	·000655	·000539	·000438	*2·5*
2·6	·001370	·001198	·001038	·000890	·000757	·000637	·000531	·000438	·000358	*2·6*

k	*r* = ·50									*k*
	h = *0·0*	*h* = *0·1*	*h* = *0·2*	*h* = *0·3*	*h* = *0·4*	*h* = *0·5*	*h* = *0·6*	*h* = *0·7*	*h* = *0·8*	
0·0	·333333	·312961	·291886	·270344	·248589	·226878	·205468	·184605	·164512	*0·0*
0·1	·312961	·294422	·275161	·255392	·235345	·215260	·195377	·175927	·157126	*0·1*
0·2	·291886	·275161	·257709	·239718	·221397	·202965	·184644	·166650	·149190	*0·2*
0·3	·270344	·255392	·239718	·223488	·206888	·190114	·173370	·156858	·140769	*0·3*
0·4	·248589	·235345	·221397	·206888	·191979	·176847	·161676	·146649	·131946	*0·4*
0·5	·226878	·215260	·202965	·190114	·176847	·163320	·149694	·136139	·122816	*0·5*
0·6	·205468	·195377	·184644	·173370	·161676	·149694	·137570	·125451	·113486	*0·6*
0·7	·184605	·175927	·166650	·156858	·146649	·136139	·125451	·114718	·104071	*0·7*
0·8	·164512	·157126	·149190	·140769	·131946	·122816	·113486	·104071	·094686	*0·8*
0·9	·145388	·139168	·132448	·125281	·117733	·109882	·101819	·093640	·085448	*0·9*
1·0	·127398	·122215	·116586	·110550	·104159	·097477	·090578	·083546	·076465	*1·0*
1·1	·110671	·106398	·101733	·096704	·091350	·085722	·079882	·073896	·067839	*1·1*
1·2	·095297	·091814	·087990	·083844	·079407	·074718	·069825	·064784	·059656	*1·2*
1·3	·081329	·078522	·075421	·072042	·068404	·064539	·060485	·056284	·051988	*1·3*
1·4	·068785	·066546	·064061	·061337	·058388	·055237	·051913	·048451	·044891	*1·4*
1·5	·057646	·055882	·053912	·051740	·049377	·046836	·044142	·041320	·038401	*1·5*
1·6	·047867	·046493	·044950	·043238	·041365	·039340	·037180	·034905	·032539	*1·6*
1·7	·039379	·038321	·037126	·035793	·034325	·032729	·031018	·029204	·027307	*1·7*
1·8	·032095	·031290	·030375	·029348	·028211	·026968	·025627	·024198	·022695	*1·8*
1·9	·025912	·025307	·024615	·023834	·022963	·022006	·020968	·019854	·018677	*1·9*
2·0	·020724	·020274	·019756	·019169	·018510	·017782	·016987	·016130	·015218	*2·0*
2·1	·016417	·016087	·015704	·015268	·014776	·014228	·013627	·012974	·012276	*2·1*
2·2	·012882	·012642	·012363	·012043	·011679	·011272	·010823	·010332	·009804	*2·2*
2·3	·010011	·009840	·009638	·009406	·009141	·008842	·008510	·008145	·007750	*2·3*
2·4	·007706	·007585	·007441	·007275	·007083	·006867	·006624	·006357	·006065	*2·4*
2·5	·005875	·005790	·005689	·005571	·005435	·005280	·005105	·004911	·004698	*2·5*
2·6	·004436	·004377	·004307	·004225	·004129	·004019	·003894	·003755	·003602	*2·6*

k	*r* = ·50									*k*
	h = *0·9*	*h* = *1·0*	*h* = *1·1*	*h* = *1·2*	*h* = *1·3*	*h* = *1·4*	*h* = *1·5*	*h* = *1·6*	*h* = *1·7*	
0·0	·145388	·127398	·110671	·095297	·081329	·068785	·057646	·047867	·039379	*0·0*
0·1	·139168	·122215	·106398	·091814	·078522	·066546	·055882	·046493	·038321	*0·1*
0·2	·132448	·116586	·101733	·087990	·075421	·064061	·053912	·044950	·037126	*0·2*
0·3	·125281	·110550	·096704	·083844	·072042	·061337	·051740	·043238	·035793	*0·3*
0·4	·117733	·104159	·091350	·079407	·068404	·058388	·049377	·041365	·034325	*0·4*
0·5	·109882	·097477	·085722	·074718	·064539	·055237	·046836	·039340	·032729	*0·5*
0·6	·101819	·090578	·079882	·069825	·060485	·051913	·044142	·037180	·031018	*0·6*
0·7	·093640	·083546	·073896	·064784	·056284	·048451	·041320	·034905	·029204	*0·7*
0·8	·085448	·076465	·067839	·059656	·051988	·044891	·038401	·032539	·027307	*0·8*
0·9	·077344	·069426	·061786	·054504	·047649	·041275	·035421	·030110	·025349	*0·9*
1·0	·069426	·062514	·055812	·049394	·043323	·037651	·032418	·027648	·023353	*1·0*
1·1	·061786	·055812	·049991	·044388	·039062	·034063	·029428	·025184	·021345	*1·1*
1·2	·054504	·049394	·044388	·039545	·034920	·030556	·026490	·022749	·019349	*1·2*
1·3	·047649	·043323	·039062	·034920	·030942	·027170	·023639	·020373	·017391	*1·3*
1·4	·041275	·037651	·034063	·030556	·027170	·023944	·020907	·018085	·015494	*1·4*
1·5	·035421	·032418	·029428	·026490	·023639	·020907	·018323	·015909	·013681	*1·5*
1·6	·030110	·027648	·025184	·022749	·020373	·018085	·015909	·013864	·011969	*1·6*
1·7	·025349	·023353	·021345	·019349	·017391	·015494	·013681	·011969	·010372	*1·7*
1·8	·021134	·019534	·017915	·016297	·014701	·013146	·011652	·010233	·008902	*1·8*
1·9	·017447	·016178	·014888	·013591	·012304	·011044	·009826	·008663	·007566	*1·9*
2·0	·014260	·013266	·012249	·011221	·010196	·009186	·008204	·007261	·006367	*2·0*
2·1	·011539	·010769	·009977	·009171	·008363	·007563	·006780	·006024	·005304	*2·1*
2·2	·009242	·008654	·008043	·007420	·006790	·006163	·005546	·004947	·004373	*2·2*
2·3	·007328	·006883	·006419	·005941	·005457	·004971	·004490	·004021	·003568	*2·3*
2·4	·005751	·005418	·005069	·004708	·004339	·003968	·003598	·003234	·002882	*2·4*
2·5	·004468	·004222	·003962	·003692	·003415	·003134	·002852	·002574	·002303	*2·5*
2·6	·003435	·003255	·003065	·002865	·002659	·002449	·002237	·002027	·001820	*2·6*

k	r = ·50									k
	h = 1·8	h = 1·9	h = 2·0	h = 2·1	h = 2·2	h = 2·3	h = 2·4	h = 2·5	h = 2·6	
0·0	·032095	·025912	·020724	·016417	·012882	·010011	·007706	·005875	·004436	0·0
0·1	·031290	·025307	·020274	·016087	·012642	·009840	·007585	·005790	·004377	0·1
0·2	·030375	·024615	·019756	·015704	·012363	·009638	·007441	·005689	·004307	0·2
0·3	·029348	·023834	·019169	·015268	·012043	·009406	·007275	·005571	·004225	0·3
0·4	·028211	·022963	·018510	·014776	·011679	·009141	·007083	·005435	·004129	0·4
0·5	·026968	·022006	·017782	·014228	·011272	·008842	·006867	·005280	·004019	0·5
0·6	·025627	·020968	·016987	·013627	·010823	·008510	·006624	·005105	·003894	0·6
0·7	·024198	·019854	·016130	·012974	·010332	·008145	·006357	·004911	·003755	0·7
0·8	·022695	·018677	·015218	·012276	·009804	·007750	·006065	·004698	·003602	0·8
0·9	·021134	·017447	·014260	·011539	·009242	·007328	·005751	·004468	·003435	0·9
1·0	·019534	·016178	·013266	·010769	·008654	·006883	·005418	·004222	·003255	1·0
1·1	·017915	·014888	·012249	·009977	·008043	·006419	·005069	·003962	·003065	1·1
1·2	·016297	·013591	·011221	·009171	·007420	·005941	·004708	·003692	·002865	1·2
1·3	·014701	·012304	·010196	·008363	·006790	·005457	·004339	·003415	·002659	1·3
1·4	·013146	·011044	·009186	·007563	·006163	·004971	·003968	·003134	·002449	1·4
1·5	·011652	·009826	·008204	·006780	·005546	·004490	·003598	·002852	·002237	1·5
1·6	·010233	·008663	·007261	·006024	·004947	·004021	·003234	·002574	·002027	1·6
1·7	·008902	·007566	·006367	·005304	·004373	·003568	·002882	·002303	·001820	1·7
1·8	·007671	·006546	·005531	·004626	·003830	·003138	·002544	·002041	·001620	1·8
1·9	·006546	·005608	·004758	·003996	·003322	·002733	·002225	·001793	·001429	1·9
2·0	·005531	·004758	·004053	·003418	·002853	·002358	·001928	·001560	·001249	2·0
2·1	·004626	·003996	·003418	·002895	·002427	·002014	·001654	·001344	·001081	2·1
2·2	·003830	·003322	·002853	·002427	·002043	·001703	·001405	·001146	·000926	2·2
2·3	·003138	·002733	·002358	·002014	·001703	·001426	·001181	·000968	·000785	2·3
2·4	·002544	·002225	·001928	·001654	·001405	·001181	·000983	·000809	·000659	2·4
2·5	·002041	·001793	·001560	·001344	·001146	·000968	·000809	·000669	·000548	2·5
2·6	·001620	·001429	·001249	·001081	·000926	·000785	·000659	·000548	·000451	2·6

k	r = ·55									k
	h = 0·0	h = 0·1	h = 0·2	h = 0·3	h = 0·4	h = 0·5	h = 0·6	h = 0·7	h = 0·8	
0·0	·342686	·322250	·300984	·279133	·256963	·234747	·212762	·191272	·170523	0·0
0·1	·322250	·303714	·284328	·264313	·243907	·223364	·202942	·182893	·163452	0·1
0·2	·300984	·284328	·266819	·248648	·230030	·211196	·192384	·173829	·155757	0·2
0·3	·279133	·264313	·248648	·232305	·215474	·198360	·181181	·164155	·147493	0·3
0·4	·256963	·243907	·230030	·215474	·200401	·184994	·169449	·153964	·138735	0·4
0·5	·234747	·223364	·211196	·198360	·184994	·171258	·157324	·143371	·129577	0·5
0·6	·212762	·202942	·192384	·181181	·169449	·157324	·144956	·132503	·120128	0·6
0·7	·191272	·182893	·173829	·164155	·153964	·143371	·132503	·121500	·110505	0·7
0·8	·170523	·163452	·155757	·147493	·138735	·129577	·120128	·110505	·100834	0·8
0·9	·150733	·144834	·138373	·131391	·123947	·116116	·107987	·099659	·091241	0·9
1·0	·132086	·127220	·121858	·116026	·109769	·103146	·096229	·089099	·081849	1·0
1·1	·114725	·110759	·106360	·101545	·096345	·090807	·084986	·078950	·072774	1·1
1·2	·098756	·095562	·091995	·088065	·083794	·079216	·074374	·069320	·064118	1·2
1·3	·084240	·081698	·078840	·075671	·072205	·068464	·064482	·060300	·055967	1·3
1·4	·071200	·069202	·066940	·064415	·061634	·058614	·055378	·051957	·048390	1·4
1·5	·059623	·058071	·056303	·054315	·052112	·049703	·047104	·044339	·041436	1·5
1·6	·049463	·048273	·046908	·045363	·043638	·041739	·039678	·037469	·035135	1·6
1·7	·040650	·039749	·038708	·037522	·036188	·034710	·033095	·031352	·029498	1·7
1·8	·033093	·032419	·031635	·030736	·029718	·028581	·027331	·025972	·024518	1·8
1·9	·026686	·026188	·025605	·024932	·024164	·023302	·022346	·021300	·020173	1·9
2·0	·021314	·020952	·020524	·020026	·019455	·018808	·018087	·017292	·016429	2·0
2·1	·016862	·016601	·016291	·015928	·015508	·015030	·014492	·013895	·013243	2·1
2·2	·013213	·013027	·012806	·012544	·012240	·011890	·011494	·011052	·010565	2·2
2·3	·010254	·010124	·009968	·009782	·009564	·009311	·009024	·008700	·008342	2·3
2·4	·007882	·007792	·007683	·007552	·007398	·007219	·007012	·006779	·006518	2·4
2·5	·006000	·005939	·005864	·005773	·005666	·005540	·005394	·005227	·005040	2·5
2·6	·004524	·004483	·004432	·004370	·004296	·004209	·004107	·003990	·003857	2·6

k	$r = \cdot55$									k
	$h=0\cdot9$	$h=1\cdot0$	$h=1\cdot1$	$h=1\cdot2$	$h=1\cdot3$	$h=1\cdot4$	$h=1\cdot5$	$h=1\cdot6$	$h=1\cdot7$	
0·0	·150733	·132086	·114725	·098756	·084240	·071200	·059623	·049463	·040650	0·0
0·1	·144834	·127220	·110759	·095562	·081698	·069202	·058071	·048273	·039749	0·1
0·2	·138373	·121858	·106360	·091995	·078840	·066940	·056303	·046908	·038708	0·2
0·3	·131391	·116026	·101545	·088065	·075671	·064415	·054315	·045363	·037522	0·3
0·4	·123947	·109769	·096345	·083794	·072205	·061634	·052112	·043638	·036188	0·4
0·5	·116116	·103146	·090807	·079216	·068464	·058614	·049703	·041739	·034710	0·5
0·6	·107987	·096229	·084986	·074374	·064482	·055378	·047104	·039678	·033095	0·6
0·7	·099659	·089099	·078950	·069320	·060300	·051957	·044339	·037469	·031352	0·7
0·8	·091241	·081849	·072774	·064118	·055967	·048390	·041436	·035135	·029498	0·8
0·9	·082843	·074574	·066539	·058833	·051537	·044720	·038431	·032703	·027553	0·9
1·0	·074574	·067369	·060327	·053536	·047070	·040995	·035360	·030202	·025540	1·0
1·1	·066539	·060327	·054221	·048296	·042624	·037265	·032266	·027665	·023484	1·1
1·2	·058833	·053536	·048296	·043183	·038259	·033579	·029190	·025127	·021414	1·2
1·3	·051537	·047070	·042624	·038259	·034030	·029987	·026172	·022620	·019357	1·3
1·4	·044720	·040995	·037265	·033579	·029987	·026531	·023251	·020180	·017340	1·4
1·5	·038431	·035360	·032266	·029190	·026172	·023251	·020462	·017834	·015390	1·5
1·6	·032703	·030202	·027665	·025127	·022620	·020180	·017834	·015610	·013530	1·6
1·7	·027553	·025540	·023484	·021414	·019357	·017340	·015390	·013530	·011778	1·7
1·8	·022981	·021380	·019735	·018066	·016397	·014751	·013148	·011609	·010151	1·8
1·9	·018974	·017716	·016414	·015086	·013748	·012420	·011118	·009860	·008661	1·9
2·0	·015505	·014529	·013512	·012467	·011408	·010349	·009304	·008288	·007313	2·0
2·1	·012540	·011792	·011007	·010195	·009367	·008533	·007704	·006893	·006110	2·1
2·2	·010037	·009471	·008873	·008250	·007610	·006961	·006312	·005673	·005051	2·2
2·3	·007950	·007527	·007077	·006605	·006116	·005618	·005116	·004618	·004130	2·3
2·4	·006231	·005919	·005585	·005231	·004863	·004485	·004102	·003719	·003341	2·4
2·5	·004832	·004605	·004360	·004099	·003825	·003542	·003253	·002962	·002673	2·5
2·6	·003709	·003545	·003368	·003177	·002976	·002767	·002552	·002333	·002115	2·6

k	$r = \cdot55$									k
	$h=1\cdot8$	$h=1\cdot9$	$h=2\cdot0$	$h=2\cdot1$	$h=2\cdot2$	$h=2\cdot3$	$h=2\cdot4$	$h=2\cdot5$	$h=2\cdot6$	
0·0	·033093	·026686	·021314	·016862	·013213	·010254	·007882	·006000	·004524	0·0
0·1	·032419	·026188	·020952	·016601	·013027	·010124	·007792	·005939	·004483	0·1
0·2	·031635	·025605	·020524	·016291	·012806	·009968	·007683	·005864	·004432	0·2
0·3	·030736	·024932	·020026	·015928	·012544	·009782	·007552	·005773	·004370	0·3
0·4	·029718	·024164	·019455	·015508	·012240	·009564	·007398	·005666	·004296	0·4
0·5	·028581	·023302	·018808	·015030	·011890	·009311	·007219	·005540	·004209	0·5
0·6	·027331	·022346	·018087	·014492	·011494	·009024	·007012	·005394	·004107	0·6
0·7	·025972	·021300	·017292	·013895	·011052	·008700	·006779	·005227	·003990	0·7
0·8	·024518	·020173	·016429	·013243	·010565	·008342	·006518	·005040	·003857	0·8
0·9	·022981	·018974	·015505	·012540	·010037	·007950	·006231	·004832	·003709	0·9
1·0	·021380	·017716	·014529	·011792	·009471	·007527	·005919	·004605	·003545	1·0
1·1	·019735	·016414	·013512	·011007	·008873	·007077	·005585	·004360	·003368	1·1
1·2	·018066	·015086	·012467	·010195	·008250	·006605	·005231	·004099	·003177	1·2
1·3	·016397	·013748	·011408	·009367	·007610	·006116	·004863	·003825	·002976	1·3
1·4	·014751	·012420	·010349	·008533	·006961	·005618	·004485	·003542	·002767	1·4
1·5	·013148	·011118	·009304	·007704	·006312	·005116	·004102	·003253	·002552	1·5
1·6	·011609	·009860	·008288	·006893	·005673	·004618	·003719	·002962	·002333	1·6
1·7	·010151	·008661	·007313	·006110	·005051	·004130	·003341	·002673	·002115	1·7
1·8	·008789	·007532	·006389	·005363	·004454	·003659	·002974	·002390	·001900	1·8
1·9	·007532	·006484	·005527	·004661	·003889	·003210	·002621	·002117	·001691	1·9
2·0	·006389	·005527	·004732	·004010	·003362	·002788	·002288	·001856	·001490	2·0
2·1	·005363	·004661	·004010	·003414	·002876	·002397	·001977	·001612	·001300	2·1
2·2	·004454	·003889	·003362	·002876	·002435	·002040	·001690	·001385	·001123	2·2
2·3	·003659	·003210	·002788	·002397	·002040	·001717	·001430	·001178	·000960	2·3
2·4	·002974	·002621	·002288	·001977	·001690	·001430	·001197	·000991	·000811	2·4
2·5	·002390	·002117	·001856	·001612	·001385	·001178	·000991	·000825	·000679	2·5
2·6	·001900	·001691	·001490	·001300	·001123	·000960	·000811	·000679	·000562	2·6

k	r = ·60									k
	h = 0·0	h = 0·1	h = 0·2	h = 0·3	h = 0·4	h = 0·5	h = 0·6	h = 0·7	h = 0·8	
0·0	·352416	·331907	·310427	·288230	·265596	·242818	·220195	·198017	·176552	0·0
0·1	·331907	·313382	·293864	·273578	·252775	·231727	·210712	·190003	·169863	0·1
0·2	·310427	·293864	·276306	·257945	·239006	·219734	·200384	·181214	·162473	0·2
0·3	·288230	·273578	·257945	·241495	·224423	·206945	·189295	·171710	·154423	0·3
0·4	·265596	·252775	·239006	·224423	·209191	·193500	·177558	·161579	·145781	0·4
0·5	·242818	·231727	·219734	·206945	·193500	·179560	·165307	·150932	·136634	0·5
0·6	·220195	·210712	·200384	·189295	·177558	·165307	·152698	·139901	·127090	0·6
0·7	·198017	·190003	·181214	·171710	·161579	·150932	·139901	·128629	·117273	0·7
0·8	·176552	·169863	·162473	·154423	·145781	·136634	·127090	·117273	·107315	0·8
0·9	·156043	·150528	·144390	·137654	·130369	·122604	·114443	·105989	·097355	0·9
1·0	·136692	·132203	·127168	·121601	·115536	·109022	·102127	·094932	·087532	1·0
1·1	·118663	·115055	·110977	·106434	·101446	·096050	·090294	·084245	·077977	1·1
1·2	·102072	·099210	·095950	·092290	·088240	·083824	·079080	·074055	·068811	1·2
1·3	·086991	·084751	·082178	·079267	·076021	·072455	·068592	·064471	·060137	1·3
1·4	·073448	·071718	·069715	·067431	·064863	·062019	·058915	·055578	·052040	1·4
1·5	·061433	·060115	·058576	·056807	·054802	·052564	·050102	·047434	·044585	1·5
1·6	·050899	·049908	·048742	·047391	·045846	·044109	·042182	·040077	·037810	1·6
1·7	·041773	·041038	·040167	·039148	·037974	·036643	·035155	·033516	·031737	1·7
1·8	·033957	·033420	·032778	·032020	·031141	·030134	·029001	·027741	·026364	1·8
1·9	·027341	·026954	·026487	·025932	·025281	·024531	·023679	·022725	·021673	1·9
2·0	·021804	·021529	·021194	·020793	·020319	·019768	·019136	·018423	·017629	2·0
2·1	·017223	·017030	·016793	·016507	·016167	·015767	·015305	·014779	·014190	2·1
2·2	·013474	·013341	·013176	·012975	·012734	·012448	·012115	·011733	·011300	2·2
2·3	·010441	·010350	·010237	·010098	·009929	·009728	·009491	·009217	·008904	2·3
2·4	·008013	·007952	·007876	·007781	·007664	·007524	·007359	·007165	·006942	2·4
2·5	·006092	·006051	·006000	·005936	·005857	·005761	·005647	·005512	·005355	2·5
2·6	·004586	·004560	·004526	·004484	·004431	·004366	·004288	·004196	·004087	2·6

k	r = ·60									k
	h = 0·9	h = 1·0	h = 1·1	h = 1·2	h = 1·3	h = 1·4	h = 1·5	h = 1·6	h = 1·7	
0·0	·156043	·136692	·118663	·102072	·086991	·073448	·061433	·050899	·041773	0·0
0·1	·150528	·132203	·115055	·099210	·084751	·071718	·060115	·049908	·041038	0·1
0·2	·144390	·127168	·110977	·095950	·082178	·069715	·058576	·048742	·040167	0·2
0·3	·137654	·121601	·106434	·092290	·079267	·067431	·056807	·047391	·039148	0·3
0·4	·130369	·115536	·101446	·088240	·076021	·064863	·054802	·045846	·037974	0·4
0·5	·122604	·109022	·096050	·083824	·072455	·062019	·052564	·044109	·036643	0·5
0·6	·114443	·102127	·090294	·079080	·068592	·058915	·050102	·042182	·035155	0·6
0·7	·105989	·094932	·084245	·074055	·064471	·055578	·047434	·040077	·033516	0·7
0·8	·097355	·087532	·077977	·068811	·060137	·052040	·044585	·037810	·031737	0·8
0·9	·088661	·080028	·071576	·063415	·055645	·048346	·041585	·035406	·029835	0·9
1·0	·080028	·072526	·065131	·057944	·051055	·044544	·038474	·032894	·027832	1·0
1·1	·071576	·065131	·058734	·052474	·046434	·040687	·035295	·030305	·025752	1·1
1·2	·063415	·057944	·052474	·047083	·041846	·036830	·032091	·027678	·023624	1·2
1·3	·055645	·051055	·046434	·041846	·037358	·033028	·028910	·025049	·021479	1·3
1·4	·048346	·044544	·040687	·036830	·033028	·029335	·025797	·022457	·019348	1·4
1·5	·041585	·038474	·035295	·032091	·028910	·025797	·022794	·019938	·017261	1·5
1·6	·035406	·032894	·030305	·027678	·025049	·022457	·019938	·017526	·015247	1·6
1·7	·029835	·027832	·025752	·023624	·021479	·019348	·017261	·015247	·013332	1·7
1·8	·024880	·023303	·021653	·019952	·018224	·016494	·014787	·013127	·011536	1·8
1·9	·020529	·019305	·018014	·016672	·015298	·013911	·012532	·011181	·009876	1·9
2·0	·016761	·015823	·014826	·013781	·012702	·011605	·010506	·009420	·008363	2·0
2·1	·013538	·012830	·012070	·011267	·010432	·009576	·008711	·007849	·007005	2·1
2·2	·010819	·010291	·009720	·009112	·008474	·007814	·007142	·006468	·005802	2·2
2·3	·008554	·008165	·007742	·007287	·006806	·006305	·005791	·005270	·004751	2·3
2·4	·006690	·006408	·006099	·005764	·005406	·005031	·004642	·004245	·003846	2·4
2·5	·005176	·004975	·004752	·004508	·004246	·003969	·003679	·003381	·003079	2·5
2·6	·003962	·003821	·003662	·003487	·003298	·003096	·002883	·002662	·002436	2·6

k	r = ·60									*k*
	h = *1·8*	*h* = *1·9*	*h* = *2·0*	*h* = *2·1*	*h* = *2·2*	*h* = *2·3*	*h* = *2·4*	*h* = *2·5*	*h* = *2·6*	
0·0	·033957	·027341	·021804	·017223	·013474	·010441	·008013	·006092	·004586	*0·0*
0·1	·033420	·026954	·021529	·017030	·013341	·010350	·007952	·006051	·004560	*0·1*
0·2	·032778	·026487	·021194	·016793	·013176	·010237	·007876	·006000	·004526	*0·2*
0·3	·032020	·025932	·020793	·016507	·012975	·010098	·007781	·005936	·004484	*0·3*
0·4	·031141	·025281	·020319	·016167	·012734	·009929	·007664	·005857	·004431	*0·4*
0·5	·030134	·024531	·019768	·015767	·012448	·009728	·007524	·005761	·004366	*0·5*
0·6	·029001	·023679	·019136	·015305	·012115	·009491	·007359	·005647	·004288	*0·6*
0·7	·027741	·022725	·018423	·014779	·011733	·009217	·007165	·005512	·004196	*0·7*
0·8	·026364	·021673	·017629	·014190	·011300	·008904	·006942	·005355	·004087	*0·8*
0·9	·024880	·020529	·016761	·013538	·010819	·008554	·006690	·005176	·003962	*0·9*
1·0	·023303	·019305	·015823	·012830	·010291	·008165	·006408	·004975	·003821	*1·0*
1·1	·021653	·018014	·014826	·012070	·009720	·007742	·006099	·004752	·003662	*1·1*
1·2	·019952	·016672	·013781	·011267	·009112	·007287	·005764	·004508	·003487	*1·2*
1·3	·018224	·015298	·012702	·010432	·008474	·006806	·005406	·004246	·003298	*1·3*
1·4	·016494	·013911	·011605	·009576	·007814	·006305	·005031	·003969	·003096	*1·4*
1·5	·014787	·012532	·010506	·008711	·007142	·005791	·004642	·003679	·002883	*1·5*
1·6	·013127	·011181	·009420	·007849	·006468	·005270	·004245	·003381	·002662	*1·6*
1·7	·011536	·009876	·008363	·007005	·005802	·004751	·003846	·003079	·002436	*1·7*
1·8	·010033	·008634	·007350	·006188	·005153	·004242	·003452	·002778	·002209	*1·8*
1·9	·008634	·007469	·006392	·005411	·004529	·003749	·003068	·002481	·001984	*1·9*
2·0	·007350	·006392	·005500	·004681	·003940	·003279	·002698	·002194	·001764	*2·0*
2·1	·006188	·005411	·004681	·004006	·003390	·002837	·002348	·001920	·001552	*2·1*
2·2	·005153	·004529	·003940	·003390	·002886	·002429	·002021	·001662	·001351	*2·2*
2·3	·004242	·003749	·003279	·002837	·002429	·002056	·001720	·001423	·001164	*2·3*
2·4	·003452	·003068	·002698	·002348	·002021	·001720	·001448	·001205	·000991	*2·4*
2·5	·002778	·002481	·002194	·001920	·001662	·001423	·001205	·001009	·000835	*2·5*
2·6	·002209	·001984	·001764	·001552	·001351	·001164	·000991	·000835	·000695	*2·6*

k	r = ·65									*k*
	h = *0·0*	*h* = *0·1*	*h* = *0·2*	*h* = *0·3*	*h* = *0·4*	*h* = *0·5*	*h* = *0·6*	*h* = *0·7*	*h* = *0·8*	
0·0	·362615	·342023	·320297	·297702	·274538	·251123	·227783	·204836	·182581	*0·0*
0·1	·342023	·323519	·303856	·283267	·262017	·240399	·218716	·197271	·176354	*0·1*
0·2	·320297	·303856	·286256	·267694	·248401	·228640	·208690	·188833	·169349	*0·2*
0·3	·297702	·283267	·267694	·251145	·233818	·215944	·197774	·179568	·161588	*0·3*
0·4	·274538	·262017	·248401	·233818	·218433	·202444	·186072	·169552	·153125	*0·4*
0·5	·251123	·240399	·228640	·215944	·202444	·188305	·173717	·158889	·144039	*0·5*
0·6	·227783	·218716	·208690	·197774	·186072	·173717	·160870	·147712	·134436	*0·6*
0·7	·204836	·197271	·188833	·179568	·169552	·158889	·147712	·136173	·124440	*0·7*
0·8	·182581	·176354	·169349	·161588	·153125	·144039	·134436	·124440	·114194	*0·8*
0·9	·161285	·156232	·150493	·144080	·137024	·129383	·121237	·112687	·103850	*0·9*
1·0	·141177	·137133	·132498	·127270	·121468	·115127	·108308	·101089	·093564	*1·0*
1·1	·122437	·119247	·115556	·111355	·106648	·101459	·095827	·089813	·083489	*1·1*
1·2	·105196	·102716	·099819	·096490	·092726	·088538	·083951	·079008	·073764	*1·2*
1·3	·089536	·087635	·085395	·082795	·079828	·076495	·072812	·068805	·064516	*1·3*
1·4	·075487	·074053	·072345	·070345	·068040	·065426	·062510	·059308	·055849	*1·4*
1·5	·063040	·061974	·060692	·059176	·057411	·055391	·053115	·050594	·047843	*1·5*
1·6	·052146	·051365	·050417	·049284	·047954	·046415	·044666	·042708	·040552	*1·6*
1·7	·042723	·042161	·041470	·040637	·039649	·038495	·037169	·035672	·034007	*1·7*
1·8	·034671	·034271	·033776	·033173	·032450	·031597	·030608	·029479	·028213	*1·8*
1·9	·027868	·027589	·027239	·026809	·026288	·025668	·024941	·024103	·023154	*1·9*
2·0	·022187	·021995	·021752	·021450	·021080	·020636	·020110	·019497	·018797	*2·0*
2·1	·017497	·017366	·017200	·016991	·016733	·016420	·016045	·015604	·015094	*2·1*
2·2	·013667	·013580	·013468	·013326	·013149	·012931	·012668	·012355	·011991	*2·2*
2·3	·010574	·010517	·010443	·010348	·010228	·010079	·009897	·009679	·009422	*2·3*
2·4	·008104	·008067	·008019	·007956	·007876	·007776	·007653	·007503	·007324	*2·4*
2·5	·006152	·006129	·006098	·006057	·006005	·005939	·005856	·005755	·005633	*2·5*
2·6	·004626	·004612	·004592	·004566	·004533	·004489	·004435	·004368	·004286	*2·6*

k	$r=\cdot 65$									k
	$h=0\cdot 9$	$h=1\cdot 0$	$h=1\cdot 1$	$h=1\cdot 2$	$h=1\cdot 3$	$h=1\cdot 4$	$h=1\cdot 5$	$h=1\cdot 6$	$h=1\cdot 7$	
0·0	·161285	·141177	·122437	·105196	·089536	·075487	·063040	·052146	·042723	*0·0*
0·1	·156232	·137133	·119247	·102716	·087635	·074053	·061974	·051365	·042161	*0·1*
0·2	·150493	·132498	·115556	·099819	·085395	·072345	·060692	·050417	·041470	*0·2*
0·3	·144080	·127270	·111355	·096490	·082795	·070345	·059176	·049284	·040637	*0·3*
0·4	·137024	·121468	·106648	·092726	·079828	·068040	·057411	·047954	·039649	*0·4*
0·5	·129383	·115127	·101459	·088538	·076495	·065426	·055391	·046415	·038495	*0·5*
0·6	·121237	·108308	·095827	·083951	·072812	·062510	·053115	·044666	·037169	*0·6*
0·7	·112687	·101089	·089813	·079008	·068805	·059308	·050594	·042708	·035672	*0·7*
0·8	·103850	·093564	·083489	·073764	·064516	·055849	·047843	·040552	·034007	*0·8*
0·9	·094856	·085842	·076943	·068288	·059997	·052171	·044891	·038217	·032187	*0·9*
1·0	·085842	·078038	·070272	·062660	·055311	·048323	·041774	·035728	·030227	*1·0*
1·1	·076943	·070272	·063578	·056964	·050527	·044359	·038534	·033117	·028153	*1·1*
1·2	·068288	·062660	·056964	·051288	·045720	·040340	·035220	·030422	·025991	*1·2*
1·3	·059997	·055311	·050527	·045720	·040963	·036329	·031883	·027683	·023775	*1·3*
1·4	·052171	·048323	·044359	·040340	·036329	·032388	·028576	·024944	·021538	*1·4*
1·5	·044891	·041774	·038534	·035220	·031883	·028576	·025349	·022249	·019316	*1·5*
1·6	·038217	·035728	·033117	·030422	·027683	·024944	·022249	·019636	·017144	*1·6*
1·7	·032187	·030227	·028153	·025991	·023775	·021538	·019316	·017144	·015055	*1·7*
1·8	·026814	·025294	·023670	·021961	·020192	·018391	·016585	·014804	·013075	*1·8*
1·9	·022095	·020933	·019680	·018348	·016957	·015527	·014080	·012639	·011228	*1·9*
2·0	·018007	·017133	·016179	·015157	·014079	·012960	·011817	·010668	·009532	*2·0*
2·1	·014515	·013866	·013152	·012379	·011556	·010693	·009803	·008900	·007999	*2·1*
2·2	·011571	·011097	·010571	·009995	·009375	·008720	·008037	·007338	·006633	*2·2*
2·3	·009124	·008783	·008400	·007978	·007519	·007028	·006512	·005978	·005436	*2·3*
2·4	·007115	·006874	·006600	·006295	·005960	·005598	·005214	·004813	·004401	*2·4*
2·5	·005488	·005320	·005127	·004910	·004669	·004407	·004125	·003828	·003520	*2·5*
2·6	·004187	·004072	·003938	·003786	·003616	·003428	·003225	·003008	·002781	*2·6*

k	$r=\cdot 65$									k
	$h=1\cdot 8$	$h=1\cdot 9$	$h=2\cdot 0$	$h=2\cdot 1$	$h=2\cdot 2$	$h=2\cdot 3$	$h=2\cdot 4$	$h=2\cdot 5$	$h=2\cdot 6$	
0·0	·034671	·027868	·022187	·017497	·013667	·010574	·008104	·006152	·004626	*0·0*
0·1	·034271	·027589	·021995	·017366	·013580	·010517	·008067	·006129	·004612	*0·1*
0·2	·033776	·027239	·021752	·017200	·013468	·010443	·008019	·006098	·004592	*0·2*
0·3	·033173	·026809	·021450	·016991	·013326	·010348	·007956	·006057	·004566	*0·3*
0·4	·032450	·026288	·021080	·016733	·013149	·010228	·007876	·006005	·004533	*0·4*
0·5	·031597	·025668	·020636	·016420	·012931	·010079	·007776	·005939	·004489	*0·5*
0·6	·030608	·024941	·020110	·016045	·012668	·009897	·007653	·005856	·004435	*0·6*
0·7	·029479	·024103	·019497	·015604	·012355	·009679	·007503	·005755	·004368	*0·7*
0·8	·028213	·023154	·018797	·015094	·011991	·009422	·007324	·005633	·004286	*0·8*
0·9	·026814	·022095	·018007	·014515	·011571	·009124	·007115	·005488	·004187	*0·9*
1·0	·025294	·020933	·017133	·013866	·011097	·008783	·006874	·005320	·004072	*1·0*
1·1	·023670	·019680	·016179	·013152	·010571	·008400	·006600	·005127	·003938	*1·1*
1·2	·021961	·018348	·015157	·012379	·009995	·007978	·006295	·004910	·003786	*1·2*
1·3	·020192	·016957	·014079	·011556	·009375	·007519	·005960	·004669	·003616	*1·3*
1·4	·018391	·015527	·012960	·010693	·008720	·007028	·005598	·004407	·003428	*1·4*
1·5	·016585	·014080	·011817	·009803	·008037	·006512	·005214	·004125	·003225	*1·5*
1·6	·014804	·012639	·010668	·008900	·007338	·005978	·004813	·003828	·003008	*1·6*
1·7	·013075	·011228	·009532	·007999	·006633	·005436	·004401	·003520	·002781	*1·7*
1·8	·011422	·009867	·008427	·007114	·005935	·004892	·003984	·003205	·002547	*1·8*
1·9	·009867	·008575	·007368	·006258	·005253	·004357	·003570	·002889	·002310	*1·9*
2·0	·008427	·007368	·006370	·005444	·004598	·003838	·003164	·002577	·002073	*2·0*
2·1	·007114	·006258	·005444	·004682	·003980	·003343	·002774	·002274	·001841	*2·1*
2·2	·005935	·005253	·004598	·003980	·003405	·002878	·002404	·001983	·001616	*2·2*
2·3	·004892	·004357	·003838	·003343	·002878	·002449	·002059	·001710	·001403	*2·3*
2·4	·003984	·003570	·003164	·002774	·002404	·002059	·001743	·001457	·001204	*2·4*
2·5	·003205	·002889	·002577	·002274	·001983	·001710	·001457	·001227	·001020	*2·5*
2·6	·002547	·002310	·002073	·001841	·001616	·001403	·001204	·001020	·000854	*2·6*

k	r = ·70									k
	h = *0·0*	*h* = *0·1*	*h* = *0·2*	*h* = *0·3*	*h* = *0·4*	*h* = *0·5*	*h* = *0·6*	*h* = *0·7*	*h* = *0·8*	
0·0	·373407	·352717	·330699	·307637	·283853	·259697	·235531	·211712	·188570	*0·0*
0·1	·352717	·334247	·314423	·293485	·271717	·249439	·226988	·204702	·182909	*0·1*
0·2	·330699	·314423	·296795	·278012	·258319	·237999	·217360	·196720	·176391	*0·2*
0·3	·307637	·293485	·278012	·261373	·243773	·225456	·206696	·187786	·169017	*0·3*
0·4	·283853	·271717	·258319	·243773	·228243	·211935	·195086	·177957	·160820	*0·4*
0·5	·259697	·249439	·237999	·225456	·211935	·197602	·182658	·167331	·151864	*0·5*
0·6	·235531	·226988	·217360	·206696	·195086	·182658	·169576	·156035	·142246	*0·6*
0·7	·211712	·204702	·196720	·187786	·177957	·167331	·156035	·144229	·132095	*0·7*
0·8	·188570	·182909	·176391	·169017	·160820	·151864	·142246	·132095	·121559	*0·8*
0·9	·166407	·161907	·156668	·150675	·143940	·136503	·128434	·119829	·110808	*0·9*
1·0	·145478	·141958	·137812	·133018	·127570	·121489	·114819	·107630	·100017	*1·0*
1·1	·125983	·123274	·120047	·116272	·111934	·107038	·101609	·095695	·089364	*1·1*
1·2	·108067	·106016	·103546	·100621	·097223	·093343	·088993	·084201	·079016	*1·2*
1·3	·091817	·090291	·088430	·086203	·083583	·080558	·077127	·073306	·069125	*1·3*
1·4	·077268	·076151	·074774	·073104	·071118	·068798	·066137	·063138	·059820	*1·4*
1·5	·064404	·063601	·062599	·061369	·059889	·058139	·056108	·053793	·051201	*1·5*
1·6	·053172	·052604	·051887	·050997	·049912	·048614	·047090	·045332	·043341	*1·6*
1·7	·043481	·043088	·042583	·041950	·041169	·040222	·039098	·037785	·036281	*1·7*
1·8	·035221	·034952	·034604	·034161	·033608	·032930	·032114	·031150	·030033	*1·8*
1·9	·028260	·028080	·027843	·027539	·027154	·026677	·026095	·025400	·024584	*1·9*
2·0	·022461	·022343	·022185	·021980	·021717	·021386	·020979	·020486	·019901	*2·0*
2·1	·017685	·017608	·017505	·017369	·017192	·016968	·016687	·016344	·015931	*2·1*
2·2	·013793	·013745	·013679	·013590	·013474	·013324	·013134	·012899	·012613	*2·2*
2·3	·010658	·010628	·010586	·010529	·010454	·010356	·010230	·010072	·009877	*2·3*
2·4	·008159	·008140	·008114	·008079	·008031	·007967	·007885	·007781	·007650	*2·4*
2·5	·006187	·006176	·006160	·006138	·006108	·006068	·006016	·005948	·005862	*2·5*
2·6	·004648	·004642	·004632	·004619	·004601	·004576	·004543	·004500	·004444	*2·6*

k	r = ·70									k
	h = *0·9*	*h* = *1·0*	*h* = *1·1*	*h* = *1·2*	*h* = *1·3*	*h* = *1·4*	*h* = *1·5*	*h* = *1·6*	*h* = *1·7*	
0·0	·166407	·145478	·125983	·108067	·091817	·077268	·064404	·053172	·043481	*0·0*
0·1	·161907	·141958	·123274	·106016	·090291	·076151	·063601	·052604	·043088	*0·1*
0·2	·156668	·137812	·120047	·103546	·088430	·074774	·062599	·051887	·042583	*0·2*
0·3	·150675	·133018	·116272	·100621	·086203	·073104	·061369	·050997	·041950	*0·3*
0·4	·143940	·127570	·111934	·097223	·083583	·071118	·059889	·049912	·041169	*0·4*
0·5	·136503	·121489	·107038	·093343	·080558	·068798	·058139	·048614	·040222	*0·5*
0·6	·128434	·114819	·101609	·088993	·077127	·066137	·056108	·047090	·039098	*0·6*
0·7	·119829	·107630	·095695	·084201	·073306	·063138	·053793	·045332	·037785	*0·7*
0·8	·110808	·100017	·089364	·079016	·069125	·059820	·051201	·043341	·036281	*0·8*
0·9	·101511	·092091	·082703	·073502	·064630	·056212	·048352	·041127	·034589	*0·9*
1·0	·092091	·083979	·075818	·067742	·059884	·052361	·045276	·038709	·032719	*1·0*
1·1	·082703	·075818	·068819	·061827	·054958	·048320	·042013	·036115	·030690	*1·1*
1·2	·073502	·067742	·061827	·055857	·049934	·044156	·038612	·033382	·028528	*1·2*
1·3	·064630	·059884	·054958	·049934	·044899	·039937	·035131	·030553	·026265	*1·3*
1·4	·056212	·052361	·048320	·044156	·039937	·035738	·031629	·027676	·023937	*1·4*
1·5	·048352	·045276	·042013	·038612	·035131	·031629	·028166	·024801	·021587	*1·5*
1·6	·041127	·038709	·036115	·033382	·030553	·027676	·024801	·021978	·019253	*1·6*
1·7	·034589	·032719	·030690	·028528	·026265	·023937	·021587	·019253	·016977	*1·7*
1·8	·028761	·027339	·025778	·024095	·022313	·020460	·018568	·016668	·014796	*1·8*
1·9	·023644	·022581	·021399	·020111	·018731	·017279	·015779	·014258	·012741	*1·9*
2·0	·019217	·018435	·017556	·016586	·015535	·014416	·013247	·012047	·010838	*2·0*
2·1	·015443	·014878	·014235	·013517	·012729	·011881	·010984	·010054	·009105	*2·1*
2·2	·012271	·011869	·011407	·010884	·010304	·009672	·008996	·008285	·007553	*2·2*
2·3	·009641	·009361	·009034	·008660	·008240	·007777	·007275	·006742	·006186	*2·3*
2·4	·007491	·007298	·007072	·006809	·006510	·006176	·005811	·005418	·005002	*2·4*
2·5	·005756	·005626	·005472	·005290	·005081	·004845	·004583	·004298	·003993	*2·5*
2·6	·004375	·004289	·004185	·004062	·003919	·003755	·003570	·003367	·003147	*2·6*

k	r = ·70									k
	h = 1·8	h = 1·9	h = 2·0	h = 2·1	h = 2·2	h = 2·3	h = 2·4	h = 2·5	h = 2·6	
0·0	·035221	·028260	·022461	·017685	·013793	·010658	·008159	·006187	·004648	0·0
0·1	·034952	·028080	·022343	·017608	·013745	·010628	·008140	·006176	·004642	0·1
0·2	·034604	·027843	·022185	·017505	·013679	·010586	·008114	·006160	·004632	0·2
0·3	·034161	·027539	·021980	·017369	·013590	·010529	·008079	·006138	·004619	0·3
0·4	·033608	·027154	·021717	·017192	·013474	·010454	·008031	·006108	·004601	0·4
0·5	·032930	·026677	·021386	·016968	·013324	·010356	·007967	·006068	·004576	0·5
0·6	·032114	·026095	·020979	·016687	·013134	·010230	·007885	·006016	·004543	0·6
0·7	·031150	·025400	·020486	·016344	·012899	·010072	·007781	·005948	·004500	0·7
0·8	·030033	·024584	·019901	·015931	·012613	·009877	·007650	·005862	·004444	0·8
0·9	·028761	·023644	·019217	·015443	·012271	·009641	·007491	·005756	·004375	0·9
1·0	·027339	·022581	·018435	·014878	·011869	·009361	·007298	·005626	·004289	1·0
1·1	·025778	·021399	·017556	·014235	·011407	·009034	·007072	·005472	·004185	1·1
1·2	·024095	·020111	·016586	·013517	·010884	·008660	·006809	·005290	·004062	1·2
1·3	·022313	·018731	·015535	·012729	·010304	·008240	·006510	·005081	·003919	1·3
1·4	·020460	·017279	·014416	·011881	·009672	·007777	·006176	·004845	·003755	1·4
1·5	·018568	·015779	·013247	·010984	·008996	·007275	·005811	·004583	·003570	1·5
1·6	·016668	·014258	·012047	·010054	·008285	·006742	·005418	·004298	·003367	1·6
1·7	·014796	·012741	·010838	·009105	·007553	·006186	·005002	·003993	·003147	1·7
1·8	·012982	·011256	·009640	·008154	·006811	·005616	·004572	·003674	·002914	1·8
1·9	·011256	·009826	·008475	·007219	·006072	·005042	·004133	·003344	·002670	1·9
2·0	·009640	·008475	·007362	·006316	·005351	·004475	·003695	·003010	·002421	2·0
2·1	·008154	·007219	·006316	·005458	·004658	·003925	·003264	·002679	·002170	2·1
2·2	·006811	·006072	·005351	·004658	·004005	·003400	·002849	·002356	·001923	2·2
2·3	·005616	·005042	·004475	·003925	·003400	·002908	·002456	·002046	·001683	2·3
2·4	·004572	·004133	·003695	·003264	·002849	·002456	·002090	·001755	·001455	2·4
2·5	·003674	·003344	·003010	·002679	·002356	·002046	·001755	·001486	·001241	2·5
2·6	·002914	·002670	·002421	·002170	·001923	·001683	·001455	·001241	·001045	2·6

k	r = ·75									k
	h = 0·0	h = 0·1	h = 0·2	h = 0·3	h = 0·4	h = 0·5	h = 0·6	h = 0·7	h = 0·8	
0·0	·384973	·364160	·341785	·318152	·293618	·268576	·243436	·218600	·194450	0·0
0·1	·364160	·345745	·325735	·304380	·281991	·258925	·235563	·212292	·189486	0·1
0·2	·341785	·325735	·308095	·289064	·268902	·247922	·226469	·204906	·183595	0·2
0·3	·318152	·304380	·289064	·272350	·254447	·235619	·216170	·196431	·176743	0·3
0·4	·293618	·281991	·268902	·254447	·238784	·222126	·204732	·186897	·168931	0·4
0·5	·268576	·258925	·247922	·235619	·222126	·207607	·192274	·176381	·160203	0·5
0·6	·243436	·235563	·226469	·216170	·204732	·192274	·178963	·165005	·150640	0·6
0·7	·218600	·212292	·204906	·196431	·186897	·176381	·165005	·152934	·140366	0·7
0·8	·194450	·189486	·183595	·176743	·168931	·160203	·150640	·140366	·129540	0·8
0·9	·171321	·167488	·162875	·157434	·151146	·144026	·136123	·127523	·118347	0·9
1·0	·149499	·146595	·143049	·138808	·133838	·128133	·121715	·114639	·106989	1·0
1·1	·129205	·127047	·124373	·121129	·117273	·112784	·107665	·101944	·095677	1·1
1·2	·110594	·109022	·107044	·104610	·101674	·098208	·094200	·089657	·084613	1·2
1·3	·093759	·092635	·091201	·089410	·087218	·084592	·081511	·077970	·073983	1·3
1·4	·078729	·077942	·076922	·075630	·074024	·072073	·069750	·067041	·063949	1·4
1·5	·065481	·064941	·064231	·063316	·062164	·060742	·059024	·056992	·054637	1·5
1·6	·053949	·053586	·053101	·052467	·051656	·050640	·049395	·047899	·046141	1·6
1·7	·044031	·043792	·043467	·043037	·042478	·041766	·040881	·039802	·038514	1·7
1·8	·035601	·035447	·035234	·034948	·034570	·034082	·033465	·032702	·031778	1·8
1·9	·028517	·028420	·028284	·028097	·027847	·027519	·027098	·026569	·025919	1·9
2·0	·022632	·022572	·022486	·022367	·022205	·021989	·021708	·021348	·020900	2·0
2·1	·017796	·017760	·017707	·017633	·017530	·017390	·017206	·016967	·016664	2·1
2·2	·013865	·013843	·013811	·013766	·013702	·013614	·013495	·013340	·013139	2·2
2·3	·010702	·010690	·010671	·010644	·010605	·010551	·010476	·010377	·010247	2·3
2·4	·008186	·008179	·008168	·008152	·008129	·008096	·008050	·007988	·007905	2·4
2·5	·006203	·006199	·006193	·006184	·006171	·006151	·006124	·006086	·006034	2·5
2·6	·004658	·004656	·004652	·004647	·004639	·004628	·004612	·004589	·004558	2·6

k	$r=\cdot75$									k
	$h=0\cdot9$	$h=1\cdot0$	$h=1\cdot1$	$h=1\cdot2$	$h=1\cdot3$	$h=1\cdot4$	$h=1\cdot5$	$h=1\cdot6$	$h=1\cdot7$	
0·0	·171321	·149499	·129205	·110594	·093759	·078729	·065481	·053949	·044031	0·0
0·1	·167488	·146595	·127047	·109022	·092635	·077942	·064941	·053586	·043792	0·1
0·2	·162875	·143049	·124373	·107044	·091201	·076922	·064231	·053101	·043467	0·2
0·3	·157434	·138808	·121129	·104610	·089410	·075630	·063316	·052467	·043037	0·3
0·4	·151146	·133838	·117273	·101674	·087218	·074024	·062164	·051656	·042478	0·4
0·5	·144026	·128133	·112784	·098208	·084592	·072073	·060742	·050640	·041766	0·5
0·6	·136123	·121715	·107665	·094200	·081511	·069750	·059024	·049395	·040881	0·6
0·7	·127523	·114639	·101944	·089657	·077970	·067041	·056992	·047899	·039802	0·7
0·8	·118347	·106989	·095677	·084613	·073983	·063949	·054637	·046141	·038514	0·8
0·9	·108743	·098883	·088949	·079125	·069586	·060489	·051966	·044117	·037010	0·9
1·0	·098883	·090457	·081866	·073272	·064833	·056698	·048997	·041836	·035290	1·0
1·1	·088949	·081866	·074555	·067151	·059797	·052627	·045765	·039316	·033363	1·1
1·2	·079125	·073272	·067151	·060877	·054567	·048342	·042317	·036591	·031249	1·2
1·3	·069586	·064833	·059797	·054567	·049241	·043922	·038711	·033702	·028977	1·3
1·4	·060489	·056698	·052627	·048342	·043922	·039451	·035017	·030702	·026584	1·4
1·5	·051966	·048997	·045765	·042317	·038711	·035017	·031305	·027648	·024115	1·5
1·6	·044117	·041836	·039316	·036591	·033702	·030702	·027648	·024601	·021618	1·6
1·7	·037010	·035290	·033363	·031249	·028977	·026584	·024115	·021618	·019143	1·7
1·8	·030682	·029410	·027964	·026354	·024600	·022726	·020766	·018757	·016738	1·8
1·9	·025136	·024213	·023150	·021948	·020619	·019179	·017651	·016063	·014446	1·9
2·0	·020352	·019696	·018929	·018049	·017061	·015975	·014807	·013575	·012303	2·0
2·1	·016288	·015831	·015288	·014657	·013937	·013134	·012257	·011320	·010338	2·1
2·2	·012886	·012575	·012198	·011754	·011240	·010658	·010013	·009313	·008569	2·2
2·3	·010081	·009872	·009617	·009310	·008950	·008537	·008071	·007559	·007006	2·3
2·4	·007798	·007662	·007492	·007285	·007038	·006749	·006420	·006052	·005650	2·4
2·5	·005967	·005879	·005768	·005631	·005465	·005269	·005040	·004782	·004494	2·5
2·6	·004516	·004461	·004390	·004302	·004192	·004061	·003906	·003727	·003526	2·6

k	$r=\cdot75$									k
	$h=1\cdot8$	$h=1\cdot9$	$h=2\cdot0$	$h=2\cdot1$	$h=2\cdot2$	$h=2\cdot3$	$h=2\cdot4$	$h=2\cdot5$	$h=2\cdot6$	
0·0	·035601	·028517	·022632	·017796	·013865	·010702	·008186	·006203	·004658	0·0
0·1	·035447	·028420	·022572	·017760	·013843	·010690	·008179	·006199	·004656	0·1
0·2	·035234	·028284	·022486	·017707	·013811	·010671	·008168	·006193	·004652	0·2
0·3	·034948	·028097	·022367	·017633	·013766	·010644	·008152	·006184	·004647	0·3
0·4	·034570	·027847	·022205	·017530	·013702	·010605	·008129	·006171	·004639	0·4
0·5	·034082	·027519	·021989	·017390	·013614	·010551	·008096	·006151	·004628	0·5
0·6	·033465	·027098	·021708	·017206	·013495	·010476	·008050	·006124	·004612	0·6
0·7	·032702	·026569	·021348	·016967	·013340	·010377	·007988	·006086	·004589	0·7
0·8	·031778	·025919	·020900	·016664	·013139	·010247	·007905	·006034	·004558	0·8
0·9	·030682	·025136	·020352	·016288	·012886	·010081	·007798	·005967	·004516	0·9
1·0	·029410	·024213	·019696	·015831	·012575	·009872	·007662	·005879	·004461	1·0
1·1	·027964	·023150	·018929	·015288	·012198	·009617	·007492	·005768	·004390	1·1
1·2	·026354	·021948	·018049	·014657	·011754	·009310	·007285	·005631	·004302	1·2
1·3	·024600	·020619	·017061	·013937	·011240	·008950	·007038	·005465	·004192	1·3
1·4	·022726	·019179	·015975	·013134	·010658	·008537	·006749	·005269	·004061	1·4
1·5	·020766	·017651	·014807	·012257	·010013	·008071	·006420	·005040	·003906	1·5
1·6	·018757	·016063	·013575	·011320	·009313	·007559	·006052	·004782	·003727	1·6
1·7	·016738	·014446	·012303	·010338	·008569	·007006	·005650	·004494	·003526	1·7
1·8	·014751	·012832	·011016	·009330	·007794	·006422	·005218	·004181	·003304	1·8
1·9	·012832	·011253	·009740	·008317	·007005	·005818	·004765	·003847	·003063	1·9
2·0	·011016	·009740	·008500	·007319	·006216	·005206	·004299	·003500	·002809	2·0
2·1	·009330	·008317	·007319	·006356	·005445	·004599	·003831	·003145	·002545	2·1
2·2	·007794	·007005	·006216	·005445	·004705	·004009	·003369	·002790	·002277	2·2
2·3	·006422	·005818	·005206	·004599	·004009	·003447	·002922	·002442	·002011	2·3
2·4	·005218	·004765	·004299	·003831	·003369	·002922	·002500	·002108	·001752	2·4
2·5	·004181	·003847	·003500	·003145	·002790	·002442	·002108	·001795	·001506	2·5
2·6	·003304	·003063	·002809	·002545	·002277	·002011	·001752	·001506	·001275	2·6

k	r = ·80									k
	h = 0·0	h = 0·1	h = 0·2	h = 0·3	h = 0·4	h = 0·5	h = 0·6	h = 0·7	h = 0·8	
0·0	·397583	·376613	·353778	·329412	·303929	·277783	·251453	·225411	·200091	0·0
0·1	·376613	·358285	·338048	·316170	·293001	·268950	·244465	·220001	·195996	0·1
0·2	·353778	·338048	·320423	·301103	·280367	·258565	·236106	·213416	·190922	0·2
0·3	·329412	·316170	·301103	·284337	·266086	·246639	·226344	·205592	·184786	0·3
0·4	·303929	·293001	·280367	·266086	·250308	·233252	·215207	·196514	·177541	0·4
0·5	·277783	·268950	·258565	·246639	·233252	·218560	·202791	·186227	·169191	0·5
0·6	·251453	·244465	·236106	·226344	·215207	·202791	·189259	·174834	·159790	0·6
0·7	·225411	·220001	·213416	·205592	·196514	·186227	·174834	·162503	·149450	0·7
0·8	·200091	·195996	·190922	·184786	·177541	·169191	·159790	·149450	·138334	0·8
0·9	·175878	·172850	·169028	·164320	·158664	·152028	·144430	·135932	·126647	0·9
1·0	·153091	·150903	·148089	·144559	·140239	·135080	·129069	·122228	·114631	1·0
1·1	·131965	·130422	·128399	·125812	·122588	·118666	·114014	·108626	·102536	1·1
1·2	·112660	·111601	·110181	·108330	·105979	·103065	·099544	·095393	·090617	1·2
1·3	·095267	·094559	·093585	·092292	·090617	·088503	·085899	·082772	·079108	1·3
1·4	·079809	·079341	·078690	·077808	·076644	·075145	·073264	·070962	·068215	1·4
1·5	·066233	·065934	·065509	·064922	·064132	·063095	·061768	·060113	·058101	1·5
1·6	·054460	·054273	·054003	·053622	·053099	·052399	·051485	·050323	·048883	1·6
1·7	·044369	·044256	·044088	·043847	·043509	·043048	·042434	·041638	·040632	1·7
1·8	·035820	·035753	·035651	·035503	·035290	·034993	·034591	·034058	·033373	1·8
1·9	·028656	·028617	·028557	·028468	·028337	·028151	·027894	·027547	·027090	1·9
2·0	·022718	·022696	·022661	·022609	·022531	·022417	·022257	·022036	·021740	2·0
2·1	·017847	·017835	·017816	·017786	·017741	·017673	·017575	·017438	·017251	2·1
2·2	·013895	·013888	·013878	·013861	·013835	·013796	·013738	·013655	·013540	2·2
2·3	·010720	·010716	·010711	·010702	·010687	·010665	·010632	·010583	·010513	2·3
2·4	·008195	·008194	·008191	·008186	·008178	·008166	·008147	·008119	·008078	2·4
2·5	·006209	·006208	·006206	·006204	·006200	·006193	·006183	·006167	·006144	2·5
2·6	·004661	·004660	·004660	·004658	·004656	·004653	·004647	·004639	·004626	2·6

k	r = ·80									k
	h = 0·9	h = 1·0	h = 1·1	h = 1·2	h = 1·3	h = 1·4	h = 1·5	h = 1·6	h = 1·7	
0·0	·175878	·153091	·131965	·112660	·095267	·079809	·066233	·054460	·044369	0·0
0·1	·172850	·150903	·130422	·111601	·094559	·079341	·065934	·054273	·044256	0·1
0·2	·169028	·148089	·128399	·110181	·093585	·078690	·065509	·054003	·044088	0·2
0·3	·164320	·144559	·125812	·108330	·092292	·077808	·064922	·053622	·043847	0·3
0·4	·158664	·140239	·122588	·105979	·090617	·076644	·064132	·053099	·043509	0·4
0·5	·152028	·135080	·118666	·103065	·088503	·075145	·063095	·052399	·043048	0·5
0·6	·144430	·129069	·114014	·099544	·085899	·073264	·061768	·051485	·042434	0·6
0·7	·135932	·122228	·108626	·095393	·082772	·070962	·060113	·050323	·041638	0·7
0·8	·126647	·114631	·102536	·090617	·079108	·068215	·058101	·048883	·040632	0·8
0·9	·116733	·106383	·095813	·085253	·074919	·065017	·055714	·047143	·039393	0·9
1·0	·106383	·097637	·088568	·079372	·070247	·061385	·052953	·045091	·037904	1·0
1·1	·095813	·088568	·080933	·073072	·065156	·057357	·049835	·042733	·036160	1·1
1·2	·085253	·079372	·073072	·066480	·059739	·052997	046402	·040087	·034167	1·2
1·3	·074919	·070247	·065156	·059739	·054109	·048390	·042709	·037191	·031946	1·3
1·4	·065017	·061385	·057357	·052997	·048390	·043632	·038832	·034098	·029531	1·4
1·5	·055714	·052953	·049835	·046402	·042709	·038832	·034856	·030870	·026966	1·5
1·6	·047143	·045091	·042733	·040087	·037191	·034098	·030870	·027582	·024309	1·6
1·7	·039393	·037904	·036160	·034167	·031946	·029531	·026966	·024309	·021620	1·7
1·8	·032511	·031456	·030195	·028729	·027064	·025220	·023227	·021125	·018960	1·8
1·9	·026506	·025776	·024887	·023833	·022613	·021237	·019721	·018095	·016390	1·9
2·0	·021353	·020860	·020248	·019508	·018635	·017631	·016506	·015275	·013962	2·0
2·1	·017001	·016676	·016265	·015758	·015148	·014434	·013617	·012707	·011718	2·1
2·2	·013382	·013174	·012904	·012565	·012149	·011653	·011074	·010416	·009689	2·2
2·3	·010417	·010286	·010114	·009893	·009616	·009279	·008879	·008415	·007892	2·3
2·4	·008021	·007941	·007833	·007693	·007513	·007290	·007020	·006701	·006333	2·4
2·5	·006110	·006063	·005997	·005910	·005797	·005652	·005474	·005260	·005008	2·5
2·6	·004607	·004579	·004540	·004488	·004417	·004327	·004212	·004072	·003904	2·6

k	$r=\cdot80$									k
	$h=1\cdot8$	$h=1\cdot9$	$h=2\cdot0$	$h=2\cdot1$	$h=2\cdot2$	$h=2\cdot3$	$h=2\cdot4$	$h=2\cdot5$	$h=2\cdot6$	
0·0	·035820	·028656	·022718	·017847	·013895	·010720	·008195	·006209	·004661	*0·0*
0·1	·035753	·028617	·022696	·017835	·013888	·010716	·008194	·006208	·004660	*0·1*
0·2	·035651	·028557	·022661	·017816	·013878	·010711	·008191	·006206	·004660	*0·2*
0·3	·035503	·028468	·022609	·017786	·013861	·010702	·008186	·006204	·004658	*0·3*
0·4	·035290	·028337	·022531	·017741	·013835	·010687	·008178	·006200	·004656	*0·4*
0·5	·034993	·028151	·022417	·017673	·013796	·010665	·008166	·006193	·004653	*0·5*
0·6	·034591	·027894	·022257	·017575	·013738	·010632	·008147	·006183	·004647	*0·6*
0·7	·034058	·027547	·022036	·017438	·013655	·010583	·008119	·006167	·004639	*0·7*
0·8	·033373	·027090	·021740	·017251	·013540	·010513	·008078	·006144	·004626	*0·8*
0·9	·032511	·026506	·021353	·017001	·013382	·010417	·008021	·006110	·004607	*0·9*
1·0	·031456	·025776	·020860	·016676	·013174	·010286	·007941	·006063	·004579	*1·0*
1·1	·030195	·024887	·020248	·016265	·012904	·010114	·007833	·005997	·004540	*1·1*
1·2	·028729	·023833	·019508	·015758	·012565	·009893	·007693	·005910	·004488	*1·2*
1·3	·027064	·022613	·018635	·015148	·012149	·009616	·007513	·005797	·004417	*1·3*
1·4	·025220	·021237	·017631	·014434	·011653	·009279	·007290	·005652	·004327	*1·4*
1·5	·023227	·019721	·016506	·013617	·011074	·008879	·007020	·005474	·004212	*1·5*
1·6	·021125	·018095	·015275	·012707	·010416	·008415	·006701	·005260	·004072	*1·6*
1·7	·018960	·016390	·013962	·011718	·009689	·007892	·006333	·005008	·003904	*1·7*
1·8	·016784	·014646	·012594	·010669	·008902	·007316	·005921	·004720	·003707	*1·8*
1·9	·014646	·012904	·011204	·009584	·008074	·006698	·005470	·004400	·003484	*1·9*
2·0	·012594	·011204	·009825	·008488	·007222	·006050	·004990	·004051	·003237	*2·0*
2·1	·010669	·009584	·008488	·007408	·006367	·005389	·004489	·003681	·002970	*2·1*
2·2	·008902	·008074	·007222	·006367	·005530	·004730	·003981	·003298	·002688	*2·2*
2·3	·007316	·006698	·006050	·005389	·004730	·004088	·003478	·002912	·002399	*2·3*
2·4	·005921	·005470	·004990	·004489	·003981	·003478	·002992	·002533	·002109	*2·4*
2·5	·004720	·004400	·004051	·003681	·003298	·002912	·002533	·002168	·001826	*2·5*
2·6	·003707	·003484	·003237	·002970	·002688	·002399	·002109	·001826	·001555	*2·6*

k	$r=\cdot85$									k
	$h=0\cdot0$	$h=0\cdot1$	$h=0\cdot2$	$h=0\cdot3$	$h=0\cdot4$	$h=0\cdot5$	$h=0\cdot6$	$h=0\cdot7$	$h=0\cdot8$	
0·0	·411699	·390507	·367028	·341657	·314893	·287297	·259454	·231930	·205241	*0·0*
0·1	·390507	·372323	·351791	·329205	·304980	·279618	·253671	·227700	·202237	*0·1*
0·2	·367028	·351791	·334234	·314547	·293048	·270158	·246373	·222227	·198251	*0·2*
0·3	·341657	·329205	·314547	·297774	·279096	·258844	·237439	·215365	·193127	*0·3*
0·4	·314893	·304980	·293048	·279096	·263239	·245704	·226826	·207018	·186743	*0·4*
0·5	·287297	·279618	·270158	·258844	·245704	·230869	·214580	·197166	·179027	*0·5*
0·6	·259454	·253671	·246373	·237439	·226826	·214580	·200848	·185873	·169978	*0·6*
0·7	·231930	·227700	·222227	·215365	·207018	·197166	·185873	·173295	·159674	*0·7*
0·8	·205241	·202237	·198251	·193127	·186743	·179027	·169978	·159674	·148275	*0·8*
0·9	·179823	·177753	·174935	·171220	·166474	·160597	·153539	·145316	·136013	*0·9*
1·0	·156018	·154636	·152704	·150090	·146664	·142314	·136961	·130575	·123180	*1·0*
1·1	·134073	·133179	·131894	·130110	·127710	·124585	·120640	·115817	·110097	*1·1*
1·2	·114136	·113575	·112748	·111568	·109938	·107758	·104936	·101397	·097097	*1·2*
1·3	·096270	·095929	·095413	·094657	·093583	·092109	·090151	·087631	·084490	*1·3*
1·4	·080464	·080264	·079952	·079483	·078798	·077832	·076514	·074772	·072546	*1·4*
1·5	·066651	·066537	·066355	·066073	·065649	·065036	·064176	·063010	·061478	*1·5*
1·6	·054719	·054656	·054553	·054388	·054135	·053758	·053215	·052458	·051437	*1·6*
1·7	·044525	·044492	·044435	·044343	·044196	·043972	·043639	·043163	·042503	*1·7*
1·8	·035911	·035894	·035864	·035813	·035731	·035602	·035405	·035115	·034702	*1·8*
1·9	·028707	·028699	·028684	·028657	·028612	·028540	·028428	·028257	·028006	*1·9*
2·0	·022746	·022742	·022734	·022720	·022697	·022659	·022596	·022499	·022352	*2·0*
2·1	·017863	·017861	·017857	·017850	·017839	·017818	·017785	·017731	·017648	*2·1*
2·2	·013903	·013902	·013900	·013897	·013891	·013881	·013864	·013835	·013789	*2·2*
2·3	·010724	·010723	·010722	·010721	·010718	·010713	·010705	·010690	·010666	*2·3*
2·4	·008197	·008197	·008196	·008196	·008195	·008193	·008188	·008181	·008169	*2·4*
2·5	·006210	·006210	·006209	·006209	·006209	·006208	·006206	·006202	·006196	*2·5*
2·6	·004661	·004661	·004661	·004661	·004661	·004660	·004659	·004658	·004655	*2·6*

k	r = ·85									k
	h = 0·9	h = 1·0	h = 1·1	h = 1·2	h = 1·3	h = 1·4	h = 1·5	h = 1·6	h = 1·7	
0·0	·179823	·156018	·134073	·114136	·096270	·080464	·066651	·054719	·044525	0·0
0·1	·177753	·154636	·133179	·113575	·095929	·080264	·066537	·054656	·044492	0·1
0·2	·174935	·152704	·131894	·112748	·095413	·079952	·066355	·054553	·044435	0·2
0·3	·171220	·150090	·130110	·111568	·094657	·079483	·066073	·054388	·044343	0·3
0·4	·166474	·146664	·127710	·109938	·093583	·078798	·065649	·054135	·044196	0·4
0·5	·160597	·142314	·124585	·107758	·092109	·077832	·065036	·053758	·043972	0·5
0·6	·153539	·136961	·120640	·104936	·090151	·076514	·064176	·053215	·043639	0·6
0·7	·145316	·130575	·115817	·101397	·087631	·074772	·063010	·052458	·043163	0·7
0·8	·136013	·123180	·110097	·097097	·084490	·072546	·061478	·051437	·042503	0·8
0·9	·125791	·114870	·103518	·092030	·080696	·069787	·059532	·050104	·041619	0·9
1·0	·114870	·105798	·096173	·086238	·076254	·066476	·057135	·048420	·040472	1·0
1·1	·103518	·096173	·088208	·079814	·071209	·062623	·054275	·046359	·039031	1·1
1·2	·092030	·086238	·079814	·072894	·065650	·058276	·050969	·043917	·037278	1·2
1·3	·080696	·076254	·071209	·065650	·059701	·053516	·047262	·041110	·035213	1·3
1·4	·069787	·066476	·062623	·058276	·053516	·048456	·043230	·037982	·032855	1·4
1·5	·059532	·057135	·054275	·050969	·047262	·043230	·038971	·034602	·030244	1·5
1·6	·050104	·048420	·046359	·043917	·041110	·037982	·034602	·031055	·027441	1·6
1·7	·041619	·040472	·039031	·037278	·035213	·032855	·030244	·027441	·024518	1·7
1·8	·034133	·033376	·032398	·031178	·029703	·027976	·026016	·023860	·021559	1·8
1·9	·027652	·027167	·026525	·025701	·024679	·023451	·022022	·020411	·018648	1·9
2·0	·022138	·021838	·021428	·020890	·020203	·019356	·018345	·017174	·015863	2·0
2·1	·017523	·017343	·017090	·016749	·016302	·015735	·015041	·014216	·013269	2·1
2·2	·013719	·013614	·013463	·013254	·012972	·012605	·012142	·011579	·010914	2·2
2·3	·010627	·010568	·010481	·010357	·010184	·009954	·009656	·009283	·008831	2·3
2·4	·008148	·008116	·008068	·007996	·007894	·007754	·007568	·007328	·007030	2·4
2·5	·006185	·006169	·006142	·006102	·006044	·005962	·005849	·005700	·005510	2·5
2·6	·004650	·004641	·004627	·004606	·004574	·004527	·004461	·004371	·004253	2·6

k	r = ·85									k
	h = 1·8	h = 1·9	h = 2·0	h = 2·1	h = 2·2	h = 2·3	h = 2·4	h = 2·5	h = 2·6	
0·0	·035911	·028707	·022746	·017863	·013903	·010724	·008197	·006210	·004661	0·0
0·1	·035894	·028699	·022742	·017861	·013902	·010723	·008197	·006210	·004661	0·1
0·2	·035864	·028684	·022734	·017857	·013900	·010722	·008196	·006209	·004661	0·2
0·3	·035813	·028657	·022720	·017850	·013897	·010721	·008196	·006209	·004661	0·3
0·4	·035731	·028612	·022697	·017839	·013891	·010718	·008195	·006209	·004661	0·4
0·5	·035602	·028540	·022659	·017818	·013881	·010713	·008193	·006208	·004660	0·5
0·6	·035405	·028428	·022596	·017785	·013864	·010705	·008188	·006206	·004659	0·6
0·7	·035115	·028257	·022499	·017731	·013835	·010690	·008181	·006202	·004658	0·7
0·8	·034702	·028006	·022352	·017648	·013789	·010666	·008169	·006196	·004655	0·8
0·9	·034133	·027652	·022138	·017523	·013719	·010627	·008148	·006185	·004650	0·9
1·0	·033376	·027167	·021838	·017343	·013614	·010568	·008116	·006169	·004641	1·0
1·1	·032398	·026525	·021428	·017090	·013463	·010481	·008068	·006142	·004627	1·1
1·2	·031178	·025701	·020890	·016749	·013254	·010357	·007996	·006102	·004606	1·2
1·3	·029703	·024679	·020203	·016302	·012972	·010184	·007894	·006044	·004574	1·3
1·4	·027976	·023451	·019356	·015735	·012605	·009954	·007754	·005962	·004527	1·4
1·5	·026016	·022022	·018345	·015041	·012142	·009656	·007568	·005849	·004461	1·5
1·6	·023860	·020411	·017174	·014216	·011579	·009283	·007328	·005700	·004371	1·6
1·7	·021559	·018648	·015863	·013269	·010914	·008831	·007030	·005510	·004253	1·7
1·8	·019177	·016780	·014438	·012213	·010155	·008300	·006671	·005274	·004103	1·8
1·9	·016780	·014858	·012937	·011073	·009314	·007698	·006253	·004992	·003919	1·9
2·0	·014438	·012937	·011403	·009880	·008412	·007035	·005780	·004665	·003700	2·0
2·1	·012213	·011073	·009880	·008668	·007473	·006329	·005264	·004299	·003448	2·1
2·2	·010155	·009314	·008412	·007473	·006526	·005599	·004716	·003900	·003166	2·2
2·3	·008300	·007698	·007035	·006329	·005599	·004867	·004154	·003480	·002862	2·3
2·4	·006671	·006253	·005780	·005264	·004716	·004154	·003594	·003052	·002544	2·4
2·5	·005274	·004992	·004665	·004299	·003900	·003480	·003052	·002628	·002221	2·5
2·6	·004103	·003919	·003700	·003448	·003166	·002862	·002544	·002221	·001903	2·6

k	$r=\cdot 90$									k
	$h=0\cdot 0$	$h=0\cdot 1$	$h=0\cdot 2$	$h=0\cdot 3$	$h=0\cdot 4$	$h=0\cdot 5$	$h=0\cdot 6$	$h=0\cdot 7$	$h=0\cdot 8$	
0·0	·428214	·406669	·382169	·355243	·326566	·296908	·267038	·237662	·209396	*0·0*
0·1	·406669	·388753	·367782	·344108	·318280	·290985	·262978	·235002	·207729	*0·1*
0·2	·382169	·367782	·350403	·330201	·307559	·283038	·257324	·231146	·205211	*0·2*
0·3	·355243	·344108	·330201	·313517	·294257	·272823	·249782	·225802	·201583	*0·3*
0·4	·326566	·318280	·307559	·294257	·278406	·260232	·240150	·218720	·196585	*0·4*
0·5	·296908	·290985	·283038	·272823	·260232	·245325	·228353	·209730	·190001	*0·5*
0·6	·267038	·262978	·257324	·249782	·240150	·228353	·214481	·198793	·181701	*0·6*
0·7	·237662	·235002	·231146	·225802	·218720	·209730	·198793	·186016	·171665	*0·7*
0·8	·209396	·207729	·205211	·201583	·196585	·190001	·181701	·171665	·160017	*0·8*
0·9	·182714	·181715	·180143	·177784	·174405	·169780	·163726	·156138	·147024	*0·9*
1·0	·157949	·157378	·156440	·154974	·152788	·149675	·145439	·139928	·133064	*1·0*
1·1	·135313	·135001	·134466	·133596	·132243	·130238	·127399	·123560	·118595	*1·1*
1·2	·114899	·114738	·114448	·113954	·113155	·111920	·110099	·107537	·104093	*1·2*
1·3	·096720	·096642	·096492	·096225	·095774	·095047	·093931	·092296	·090008	*1·3*
1·4	·080721	·080685	·080611	·080473	·080230	·079822	·079169	·078170	·076716	*1·4*
1·5	·066793	·066776	·066741	·066673	·066549	·066330	·065965	·065383	·064500	*1·5*
1·6	·054792	·054786	·054771	·054739	·054678	·054567	·054372	·054048	·053536	*1·6*
1·7	·044563	·044560	·044554	·044540	·044511	·044459	·044358	·044186	·043903	*1·7*
1·8	·035928	·035928	·035926	·035920	·035907	·035882	·035834	·035747	·035598	*1·8*
1·9	·028716	·028716	·028715	·028712	·028707	·028696	·028674	·028632	·028557	*1·9*
2·0	·022749	·022749	·022749	·022749	·022746	·022742	·022732	·022713	·022677	*2·0*
2·1	·017864	·017864	·017864	·017864	·017863	·017861	·017857	·017849	·017832	*2·1*
2·2	·013903	·013903	·013903	·013903	·013903	·013903	·013901	·013897	·013890	*2·2*
2·3	·010724	·010724	·010724	·010724	·010724	·010724	·010723	·010722	·010719	*2·3*
2·4	·008198	·008198	·008198	·008198	·008198	·008198	·008197	·008197	·008195	*2·4*
2·5	·006210	·006210	·006210	·006210	·006210	·006210	·006210	·006210	·006209	*2·5*
2·6	·004661	·004661	·004661	·004661	·004661	·004661	·004661	·004661	·004661	*2·6*

k	$r=\cdot 90$									k
	$h=0\cdot 9$	$h=1\cdot 0$	$h=1\cdot 1$	$h=1\cdot 2$	$h=1\cdot 3$	$h=1\cdot 4$	$h=1\cdot 5$	$h=1\cdot 6$	$h=1\cdot 7$	
0·0	·182714	·157949	·135313	·114899	·096720	·080721	·066793	·054792	·044563	*0·0*
0·1	·181715	·157378	·135001	·114738	·096642	·080685	·066776	·054786	·044560	*0·1*
0·2	·180143	·156440	·134466	·114448	·096492	·080611	·066741	·054771	·044554	*0·2*
0·3	·177784	·154974	·133596	·113954	·096225	·080473	·066673	·054739	·044540	*0·3*
0·4	·174405	·152788	·132243	·113155	·095774	·080230	·066549	·054678	·044511	*0·4*
0·5	·169780	·149675	·130238	·111920	·095047	·079822	·066330	·054567	·044459	*0·5*
0·6	·163726	·145439	·127399	·110099	·093931	·079169	·065965	·054372	·044358	*0·6*
0·7	·156138	·139928	·123560	·107537	·092296	·078170	·065383	·054048	·044186	*0·7*
0·8	·147024	·133064	·118595	·104093	·090008	·076716	·064500	·053536	·043903	*0·8*
0·9	·136515	·124872	·112450	·099668	·086951	·074694	·063220	·052762	·043456	*0·9*
1·0	·124872	·115490	·105164	·094224	·083047	·072009	·061453	·051649	·042786	*1·0*
1·1	·112450	·105164	·096874	·087810	·078274	·068600	·059118	·050118	·041826	*1·1*
1·2	·099668	·094224	·087810	·080559	·072686	·064459	·056173	·048110	·040515	*1·2*
1·3	·086951	·083047	·078274	·072686	·066410	·059641	·052617	·045591	·038805	*1·3*
1·4	·074694	·072009	·068600	·064459	·059641	·054265	·048506	·042569	·036674	*1·4*
1·5	·063220	·061453	·059118	·056173	·052617	·048506	·043948	·039098	·034135	*1·5*
1·6	·052762	·051649	·050118	·048110	·045591	·042569	·039098	·035274	·031231	*1·6*
1·7	·043456	·042786	·041826	·040515	·038805	·036674	·034135	·031231	·028057	*1·7*
1·8	·035351	·034966	·034391	·033573	·032462	·031021	·029236	·027121	·024722	*1·8*
1·9	·028427	·028216	·027887	·027399	·026709	·025777	·024576	·023097	·021354	*1·9*
2·0	·022612	·022502	·022322	·022044	·021635	·021058	·020285	·019293	·018080	*2·0*
2·1	·017801	·017746	·017653	·017502	·017270	·016929	·016453	·015818	·015008	*2·1*
2·2	·013876	·013850	·013803	·013725	·013600	·013408	·013128	·012738	·012222	*2·2*
2·3	·010713	·010701	·010679	·010640	·010576	·010472	·010315	·010087	·009772	*2·3*
2·4	·008193	·008188	·008178	·008160	·008128	·008075	·007991	·007864	·007680	*2·4*
2·5	·006208	·006206	·006202	·006194	·006179	·006153	·006110	·006042	·005940	*2·5*
2·6	·004661	·004660	·004658	·004655	·004648	·004636	·004615	·004581	·004526	*2·6*

k	r = ·90									k
	h = 1·8	h = 1·9	h = 2·0	h = 2·1	h = 2·2	h = 2·3	h = 2·4	h = 2·5	h = 2·6	
0·0	·035928	·028716	·022749	·017864	·013903	·010724	·008198	·006210	·004661	0·0
0·1	·035928	·028716	·022749	·017864	·013903	·010724	·008198	·006210	·004661	0·1
0·2	·035926	·028715	·022749	·017864	·013903	·010724	·008198	·006210	·004661	0·2
0·3	·035920	·028712	·022749	·017864	·013903	·010724	·008198	·006210	·004661	0·3
0·4	·035907	·028707	·022746	·017863	·013903	·010724	·008198	·006210	·004661	0·4
0·5	·035882	·028696	·022742	·017861	·013903	·010724	·008198	·006210	·004661	0·5
0·6	·035834	·028674	·022732	·017857	·013901	·010723	·008197	·006210	·004661	0·6
0·7	·035747	·028632	·022713	·017849	·013897	·010722	·008197	·006210	·004661	0·7
0·8	·035598	·028557	·022677	·017832	·013890	·010719	·008195	·006209	·004661	0·8
0·9	·035351	·028427	·022612	·017801	·013876	·010713	·008193	·006208	·004661	0·9
1·0	·034966	·028216	·022502	·017746	·013850	·010701	·008188	·006206	·004660	1·0
1·1	·034391	·027887	·022322	·017653	·013803	·010679	·008178	·006202	·004658	1·1
1·2	·033573	·027399	·022044	·017502	·013725	·010640	·008160	·006194	·004655	1·2
1·3	·032462	·026709	·021635	·017270	·013600	·010576	·008128	·006179	·004648	1·3
1·4	·031021	·025777	·021058	·016929	·013408	·010472	·008075	·006153	·004636	1·4
1·5	·029236	·024576	·020285	·016453	·013128	·010315	·007991	·006110	·004615	1·5
1·6	·027121	·023097	·019293	·015818	·012738	·010087	·007864	·006042	·004581	1·6
1·7	·024722	·021354	·018080	·015008	·012222	·009772	·007680	·005940	·004526	1·7
1·8	·022113	·019390	·016660	·014023	·011568	·009357	·007428	·005794	·004445	1·8
1·9	·019390	·017269	·015069	·012878	·010777	·008834	·007097	·005594	·004330	1·9
2·0	·016660	·015069	·013361	·011602	·009862	·008205	·006683	·005333	·004173	2·0
2·1	·014023	·012878	·011602	·010242	·008850	·007482	·006188	·005009	·003970	2·1
2·2	·011568	·010777	·009862	·008850	·007778	·006688	·005623	·004624	·003719	2·2
2·3	·009357	·008834	·008205	·007482	·006688	·005851	·005006	·004187	·003422	2·3
2·4	·007428	·007097	·006683	·006188	·005623	·005006	·004360	·003712	·003087	2·4
2·5	·005794	·005594	·005333	·005009	·004624	·004187	·003712	·003218	·002726	2·5
2·6	·004445	·004330	·004173	·003970	·003719	·003422	·003087	·002726	·002353	2·6

k	r = ·95									k
	h = 0·0	h = 0·1	h = 0·2	h = 0·3	h = 0·4	h = 0·5	h = 0·6	h = 0·7	h = 0·8	
0·0	·449458	·427148	·400525	·370551	·338465	·305541	·272899	·241401	·211641	0·0
0·1	·427148	·409885	·388048	·362178	·333273	·302578	·271347	·240657	·211315	0·1
0·2	·400525	·388048	·371208	·350051	·325166	·297571	·268502	·239173	·210607	0·2
0·3	·370551	·362178	·350051	·333789	·313499	·289802	·263724	·236469	·209204	0·3
0·4	·338465	·333273	·325166	·313499	·297954	·278690	·256354	·231956	·206660	0·4
0·5	·305541	·302578	·297571	·289802	·278690	·263982	·245878	·225035	·202439	0·5
0·6	·272899	·271347	·268502	·263724	·256354	·245878	·232102	·215259	·196007	0·6
0·7	·241401	·240657	·239173	·236469	·231956	·225035	·215259	·202487	·186977	0·7
0·8	·211641	·211315	·210607	·209204	·206660	·202439	·196007	·186977	·175254	0·8
0·9	·183985	·183855	·183547	·182880	·181566	·179198	·175290	·169372	·161115	0·9
1·0	·158631	·158584	·158461	·158172	·157551	·156332	·154150	·150570	·145180	1·0
1·1	·135659	·135643	·135599	·135484	·135216	·134643	·133525	·131534	·128287	1·1
1·2	·115068	·115063	·115049	·115007	·114902	·114655	·114131	·113116	·111318	1·2
1·3	·096800	·096799	·096794	·096781	·096743	·096646	·096422	·095949	·095036	1·3
1·4	·080757	·080756	·080755	·080751	·080739	·080704	·080617	·080415	·079992	1·4
1·5	·066807	·066807	·066807	·066806	·066802	·066791	·066760	·066682	·066502	1·5
1·6	·054799	·054799	·054799	·054799	·054798	·054795	·054785	·054757	·054688	1·6
1·7	·044565	·044565	·044565	·044565	·044565	·044564	·044561	·044552	·044528	1·7
1·8	·035930	·035930	·035930	·035930	·035930	·035930	·035929	·035927	·035919	1·8
1·9	·028717	·028717	·028717	·028717	·028717	·028716	·028716	·028716	·028713	1·9
2·0	·022750	·022750	·022750	·022750	·022750	·022750	·022750	·022750	·022749	2·0
2·1	·017864	·017864	·017864	·017864	·017864	·017864	·017864	·017864	·017864	2·1
2·2	·013903	·013903	·013903	·013903	·013903	·013903	·013903	·013903	·013903	2·2
2·3	·010724	·010724	·010724	·010724	·010724	·010724	·010724	·010724	·010724	2·3
2·4	·008198	·008198	·008198	·008198	·008198	·008198	·008198	·008198	·008198	2·4
2·5	·006210	·006210	·006210	·006210	·006210	·006210	·006210	·006210	·006210	2·5
2·6	·004661	·004661	·004661	·004661	·004661	·004661	·004661	·004661	·004661	2·6

k	r = ·95									k
	h = 0·9	h = 1·0	h = 1·1	h = 1·2	h = 1·3	h = 1·4	h = 1·5	h = 1·6	h = 1·7	
0·0	·183985	·158631	·135659	·115068	·096800	·080757	·066807	·054799	·044565	0·0
0·1	·183855	·158584	·135643	·115063	·096799	·080756	·066807	·054799	·044565	0·1
0·2	·183547	·158461	·135599	·115049	·096794	·080755	·066807	·054799	·044565	0·2
0·3	·182880	·158172	·135484	·115007	·096781	·080751	·066806	·054799	·044565	0·3
0·4	·181566	·157551	·135216	·114902	·096743	·080739	·066802	·054798	·044565	0·4
0·5	·179198	·156332	·134643	·114655	·096646	·080704	·066791	·054795	·044564	0·5
0·6	·175290	·154150	·133525	·114131	·096422	·080617	·066760	·054785	·044561	0·6
0·7	·169372	·150570	·131534	·113116	·095949	·080415	·066682	·054757	·044552	0·7
0·8	·161115	·145180	·128287	·111318	·095036	·079992	·066502	·054688	·044528	0·8
0·9	·150466	·137707	·123428	·108404	·093429	·079179	·066127	·054529	·044467	0·9
1·0	·137707	·128130	·116733	·104068	·090839	·077758	·065411	·054200	·044329	1·0
1·1	·123428	·116733	·108208	·098130	·087009	·075480	·064166	·053577	·044044	1·1
1·2	·108404	·104068	·098130	·090618	·081797	·072131	·062183	·052497	·043505	1·2
1·3	·093429	·090839	·087009	·081797	·075245	·067601	·059285	·050787	·042579	1·3
1·4	·079179	·077758	·075480	·072131	·067601	·061945	·055389	·048305	·041121	1·4
1·5	·066127	·065411	·064166	·062183	·059285	·055389	·050554	·044989	·039016	1·5
1·6	·054529	·054200	·053577	·052497	·050787	·048305	·044989	·040898	·036221	1·6
1·7	·044467	·044329	·044044	·043505	·042579	·041121	·039016	·036221	·032796	1·7
1·8	·035898	·035845	·035725	·035480	·035020	·034234	·033003	·031236	·028904	1·8
1·9	·028707	·028688	·028643	·028541	·028332	·027944	·027282	·026253	·024785	1·9
2·0	·022747	·022742	·022726	·022687	·022600	·022425	·022100	·021550	·020695	2·0
2·1	·017864	·017862	·017857	·017844	·017811	·017739	·017593	·017323	·016870	2·1
2·2	·013903	·013903	·013901	·013897	·013886	·013859	·013799	·013678	·013457	2·2
2·3	·010724	·010724	·010723	·010722	·010719	·010710	·010687	·010638	·010540	2·3
2·4	·008198	·008198	·008197	·008197	·008196	·008193	·008186	·008167	·008127	2·4
2·5	·006210	·006210	·006210	·006210	·006209	·006208	·006206	·006200	·006185	2·5
2·6	·004661	·004661	·004661	·004661	·004661	·004661	·004660	·004658	·004653	2·6

k	r = ·95									k
	h = 1·8	h = 1·9	h = 2·0	h = 2·1	h = 2·2	h = 2·3	h = 2·4	h = 2·5	h = 2·6	
0·0	·035930	·028717	·022750	·017864	·013903	·010724	·008198	·006210	·004661	0·0
0·1	·035930	·028717	·022750	·017864	·013903	·010724	·008198	·006210	·004661	0·1
0·2	·035930	·028717	·022750	·017864	·013903	·010724	·008198	·006210	·004661	0·2
0·3	·035930	·028717	·022750	·017864	·013903	·010724	·008198	·006210	·004661	0·3
0·4	·035930	·028717	·022750	·017864	·013903	·010724	·008198	·006210	·004661	0·4
0·5	·035930	·028716	·022750	·017864	·013903	·010724	·008198	·006210	·004661	0·5
0·6	·035929	·028716	·022750	·017864	·013903	·010724	·008198	·006210	·004661	0·6
0·7	·035927	·028716	·022750	·017864	·013903	·010724	·008198	·006210	·004661	0·7
0·8	·035919	·028713	·022749	·017864	·013903	·010724	·008198	·006210	·004661	0·8
0·9	·035898	·028707	·022747	·017864	·013903	·010724	·008198	·006210	·004661	0·9
1·0	·035845	·028688	·022742	·017862	·013903	·010724	·008198	·006210	·004661	1·0
1·1	·035725	·028643	·022726	·017857	·013901	·010723	·008197	·006210	·004661	1·1
1·2	·035480	·028541	·022687	·017844	·013897	·010722	·008197	·006210	·004661	1·2
1·3	·035020	·028332	·022600	·017811	·013886	·010719	·008196	·006209	·004661	1·3
1·4	·034234	·027944	·022425	·017739	·013859	·010710	·008193	·006208	·004661	1·4
1·5	·033003	·027282	·022100	·017593	·013799	·010687	·008186	·006206	·004660	1·5
1·6	·031236	·026253	·021550	·017323	·013678	·010638	·008167	·006200	·004658	1·6
1·7	·028904	·024785	·020695	·016870	·013457	·010540	·008127	·006185	·004653	1·7
1·8	·026064	·022860	·019491	·016173	·013088	·010360	·008048	·006153	·004642	1·8
1·9	·022860	·020529	·017917	·015190	·012523	·010062	·007904	·006089	·004616	1·9
2·0	·019491	·017917	·016024	·013917	·011731	·009609	·007666	·005975	·004566	2·0
2·1	·016173	·015190	·013917	·012395	·010711	·008977	·007306	·005787	·004475	2·1
2·2	·013088	·012523	·011731	·010711	·009500	·008169	·006807	·005504	·004328	2·2
2·3	·010360	·010062	·009609	·008977	·008169	·007214	·006172	·005114	·004108	2·3
2·4	·008048	·007904	·007666	·007306	·006807	·006172	·005428	·004621	·003806	2·4
2·5	·006153	·006089	·005975	·005787	·005504	·005114	·004621	·004047	·003427	2·5
2·6	·004642	·004616	·004566	·004475	·004328	·004108	·003806	·003427	·002989	2·6

k	r = 1·0									k
	h = *0·0*	*h* = *0·1*	*h* = *0·2*	*h* = *0·3*	*h* = *0·4*	*h* = *0·5*	*h* = *0·6*	*h* = *0·7*	*h* = *0·8*	
0·0	·500000	·460172	·420740	·382089	·344578	·308538	·274253	·241964	·211855	*0·0*
0·1	·460172	·460172	·420740	·382089	·344578	·308538	·274253	·241964	·211855	*0·1*
0·2	·420740	·420740	·420740	·382089	·344578	·308538	·274253	·241964	·211855	*0·2*
0·3	·382089	·382089	·382089	·382089	·344578	·308538	·274253	·241964	·211855	*0·3*
0·4	·344578	·344578	·344578	·344578	·344578	·308538	·274253	·241964	·211855	*0·4*
0·5	·308538	·308538	·308538	·308538	·308538	·308538	·274253	·241964	·211855	*0·5*
0·6	·274253	·274253	·274253	·274253	·274253	·274253	·274253	·241964	·211855	*0·6*
0·7	·241964	·241964	·241964	·241964	·241964	·241964	·241964	·241964	·211855	*0·7*
0·8	·211855	·211855	·211855	·211855	·211855	·211855	·211855	·211855	·211855	*0·8*
0·9	·184060	·184060	·184060	·184060	·184060	·184060	·184060	·184060	·184060	*0·9*
1·0	·158655	·158655	·158655	·158655	·158655	·158655	·158655	·158655	·158655	*1·0*
1·1	·135666	·135666	·135666	·135666	·135666	·135666	·135666	·135666	·135666	*1·1*
1·2	·115070	·115070	·115070	·115070	·115070	·115070	·115070	·115070	·115070	*1·2*
1·3	·096801	·096801	·096801	·096801	·096801	·096801	·096801	·096801	·096801	*1·3*
1·4	·080757	·080757	·080757	·080757	·080757	·080757	·080757	·080757	·080757	*1·4*
1·5	·066807	·066807	·066807	·066807	·066807	·066807	·066807	·066807	·066807	*1·5*
1·6	·054799	·054799	·054799	·054799	·054799	·054799	·054799	·054799	·054799	*1·6*
1·7	·044565	·044565	·044565	·044565	·044565	·044565	·044565	·044565	·044565	*1·7*
1·8	·035930	·035930	·035930	·035930	·035930	·035930	·035930	·035930	·035930	*1·8*
1·9	·028717	·028717	·028717	·028717	·028717	·028717	·028717	·028717	·028717	*1·9*
2·0	·022750	·022750	·022750	·022750	·022750	·022750	·022750	·022750	·022750	*2·0*
2·1	·017864	·017864	·017864	·017864	·017864	·017864	·017864	·017864	·017864	*2·1*
2·2	·013903	·013903	·013903	·013903	·013903	·013903	·013903	·013903	·013903	*2·2*
2·3	·010724	·010724	·010724	·010724	·010724	·010724	·010724	·010724	·010724	*2·3*
2·4	·008198	·008198	·008198	·008198	·008198	·008198	·008198	·008198	·008198	*2·4*
2·5	·006210	·006210	·006210	·006210	·006210	·006210	·006210	·006210	·006210	*2·5*
2·6	·004661	·004661	·004661	·004661	·004661	·004661	·004661	·004661	·004661	*2·6*

k	r = 1·0									k
	h = *0·9*	*h* = *1·0*	*h* = *1·1*	*h* = *1·2*	*h* = *1·3*	*h* = *1·4*	*h* = *1·5*	*h* = *1·6*	*h* = *1·7*	
0·0	·184060	·158655	·135666	·115070	·096801	·080757	·066807	·054799	·044565	*0·0*
0·1	·184060	·158655	·135666	·115070	·096801	·080757	·066807	·054799	·044565	*0·1*
0·2	·184060	·158655	·135666	·115070	·096801	·080757	·066807	·054799	·044565	*0·2*
0·3	·184060	·158655	·135666	·115070	·096801	·080757	·066807	·054799	·044565	*0·3*
0·4	·184060	·158655	·135666	·115070	·096801	·080757	·066807	·054799	·044565	*0·4*
0·5	·184060	·158655	·135666	·115070	·096801	·080757	·066807	·054799	·044565	*0·5*
0·6	·184060	·158655	·135666	·115070	·096801	·080757	·066807	·054799	·044565	*0·6*
0·7	·184060	·158655	·135666	·115070	·096801	·080757	·066807	·054799	·044565	*0·7*
0·8	·184060	·158655	·135666	·115070	·096801	·080757	·066807	·054799	·044565	*0·8*
0·9	·184060	·158655	·135666	·115070	·096801	·080757	·066807	·054799	·044565	*0·9*
1·0	·158655	·158655	·135666	·115070	·096801	·080757	·066807	·054799	·044565	*1·0*
1·1	·135666	·135666	·135666	·115070	·096801	·080757	·066807	·054799	·044565	*1·1*
1·2	·115070	·115070	·115070	·115070	·096801	·080757	·066807	·054799	·044565	*1·2*
1·3	·096801	·096801	·096801	·096801	·096801	·080757	·066807	·054799	·044565	*1·3*
1·4	·080757	·080757	·080757	·080757	·080757	·080757	·066807	·054799	·044565	*1·4*
1·5	·066807	·066807	·066807	·066807	·066807	·066807	·066807	·054799	·044565	*1·5*
1·6	·054799	·054799	·054799	·054799	·054799	·054799	·054799	·054799	·044565	*1·6*
1·7	·044565	·044565	·044565	·044565	·044565	·044565	·044565	·044565	·044565	*1·7*
1·8	·035930	·035930	·035930	·035930	·035930	·035930	·035930	·035930	·035930	*1·8*
1·9	·028717	·028717	·028717	·028717	·028717	·028717	·028717	·028717	·028717	*1·9*
2·0	·022750	·022750	·022750	·022750	·022750	·022750	·022750	·022750	·022750	*2·0*
2·1	·017864	·017864	·017864	·017864	·017864	·017864	·017864	·017864	·017864	*2·1*
2·2	·013903	·013903	·013903	·013903	·013903	·013903	·013903	·013903	·013903	*2·2*
2·3	·010724	·010724	·010724	·010724	·010724	·010724	·010724	·010724	·010724	*2·3*
2·4	·008198	·008198	·008198	·008198	·008198	·008198	·008198	·008198	·008198	*2·4*
2·5	·006210	·006210	·006210	·006210	·006210	·006210	·006210	·006210	·006210	*2·5*
2·6	·004661	·004661	·004661	·004661	·004661	·004661	·004661	·004661	·004661	*2·6*

k	*r* = 1·0									*k*
	h = *1·8*	*h* = *1·9*	*h* = *2·0*	*h* = *2·1*	*h* = *2·2*	*h* = *2·3*	*h* = *2·4*	*h* = *2·5*	*h* = *2·6*	
0·0	·035930	·028717	·022750	·017864	·013903	·010724	·008198	·006210	·004661	*0·0*
0·1	·035930	·028717	·022750	·017864	·013903	·010724	·008198	·006210	·004661	*0·1*
0·2	·035930	·028717	·022750	·017864	·013903	·010724	·008198	·006210	·004661	*0·2*
0·3	·035930	·028717	·022750	·017864	·013903	·010724	·008198	·006210	·004661	*0·3*
0·4	·035930	·028717	·022750	·017864	·013903	·010724	·008198	·006210	·004661	*0·4*
0·5	·035930	·028717	·022750	·017864	·013903	·010724	·008198	·006210	·004661	*0·5*
0·6	·035930	·028717	·022750	·017864	·013903	·010724	·008198	·006210	·004661	*0·6*
0·7	·035930	·028717	·022750	·017864	·013903	·010724	·008198	·006210	·004661	*0·7*
0·8	·035930	·028717	·022750	·017864	·013903	·010724	·008198	·006210	·004661	*0·8*
0·9	·035930	·028717	·022750	·017864	·013903	·010724	·008198	·006210	·004661	*0·9*
1·0	·035930	·028717	·022750	·017864	·013903	·010724	·008198	·006210	·004661	*1·0*
1·1	·035930	·028717	·022750	·017864	·013903	·010724	·008198	·006210	·004661	*1·1*
1·2	·035930	·028717	·022750	·017864	·013903	·010724	·008198	·006210	·004661	*1·2*
1·3	·035930	·028717	·022750	·017864	·013903	·010724	·008198	·006210	·004661	*1·3*
1·4	·035930	·028717	·022750	·017864	·013903	·010724	·008198	·006210	·004661	*1·4*
1·5	·035930	·028717	·022750	·017864	·013903	·010724	·008198	·006210	·004661	*1·5*
1·6	·035930	·028717	·022750	·017864	·013903	·010724	·008198	·006210	·004661	*1·6*
1·7	·035930	·028717	·022750	·017864	·013903	·010724	·008198	·006210	·004661	*1·7*
1·8	·035930	·028717	·022750	·017864	·013903	·010724	·008198	·006210	·004661	*1·8*
1·9	·028717	·028717	·022750	·017864	·013903	·010724	·008198	·006210	·004661	*1·9*
2·0	·022750	·022750	·022750	·017864	·013903	·010724	·008198	·006210	·004661	*2·0*
2·1	·017864	·017864	·017864	·017864	·013903	·010724	·008198	·006210	·004661	*2·1*
2·2	·013903	·013903	·013903	·013903	·013903	·010724	·008198	·006210	·004661	*2·2*
2·3	·010724	·010724	·010724	·010724	·010724	·010724	·008198	·006210	·004661	*2·3*
2·4	·008198	·008198	·008198	·008198	·008198	·008198	·008198	·006210	·004661	*2·4*
2·5	·006210	·006210	·006210	·006210	·006210	·006210	·006210	·006210	·004661	*2·5*
2·6	·004661	·004661	·004661	·004661	·004661	·004661	·004661	·004661	·004661	*2·6*

DOES INSULIN IMPROVE CARBOHYDRATE TOLERANCE IN DIABETES?

(A STATISTICAL STUDY.)

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It is hardly necessary to dilate upon the value of insulin in the treatment of *Diabetes mellitus.* The physical state and mental attitude of the diabetic, uncontrolled by diet alone, speak for themselves. Death from *uncomplicated* diabetes is and should now be an uncommon occurrence. Diabetics can now be exposed to major surgical procedures of long duration, using anaesthetics of different kinds, and the mortality rate is relatively low. With proper preoperative and postoperative medical care, death due to diabetic coma, and brought on by anaesthesia, is, and should be, a rare occurrence. Insulin has, however, been employed in a sufficiently large number of cases and for a sufficient length of time to enable us to answer a question just as important as, if not of greater importance than, whether the diabetic can be kept alive. Can insulin treatment result in a cure? If not, can it result in a partial cure? And if not, can it at least improve carbohydrate tolerance?

There are no records available, clinical or experimental, which demonstrate a cure. As far as I could ascertain from the literature, there is only one record of an investigation into the question as to whether insulin treatment can result in a partial cure. Harrison (1), at King's College Hospital, London, has been unable to demonstrate a partial remission of the disease in a careful study of five selected cases with observations lasting eleven to eighteen months. All of the five patients needed as much or slightly more insulin at the end of the observation periods. Such fluctuations in dosage of insulin as occurred were attributed to difficulties in balancing the dose of insulin accurately against the diet, variations in the potency of the insulin, and alterations in metabolic activity of the tissues.

We are therefore left with the question as to whether insulin can, at least, *improve* tolerance for carbohydrates. This question is, obviously, just as important as that just discussed. There is no sharp line of demarcation between improvement of carbohydrate tolerance and cure. It is only a matter of degree. If improvement of carbohydrate tolerance could be definitely demonstrated it would be sufficiently encouraging in that partial or real cure may be a function of time. An investigation along these lines was made at the Montreal General Hospital, and it is the purpose of this paper to record the results.

Differing from the unanimity of opinion that insulin does not effect cures, there appear to be differences of opinion as to its ability to improve carbohydrate tolerance. One's experience with diabetes need not be very large before one notices instances which do suggest improvement of tolerance. Diabetics are known to require insulin on admission to the hospital, and before discharge or some time after, are able to reduce the amounts taken or do without it entirely. Also there appears to be anatomical evidence in favour of improvement in tolerance. Boyd and Robinson (2) have reported a case of a child killed in an accident shortly after having been discharged from the hospital, who had improved markedly following insulin treatment. The clinical response to insulin, associated with the anatomical changes found, were strongly suggestive of regeneration of pancreatic tissue. Whether, however, anatomical integrity necessarily parallels functional efficiency is an unsettled problem. We know that such parallelism is not always found in the case of the kidneys.

The greatest difficulties one encounters in interpreting results of an investigation of the kind we have attempted are the possible variations of the disease itself, and the numerous factors which may influence the courses of the different types. One may here repeat an often quoted remark. Sir Rose Bradford put it very well when he stated that "diabetes is not an entity, but a clinical label attached to a number of different conditions with varied origins, different morbid anatomy and liable to follow different courses." The truth of this statement becomes only too evident in the course of the treatment of a series of such patients. For example, there is the acute diabetic. This individual, known to have had no glycosuria, very shortly after some illness, probably of a very mild nature, rapidly develops glycosuria and acidosis and progresses to the stage of coma. With relatively small doses of insulin he is brought out of coma, and in a very short time requires no insulin and takes a fair amount of carbohydrate. In our last case of this type (4789—26), this was three weeks. Such patients observed over a period of two or three years do not appear to have made any downward progress. One case I have in mind is now able to take about 250 grams of carbohydrate a day. As is obvious to anyone familiar with food values, this represents a very liberal carbohydrate diet.

There is the "juvenile diabetic," who may run a rather acute course, rapidly develop severe acidosis, and progress to the stage of coma, and, as in the acute diabetes of adults just mentioned, following insulin treatment, may be brought out of coma and eventually require little or no insulin while on fairly liberal diet. One striking example of this type of case was a child of twelve years of age (6109—22). Eight days following her recovery from coma, the urine was sugar free, the blood sugar was normal, and it was possible to discontinue the use of insulin.

Then there is the very mild diabetic whose tolerance is suddenly lowered by an infection, and progresses rapidly to the stage of severe acidosis and coma. One such patient developed pneumonia (4271—25), and, in spite of taking sixty

units of insulin a day, could barely be kept free of acidosis. Hyperglycaemia and glycosuria were very difficult to control. Following recovery from the pneumonia it was possible to discontinue the use of insulin, and the patient now takes sufficient food to maintain his normal body weight, the urine sugar free and the blood sugar normal.

In view of the very limited knowledge we have of the peculiar metabolism of these three types of the disease just mentioned, and in view of such rapid fluctuations in carbohydrate tolerance as they may show, it is obvious that such patients are not suitable for an investigation of the kind which we have attempted. We are therefore left with the other form of the disease, which, fortunately for this purpose, represents by far the great majority of patients, namely, the chronic (and in the absence of treatment) progressive diabetic. The course of the majority of these patients is very clear. At first the glycosuria which develops is transient, and may be absent for intervals of weeks. As the carbohydrate tolerance is lowered, the glycosuria appears more frequently, and then becomes post-prandial, appearing regularly a few hours after meals. As the disease progresses, the glycosuria becomes persistent, being found throughout the day. With further lowering of carbohydrate tolerance acetone bodies make their appearance, and in time, in the absence of treatment, these patients slowly but surely progress to the stage of coma.

Because of the slow progress and the relative ease with which it is possible to determine the carbohydrate tolerance of these individuals, they make ideal subjects for this study. The problem is, however, not simple. Though we may be dealing with a condition which runs a fairly slow and steady course, and the metabolism of which is fairly understood, there are many variables to consider.

In view of previous observations on the influence of infection, diabetics, though belonging to this group, having foci of infection, cannot be included in such a study. From this group must also be excluded those subjects with a history of gall-bladder disease, i.e. those patients known to have diabetes, definitely pancreatic in origin.

Having selected the subjects, a most difficult factor to exclude is the influence of fasting. The value of starvation in improving carbohydrate tolerance is well known. Though one can avoid fasting, when calculating diets for subjects at rest in bed, by adding to the basal diets the equivalent of about 10 per cent. of the total basal heat production, it is necessary to make additional allowances for the amount, frequency and regularity of exercise when the subjects are out of bed. It is obvious that with an individual not at rest in bed, and on a diet of constant composition, the degree of fasting will depend upon the degree of muscular activity.

Patients whose diets are constructed on the basis of calculated, rather than experimentally determined, caloric requirement may actually be fasting, though the degree of undernutrition may be mild. Continued, however, over a long period

of time such diets will influence the course of the diabetes, in that they will tend to improve carbohydrate tolerance regardless of the insulin administered.

In addition to this there is the difficulty of determining *exactly* how much food patients are receiving. In spite of accurately weighing food we have previously shown (3) that patients may be receiving as much as 25 per cent. either more or less of carbohydrate than is intended unless the diets consist of dairy products only—milk, cream, butter, etc.—of which a large number of analyses have been made and the percentage compositions are fairly accurately known; the compositions of other foods vary widely*.

Then we have the influence of acidosis. Many patients on admission to the hospital have this complication. An increased metabolic rate generally accompanies it. Diets constructed on calculated rather than experimentally determined caloric requirements may therefore fall short of the needs of the individual, *at least temporarily*, and fasting may unintentionally play its part.

Insulin improves the general attitude of the individual, and gives him a brighter outlook on life in general. Muscular tone is increased. This increases metabolism and, providing the diet remains the same, will obviously result in some degree of fasting.

It is obvious from the foregoing observations that such a study as this is rather complicated. It is, at times, possible however, by the application of modern statistical methods, to draw approximately accurate conclusions from what appears to be a very complicated or even at first sight an insolvable problem. Such methods were adopted for this investigation.

Our object was to attempt to demonstrate whether insulin does or does not improve carbohydrate tolerance. Proof of such improvement shall be regarded as having been demonstrated by either one of the following results:

(*a*) The patient, after having been proven to require insulin and having had it for a period of time, can increase the total caloric value and carbohydrate content of the diet and at the same time not increase the dose of insulin, or,

* This fact is recognized by the U.S. Department of Agriculture in publishing the Atwater-Bryant food tables, Bulletin No. 28, but apparently not generally by those employing the same tables in constructing diets. In the introductory remarks to this publication it is stated "In the present publication it is the intention to give the maximum, minimum, and average of all available analyses of American food products up to January 1, 1899, with the exception of milk, butter and other dairy products, and sugars. The number of analyses of such products is so great and the literature of the subject so large that a compilation of the results might appropriately form the subject of a special publication." And later "As a necessary basis of this tabulation, the individual analyses have been collated in detail. In many cases the number of analyses of a single product was considerable, and it is believed that the averages which are given in the tables may be advantageously used in computing the composition of foods used in dietary studies, etc." It will thus be noted that only in the case of dairy products are the average values recommended. This recommendation is obviously based upon one of the theorems of statistics, namely, "the arithmetical mean of a large series of observed values is the most probable value of the quantity measured." Where the number of analyses is small—meats, vegetables, etc.—it is necessary to know more than the average compositions. In spite of these facts, it is generally assumed that the average values given in these tables are the most probable for *all* foods, and they are accepted at face value in calculating diets. [The statistical theorem referred to is, of course, only true for a limited type, the symmetrical, of frequency distributions. K.P.]

(*b*) The patient, after having been proven to require insulin and having had it for a period of time can, without altering the carbohydrate or caloric content of the diet, reduce the amount of insulin taken.

If, when either the dose of insulin has been decreased, and the same diet maintained, or the diet has been increased, on the same dosage of insulin, the urine fails to remain sugar free and the blood sugar normal, this shall be proof that no increase of tolerance has taken place.

Proof of requiring insulin shall be the fact that the individual cannot maintain his body weight, and at the same time keep the blood sugar normal and the urine sugar free on diet alone.

In either case, no other factors, such as fasting, must operate which, by themselves, would tend to improve carbohydrate tolerance.

The first problem with which we were confronted was to eliminate the influence of the diets given insulin patients since *diets resulting in starvation tend to improve carbohydrate tolerance regardless of insulin.* In our clinic, as most likely in all others, the general plans followed in the dietetic management of patients have been those involving either "ladder" or "basal" diets. For insulin patients we at first confined ourselves to basal diets as outlined by the Toronto workers. Since such basal diets, *in the early period of management*, do not allow for energy expended in the usual muscular activities of bed patients, nor for that due to the specific dynamic action of food, it is obvious that though they are not rigid starvation diets, they may result, over a period of time, in some degree of fasting.

The ladder diet which we generally employ is shown in Table I. It will be

TABLE I.

Ladder Diet.

Day	Carbohydrate	Fat	Protein	Calories	Total Calories
1	15	20	25	340	340
2	15	20	25	340	680
3	15	20	25	340	1020
4	20	50	30	650	1670
5	25	75	35	915	2585
6	30	100	40	1180	3765
7	35	125	45	1445	5210
8	40	150	50	1710	6920
9	45	150	50	1730	8650
10	50	150	50	1750	10400

noted that after three "green days" the diet is gradually increased to basal caloric requirements. During the period in which the amounts of food are being increased, and the total amounts are still below the basal requirements, it is obvious that the degree of fasting would be much greater than reached with the "basal" type of

diet. With the latter, the degree of fasting is represented only by the deficiency of calories necessary to meet the energy expenditure as a result of muscular activity, and the specific dynamic action of food.

For the purpose of determining quantitatively the difference between the two diets, the records of 500 non-insulin diabetics were investigated. Two hundred and fifty represented "ladder" diets, and two hundred and fifty represented "basal" diets. The average time taken for the patients of each group to become sugar free was calculated. The data necessary for the calculations are shown in Table II. The first column represents the days necessary to render the urines sugar free. The second and third columns represent the incidences of such periods with the

TABLE II.

Day	Ladder Diet	Basal Diet
1	16	1
2	46	4
3	62	5
4	34	3
5	17	8
6	10	16
7	12	7
8	13	18
9	16	35
10	4	56
11	2	29
12	3	16
13	5	4
14	2	11
15	4	9
16	2	6
17	0	12
18	1	5
19	1	1
20	—	3
21	—	1
Totals	250	250
A.M.	4·9	10·3
S.D.	3·6	3·7
P.E. of Mean	0·15	0·16

Difference between means = 5·4

Probable error of difference $= \sqrt{(0{\cdot}15)^2 + (0{\cdot}16)^2} = 0{\cdot}22$

$$\frac{\text{Difference}}{\text{Probable error of difference}} = 24{\cdot}5$$

"ladder" and "basal" diets respectively. For each group were then calculated, in order, the standard deviation* and the probable error of the mean. The difference between the means of both groups and the probable error of the difference were then obtained. From the ratio of the difference to the probable error of the difference was determined whether the differences noted between the two diets were due to chance or actually the result of diet. Thus

Number of observations of "ladder" diet	250	(N)
Number of observations of "basal" diet	250	(N')
Average number of days necessary to render urine sugar free with "ladder" diet	4·9	(M)
Average number of days necessary to render urine sugar free with "basal" diet...	10·3	(M')
Standard deviation of "ladder" diet	3·6	(σ)
Standard deviation of "basal" diet	3·7	(σ_1)
Probable error of M	0·15	(p)
Probable error of M'	0·16	(p')
Difference between means ($M'-M$)	5·4	
Probable error of the difference	0·22	

$$\frac{\text{Difference}}{\text{Probable error of difference}}=24.$$

It is obvious that, in this group of subjects, the individuals, though selected by excluding the three types previously discussed, will still represent the disease in different degrees of severity. This is a complicating variable. They are, however, known not to be very mild, and also known not to require insulin. They belong to what are known in this hospital as Types 3 and 4 previously described (4). That is, they have persistent glycosuria with and without acetonuria. There is no reason, however, on the basis of probabilities, for a larger number representing any particular severity being found in one group more than the other. If a difference should be found between the results of the two diets, statistical treatment of the average values by the above method should demonstrate the part that *chance* may have played in having produced it.

It will be observed that the average period necessary to render the urines sugar free of patients on the "ladder" diet was 4·9 days, whereas on the "basal" diet it was 10·3 days. Now was this difference of five days between these averages caused by the different diets or the result of chance? From the statistical data given it is seen that the difference between these two averages, namely, 5·4, was 24 times the probable error of the difference (i.e. 5·4 ÷ 0·22). In other words it may be stated the chances of the difference being fortuitous or the result of chance were so slight that it is practically certain that the difference is not the result of *random* sampling.

* Standard deviation $=\sqrt{\frac{\Sigma d^2}{N}}$, where Σd^2 is the sum of the squares of the residual errors and N is the number of observations.

Probable error of the mean $=0{\cdot}67449\,\frac{\sigma}{\sqrt{N}}$, where σ is the standard deviation.

Probable error of the difference $=\sqrt{(P)^2+(P_1)^2}$, where P and P_1 are the probable errors of the means.

Ratio of difference between means to the probable error of the difference $=\frac{M_1-M}{\sqrt{(P)^2+(P_1)^2}}$.

Can this difference be attributed to the effects of starvation only? The average basal food requirement of an individual in this group was 1463 calories per 24 hours. If we allow an additional 10 per cent. of this for the muscular activity of bed patients, the daily average requirement is about 1600 calories. It is known that such non-insulin patients become sugar free in an average of three days of *complete starvation.* They therefore become sugar free following a deficit of about 4800 calories. In Table III are shown the daily and total deficits of patients on

TABLE III.

Day	Food				Calorie Deficit	
	C	F	P	Calories	Daily	Total
1	15	20	25	340	1415	1415
2	15	20	25	340	1415	2830
3	15	20	25	340	1415	4245
4	20	50	30	650	1105	5350
5	25	75	35	915	840	6190
6	30	100	40	1180	575	6765
7	35	125	45	1445	310	7075
8	40	150	50	1710	45	7120
9	45	150	50	1730	25	7145
10	50	150	50	1750	5	7150

the ladder diet. In calculating total requirement of individuals taking food, an additional 10 per cent. must be added to the basal requirement for the specific dynamic action of food. The total daily requirement is therefore 1755 calories. It will be observed that a deficit necessary to render the urine sugar free, that is, a deficit of 5265 calories, occurs about the fourth day of treatment. The actual period observed as noted above was about five days.

That caloric deficiency is probably not the only influencing factor is suggested from the data of the basal diets. Assuming an average requirement of 1755 calories per day, the patients on the "basal" diets were fasting to the extent of about 300 calories per day. Since an average deficit of 4800 calories is necessary to render the urine of such patients sugar free, it should have taken about 16 days to accomplish this. The actual time taken was ten days.

The object of the above investigation was to determine, quantitatively, the difference between the effects of diets from the point of view of their ability to render urine sugar free. It will be noted that the "ladder" diet was more effective, but the important fact is that the "basal" diet had also the same effect but required a longer period of time. *The possible influence of such diets improving carbohydrate tolerance must therefore be considered in the interpretation of possible reduction in insulin dosage.*

In our clinic there were 69 cases observed in which after varying periods of time it was possible to reduce the doses of insulin given. A statistical study was

made in order to determine if possible the cause or causes of these reductions. Some of these patients were kept in bed, and others allowed to walk about. Some had acidosis. An attempt was made to determine the influence of these factors. The subjects were therefore divided into the following groups and sub-groups.

Group 1. (*a*) Walking about. (*b*) In bed.

„ 2. (*a*) With acidosis. (*b*) No acidosis.

„ 3. (*a*) No acidosis and walking about. (*b*) No acidosis and in bed.

„ 4. (*a*) Acidosis and walking about. (*b*) Acidosis and in bed.

For each group and sub-group were calculated the average number of days taken before the doses of insulin could be reduced *from the time the diets and doses of insulin were fairly well regulated.* The differences between the average values of the sub-groups of each group were then treated statistically, and the probable error determined. By means of this procedure it was possible somewhat to clarify the picture. The combined data are shown in Table IV.

TABLE IV.

Group	Number of cases (N)	Arithmetical mean days (M) noted before insulin could be reduced	Standard deviation (σ)	$\sqrt{N}$	$\frac{\sigma}{\sqrt{N}}$	Probable error of $(M) = 0{\cdot}67449 \frac{\sigma}{\sqrt{N}}$	Probable error of difference between means	$\frac{\text{Difference between means}}{\text{Probable error of difference}}$
1. (*a*) Walking	30	19·0	7·8	5·47	1·43	·95	1·77	10·9
(*b*) In bed	39	38·4	14	6·24	2·24	1·50		
2. (*a*) With acidosis	40	24·1	11·1	6·32	1·75	1·17	2·30	6·0
(*b*) No acidosis	29	37·9	16·0	5·38	2·96	1·98		
3 (*a*) No acidosis. Walking	11	23	6·5	3·31	1·96	1·31	2·39	10·1
(*b*) „ „ In bed	18	47	12·8	4·24	3·00	2·01		
4. (*a*) Acidosis. Walking ...	19	16·6	7·2	4·35	1·65	1·10	1·86	7·7
(*b*) „ In bed ...	21	31·0	10·2	4·59	2·24	1·50		

Here again, as with those cases not requiring insulin, the disease was probably represented by various degrees of severity. The argument, however, holds here also that, on the basis of probabilities, there is no reason for a larger number representing any particular severity selecting one group more than another.

It will be noted from the average values in Table IV that in all the groups, "*walking*" and "*acidosis*" appear to be factors influencing the reductions of the amounts of insulin. Since walking and acidosis increase the metabolism of the individual, and since all these subjects were on "basal" diets, they were obviously fasting. The probabilities of the differences in the averages found being the result of chance are shown in the last column of the table. These demonstrate that the differences found were not the result of *random* sampling, but actually due to fasting. For example, the lowest ratio of the difference between the means to the probable error of the difference was 6·0 (Group 2). With this value it can readily be shown that the odds against the occurrence of a deviation as great or greater than 6, as a result of random sampling, are over 1000 to 1 (actually 1341 to 1).

The results of this investigation, as far as it has proceeded, may be briefly summarized.

An attempt was made to determine whether insulin does or does not improve carbohydrate tolerance. In view of the number of variables entering into the problem, the method of approach was statistical.

The general method employed to determine the influence of any particular variable was to divide the subjects into two groups, in one of which the variable studied was excluded as much as possible. The average result in each group was calculated and the difference between the averages noted.

The probable errors of the averages were then calculated and conclusions were drawn from the ratios of the differences between the averages of each group to the probable errors of the differences.

The three variables observed so far were diet, exercise, and acidosis.

In view of the fact that insulin patients were given during the early period of treatment diets calculated on the basis of caloric requirement in the basal metabolic state, the possible influence of such diets on carbohydrate tolerance was investigated. For this purpose the effects of "basal" diets were firstly compared with those of the "ladder" or starvation type in rendering the urines sugar free. It was found that the quantitative effects of "basal" diets continued over a period of time could be measured approximately. The time taken to render urines sugar free on such diets was greatly a function of their deficiencies in caloric values, i.e. a function of the degree of fasting. Since fasting tends to improve carbohydrate tolerance the effects of such basal diets when given in combination with insulin must be considered in the interpretations of possible reductions in the amounts of the latter.

A series of diabetics requiring insulin but who were able in time to reduce the amounts taken were then observed. The subjects were grouped in order to study the influences of some factors, the presence of which would tend to result in fasting and improve carbohydrate tolerance regardless of insulin. The effects of muscular activity and increased metabolism accompanying acidosis, when not

allowed for in the construction of diets, were observed. Though the whole investigation is incomplete, the findings of the effects of the above factors appear to be of sufficient clinical importance to warrant publication. They lead to certain conclusions.

Though one cannot be dogmatic, especially when an inquiry is only in progress, the following interpretation of the above data is suggested:

Insulin does not improve the carbohydrate tolerance of the diabetic of the chronic progressive group. In the majority of cases wherever it was possible to reduce the quantities of insulin taken, other conditions, which are known to lead to improvement of carbohydrate tolerance (fasting, increased muscular activity, etc.), were found.

Though it has been demonstrated statistically that "basal" diets, "exercise" or "increased" metabolism in general could account largely for reductions in the amounts of insulin taken, these do not appear to be the only influencing factors. The individuals in the control groups, in which exercise and acidosis were excluded, were also able to reduce the amounts of insulin taken, but after a longer period of time. Were these reductions due to insulin?

This study emphasizes further, if further emphasis is necessary, the great underlying principle to be followed in the management of the diabetic, namely, undernutrition. It is most tempting, at times, to allow the diabetic more insulin and, thereby, more liberal quantities of food. To do so, in view of our present knowledge, depends upon whether we are seeking immediate results only or those of a more lasting character. *A propos* of this practice I may cite an observation made in a recent editorial (6) by one responsible for insulin: "when depancreatinized dogs treated with insulin are made fat, by feeding them with excess of carbohydrate, they exhibit much more acute symptoms of diabetes when insulin is withdrawn than are observed under the same circumstances in the case of thin dogs. The hyperglycaemia, ketonemia and glycosuria are all more intense, but most striking of all, the general symptoms are extremely acute and a fat animal seldom lives for more than four days after discontinuing the insulin, whereas a thin one may live several weeks."

In our clinic, at the time this paper is being written, there are over 1200 diabetics. The average age of this group is 47 years. At this age the expectation of life is about twenty-four years. That is a long way to go, and our object should be to reach that goal if possible. Until clinical or experimental data are available definitely demonstrating that insulin does improve carbohydrate tolerance, treatment, at least in my opinion, should aim in that direction which tends to do so—namely, construction of diets not only low in carbohydrate but also caloric content, whether the individual does or does not require insulin.

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A STUDY OF THE AUSTRALIAN AND TASMANIAN SKULLS, BASED ON PREVIOUSLY PUBLISHED MEASUREMENTS.

By G. M. MORANT, D.Sc.

THE most immediate need of biometric craniology is the metrical description of a sufficient number of adequately long series of skulls of different races. Such a description is best furnished by the detailed measurements of a single collection of the same period and from a single restricted locality, but for many racial types no such material is available and another method of proceeding must be used. That is to collect data relating to a larger territory and to pool them together if the evidence warrants the assumption that only one racial type is represented. The present paper deals with the craniology of the natives of Australia and Tasmania in this manner. The only individual measurements considered are those which have been published previously and from them new means and constants of variation have been deduced.

1. *Definitions of Measurements.* Most of the following index letters denoting measurements are ones normally used by workers in the Biometric Laboratory. C = capacity. L = maximum glăbella-occipital length in the median sagittal plane. L' = Frankfort projective glabellar horizontal length. F = Flower's ophryo-occipital length. B = maximum calvarial breadth (see p. 436 below). B' = minimum breadth between the temporal lines. B'' = greatest breadth of the frontal bone. Bi-stephanic B = chord from stephanion R to stephanion L. Bi-asterionic B = chord from asterion R to asterion L. H' = basio-bregmatic height. H = Frankfort vertical height from the basion. OH = Frankfort vertical auricular height. βOH = auricular height similar to OH but to the bregma. LB = chord nasion to basion. Q' = Frankfort vertical transverse arc from auricular point to auricular point. $\beta Q'$ = transverse arc similar to Q' through the bregma; Broca's transverse arc = arc terminating at the points *sus-auriculaires* and passing through the bregma. S = median sagittal arc from nasion to opisthion. S_1 = arc nasion to bregma. S_2 = arc bregma to lambda. S_3 = arc lambda to opisthion. Arc lambda to inion. Arc inion to opisthion. S_1' = chord nasion to bregma. S_2' = chord bregma to lambda. S_3' = chord lambda to opisthion. Chord lambda to inion. Chord inion to opisthion. U = maximum horizontal circumference passing above the super-ciliary ridges and through the ophryon. Glabella U = maximum horizontal circumference through the glabella. PH = chord from tip of anterior nasal spine to alveolar point. $G'H$ = chord nasion to alveolar point. GB = chord between the lowest points R and L of the zygomatic-maxillary sutures. J = maximum bi-zygomatic breadth. External bi-orbital B = maximum breadth between the external orbital processes of the frontal bone at the fronto-malar sutures. Internal

bi-orbital B = chord betweenp oints R and L where the external] borders of the orbits meet the fronto-malar sutures. NH' = nasal height from nasion to base of anterior nasal spine. NH, R and L = Frankfort nasal height from nasion to lowest edge, R and L of the pyriform aperture. NB = maximum breadth of the pyriform aperture. Height pyriform aperture = from the lowest median point of the nasal bones to the base of the anterior nasal spine. DC = chord from dacryon R to dacryon L. Flower's inter-orb. B = chord between the points R and L "where the ridge which forms the posterior boundary of the lacrymal groove meets the fronto-lacrymal suture." SC = minimum chord between naso-maxillary sutures. Flower's O_1 = maximum orbital breadth from the point "where the ridge which forms the posterior boundary of the lacrymal groove meets the fronto-lacrymal suture." O_1' = Broca's orbital breadth from the dacryon. O_1 = maximum breadth of the orbit. O_2 = orbital height whether taken perpendicular to O_1 to Flower's O_1, or to O_1'. (N.B. It was not possible to separate the measurements of the right and left orbits.) Palato-max. L = Flower's maxillary length (*Journal of the Anthropological Institute*, Vol. x. 1880, p. 161) from the alveolar point to "a line drawn across the hinder borders of the maxillary bones." Turner re-named this measurement the "palato-maxillary length" and the title has been generally adopted since. Broca's G_1 = palate length from the tip of the posterior nasal spine to a point "derrière les incisives médianes, sur la lèvre postérieure du bord alvéolaire." G_1' = palate length from the base of the posterior nasal spine to an imaginary line tangential to the inner rims of the alveoli of the middle incisors. Palato-max. B = Flower's maxillary width (*loc. cit.*) "between the outer borders of the alveolar arch immediately above the middle of the second molar tooth." Turner re-named this measurement the "palato-maxillary breadth." Broca's G_2 = "le maximum d'écartement de la courbe intérieure de l'arcade alvéolaire." G_2 = breadth of palate between inner alveolar walls at second molars. GL = chord from basion to alveolar point. *fml* = chord from basion to opisthion. *fmb* = greatest breadth of *foramen magnum*. Nasion-inion chord. Glabella-inion chord. Sub. to gb.-inion chord = the maximum subtense from the glabella-inion chord to the vault in the median sagittal plane.

Various indices were calculated from the above direct measurements. The Occipital Index (Oc. I.), defined to be $100\frac{S_3}{S_3'}\sqrt{\frac{S_3}{24(S_3-S_3')}}$, was found with the aid of Tildesley's table of the function (*Biometrika*, Vol. XIII. 1921, pp. 261—262). Schwalbe's index is the subtense to the glabella-inion chord expressed as a percentage of that chord. $P\angle$ is the Frankfort profile-angle. $N\angle$, $A\angle$ and $B\angle$ are the nasal, alveolar and basal angles of the triangle of which the nasion, alveolar point and basion are the apices. They were found from the chords $G'H$, GL and LB with the aid of Pearson's Trigonometer in the manner described by Fawcett (*Biometrika*, Vol. I. p. 418). The foraminal angle is that between the Frankfort horizontal plane and the chord joining basion to opisthion which forms the sagittal axis of the *foramen magnum*. It is negative if the opisthion is lower than the basion with respect to the horizontal.

2. *Measurements of Australian Skulls.* Published measurements of native Australian skulls were found in the following papers. *A.S.D.* is used as an abbreviation for *Die anthropologischen Sammlungen Deutschlands*, which was published in parts as supplements to the *Archiv für Anthropologie*. Only sexed and adult specimens are referred to.

(*a*) Pruner-Bey: Résultats de craniométrie. *Mémoires de la Société d'Anthropologie de Paris.* T. II. 1865, pp. 417—432. Means are given of 7 ♂ and 3 ♀ skulls in the Muséum d'Histoire Naturelle at Paris, but few of the measurements can be used.

(*b*) J. W. Spengel: *A.S.D.* Göttingen section, 1874. Individual measurements of 19 skulls, pp. 76—79.

(*c*) A. Ecker: *A.S.D.* Freiburg section, 1878. Individual measurements of 4 skulls (Nos. 9—12) on p. 54. The 6 specimens (Nos. 1—6) on pp. 53—54 were omitted in error.

(*d*) C. V. Cauvin: Rapport sur les mensurations et les caractères morphologiques d'une série de crânes australiens. *Archives des Missions Scientifiques et Littéraires. Choix de Rapports et Instructions publié sous les auspices du Ministère de l'Instruction Publique et des Beaux-Arts.* 3ième série. T. VII. 1881, pp. 191—211. Individual indices and horizontal circumferences only of 23 skulls in the Museum at Sydney.

(*e*) W. Turner: Report on the human crania and other bones of the skeletons collected during the voyage of H.M.S. *Challenger*, in the years 1873—1876. *Challenger Reports.* Vol. X. 1884. Individual measurements of 31 skulls in the Anatomical Museum of Edinburgh University are given in Tables III, IV and V.

(*f*) N. de Miklouho-Maclay: Remarks on a skull of an Australian aboriginal from the Lachan District. *Proceedings of the Linnean Society of New South Wales.* Vol. VIII. 1883, pp. 395—396. Another skull of which measurements are given on pp. 401—403 is clearly scaphocephalic.

(*g*) E. Schmidt: *A.S.D.* Leipzig section, 1886. Individual measurements of 5 skulls on pp. 148—149.

(*h*) N. Rüdinger: *A.S.D.* München section, 1892. Individual measurements of 2 skulls on pp. 112—113.

(*i*) C. D. Cooper: Notes on the skull of an aboriginal Australian. *Journal of the Anthropological Institute.* Vol. XXIII. 1893, pp. 153—156.

(*j*) W. L. H. Duckworth: A critical study of the collection of crania of aboriginal Australians in the Cambridge University Museum. *Ibid.* pp. 284—314. Individual measurements of 34 adult skulls in Tables I and II.

(*k*) W. L. H. Duckworth: Notes on skulls from Queensland and South Australia. *Ibid.* Vol. XXIV. 1894, pp. 213—218. Measurements of 3 skulls on pp. 217—218. They are in the Cambridge University Museum.

(*l*) J. T. Wilson: Craniological report upon two skulls collected by the Horn Expedition. Appendix III (pp. 142—148) in: *Report on the work of the Horn*

scientific Expedition to Central Australia. Part IV. *Anthropology.* Edited by Baldwin Spencer. London and Melbourne, 1896.

(*m*) W. L. H. Duckworth: Notes on Crania of Australian aborigines. *Journal of the Anthropological Institute.* Vol. XXVII. 1897, pp. 204—208: Measurements of 3 skulls on pp. 206—207. Two were at that time in private collections and the other was in the Cambridge University Museum.

(*n*) V. Giuffrida-Ruggeri: Crani dell' Australia, della Nuova Caledonia e delle isole Salomone. *Atti della Società Romana di Antropologia.* Vol. XII. 1906, pp. 7—35. Measurements of 2 skulls, pp. 13—14.

(*o*) W. H. Flower: *Catalogue of the Specimens illustrating the Osteology and Dentition of vertebrated Animals, recent and extinct, contained in the Museum of the Royal College of Surgeons of England.* Part I, *Man.* 2nd edition, 1907. Individual measurements of 79 adult and sexed skulls are given on pp. 314—337. Measurements are also recorded of a number of specimens for which no sex is given. The first edition of the catalogue (1879) contains part of these data.

(*p*) H. Basedow: Der Tasmanierschädel, ein Insulartypus. *Zeitschrift für Ethnologie.* 42 Jahrgang, 1910, pp. 176—227. Individual measurements of 121 skulls in the Museum of the Royal College of Surgeons are given on pp. 212—223. They include all the adult and sexed ones of which measurements were provided in Flower's Catalogue of 1907 except Nos. 1070, 1071, 1072, 1074, 1075 and 1093. Basedow determined all Flower's measurements and several additional ones.

(*q*) R. Pöch: Studien an Eingeborenen von Neu-Südwales und an australischen Schädeln. *Mitteilungen der anthropologischen Gesellschaft in Wien.* Band XLV. 1915, pp. 12—94. Individual measurements of 32 adult and sexed skulls belonging to the writer's private collection are given in Tables I—XI. There are 9 adult unsexed and 3 juvenile specimens in the same tables. The measurements are more complete than any other published ones of an Australian series.

(*r*) A. Schultz: Anthropologische Untersuchungen an der Schädelbasis. *Archiv für Anthropologie.* Neue Folge, Band XVI. 1918, pp. 1—103. Individual measurements of 38 skulls are given on pp. 56—63. The specimens are in the following museums: Dresden (Zool.-antr.-ethn. Mus.) 27, Berlin (Anatomie) 7, München (Anthrop. Institut) 2, Zürich (Anthrop. Institut) 2.

3. *A Comparison of Australian Skulls from different Localities.* From the sources given in Section 2 above it was possible to collect measurements of about 300 sexed Australian skulls. In the case of the greater number of them the localities from which they had been obtained were also recorded. The smallest groupings which could be used conveniently were those given by the six territorial divisions of the continent and more than three-quarters of the skulls could be assigned to one or other of those divisions. The most reliable male means that can be given for the groups are in Table I. Before pooling the measurements provided by different workers great care was taken to ensure that they had been

determined by using precisely similar methods of technique and many insufficiently defined ones had to be excluded. Exceptions to that practice were only made in the case of two characters: all capacities were accepted, although the different methods used to determine them might well have led to substantially different results, and all orbital heights were pooled whether they had been taken perpendicular to Flower's, the dacryal or some other orbital breadth. This latter pooling may be considered a fairly safe procedure. In general, the importance of comparing only those measurements which have been determined by identical methods is evident enough, but, as so often in other fields, that precaution has been neglected by many who have contributed to the craniometry of Australian peoples.

The means of each of the divisions of Table I were calculated from the individual measurements provided by several of the writers cited. The numbers of skulls are as small as any that could be used with safety, but, in spite of that

TABLE I.

Mean Measurements (Males) of Groups of Australian Skulls.

Character	Queensland	New South Wales	Victoria	South Australia	Western Australia	Northern Territory
C ...	1287·7 (19)	1287·7 (38)	1311·3 (18)	1319·6 (39)	1255·5 (20)	1224·2 (18)
F ...	185·1 (17)	187·0 (22)	186·9 (16)	187·0 (43)	186·9 (15)	180·2 (17)
B ...	131·1 (19)	132·6 (43)	132·3 (18)	132·8 (47)	130·8 (22)	128·3 (18)
B' ...	95·2 (20)	96·7 (36)	96·7 (17)	96·1 (37)	96·8 (15)	94·3 (17)
LB ...	101·3 (19)	102·7 (27)	103·8 (16)	102·4 (41)	100·7 (22)	103·8 (18)
H' ...	133·3 (20)	136·1 (29)	134·3 (16)	131·5 (45)	132·6 (22)	132·8 (18)
S ...	375·7 (9)	372·8 (26)	379·2 (6)	369·4 (22)	373·0 (12)	361·1 (14)
U ...	510·2 (22)	518·1 (35)	517·5 (14)	518·2 (43)	507·3 (11)	501·9 (16)
fml ...	34·4 (18)	34·7 (27)	35·3 (15)	35·8 (36)	35·1 (22)	35·6 (18)
fmb ...	29·1 (15)	29·3 (22)	30·4 (11)	30·9 (34)	30·2 (17)	30·4 (17)
J ...	131·9 (19)	133·4 (36)	135·8 (15)	134·6 (35)	133·2 (22)	134·2 (17)
NH' ...	49·1 (19)	49·6 (19)	49·8 (16)	50·1 (37)	48·1 (15)	49·0 (17)
NB ...	26·4 (20)	26·5 (20)	27·5 (16)	27·1 (37)	26·1 (15)	27·6 (17)
O_2 ...	34·0 (20)	33·6 (23)	33·6 (16)	33·4 (32)	33·3 (15)	33·4 (17)
$100\,B/F$ ...	70·5 (17)	70·8 (21)	70·6 (16)	71·2 (43)	70·4 (16)	71·3 (17)
$100\,H'/F$ ...	72·3 (17)	73·0 (21)	72·2 (16)	70·6 (41)	70·6 (15)	74·0 (17)
$100\,B/H'$ ...	98·6 (25)	97·6 (34)	98·3 (16)	101·1 (47)	98·7 (22)	96·7 (18)
$100 fmb/fml$	84·5 (16)	84·2 (22)	85·7 (11)	86·4 (34)	85·4 (17)	85·3 (17)
$100 NB/NH'$	54·7 (25)	54·2 (24)	55·3 (16)	53·8 (39)	54·7 (16)	56·7 (17)

fact, the means appear to show extraordinarily small differences. The cephalic indices ($100\,B/F$) are remarkably constant and the most discordant measurement would seem to be the small length (F) of the Northern Territory skulls. But little can be said without a knowledge of whether the differences are significant or not and, in any case, the evidence provided by a single measurement can be of little value. Although the material is not ample, a trustworthy comparison may be made by using the biometric method of the coefficient of racial likeness*. By taking account of a number of characters at the same time—measuring the differences in terms of their statistical significance—that criterion

* Karl Pearson: On the Coefficient of Racial Likeness. *Biometrika*, Vol. XVIII. 1926, pp. 105—117.

largely compensates for the small sizes of the samples. The standard deviations of characters are needed for that purpose and, as none have as yet been given for Australian skulls, the values for the 26th—30th Dynasty series provided by Davin and Pearson* may be used provisionally. They are the most reliable constants of variability available and also almost the smallest recorded as yet for a racial series of skulls. By using the Egyptian standard deviations for Australian groups the resulting coefficients are likely to be larger than the true ones, so we shall be erring on the side of caution. The coefficients in Table II were calculated for the 19 characters given in Table I. When it is remembered that the samples were drawn from exceedingly large areas, the lowness of the coefficients is seen to be a most noteworthy fact. There is perfect statistical justification for considering that the three eastern groups (Queensland, New South Wales and Victoria) represent a single homogeneous population. The skulls from Western Australia cannot be distinguished from that population and the South Australian group is but slightly differentiated from it. The latter stands closest to the groups from the adjoining regions of Western Australia and Victoria, but its bond with the New South Wales and Queensland skulls is close enough to suggest that we are only dealing with variants of a single racial type and not with discrete races. The Northern Territory skulls, however, are sufficiently removed from all the others to suggest a racial difference. Even so, that difference is not greater than one which may be found between two neighbouring and closely related races in an ethnically heterogeneous area such as Europe. It may be noticed that the group furthest removed from the Northern Territory one is that from South Australia, so the slight divergence of the latter from the type found in Queensland, New South Wales, Victoria and Western Australia cannot be supposed due to admixture with the differing type found in the north. From a comparison of the α's† for

TABLE II.

Coefficients of Racial Likeness between Groups of Male Australian Skulls[1].

	Queensland (18·7)	New South Wales (27·6)	Victoria (15·1)	South Australia (38·5)	Western Australia (17·4)	Northern Territory (17·1)
Queensland (18·7)[2] ...	—	−0·06 ± ·21	0·26 ± ·21	1·43 ± ·21	−0·34 ± ·21	1·51 ± ·21
New South Wales (27·6)	−0·06 ± ·21	—	−0·18 ± ·21	1·96 ± ·21	0·61 ± ·21	2·89 ± ·21
Victoria (15·1)	0·26 ± ·21	−0·18 ± ·21	—	0·18 ± ·21	0·55 ± ·21	1·99 ± ·21
South Australia (38·5) ...	1·43 ± ·21	1·96 ± ·21	0·18 ± ·21	—	0·86 ± ·21	4·40 ± ·21
Western Australia (17·4)	−0·34 ± ·21	0·61 ± ·21	0·55 ± ·21	0·86 ± ·21	—	1·83 ± ·21
Northern Territory (17·1)	1·51 ± ·21	2·89 ± ·21	1·99 ± ·21	4·40 ± ·21	1·83 ± ·21	—

[1] All the coefficients are based on the 19 characters given in Table I.

[2] The numbers in brackets are the mean numbers of skulls available for the characters used in computing the coefficients.

* Karl Pearson and Adelaide G. Davin: On the Biometric Constants of the Human Skull. *Biometrika*, Vol. XVI. 1924, pp. 328—368.

† With the usual notation $\alpha = \frac{n_s n_s'}{n_s + n_s'}\left(\frac{M_s - M_s'}{\sigma_s}\right)^2$.

individual characters, it is seen that the divergence of the Northern Territory type from all the others is due chiefly to the smaller size of nearly all its calvarial direct measurements and particularly F, B, U and S. The Northern Territory means are also smallest for the capacity (C), but not for LB and H'. The aberrant type has extreme values for the indices 100 H'/F and 100 B/H' and they show some significant differences, but none whatever are found between facial measurements. The divergence between the South Australian and New South Wales series is due to H' and the indices involving H' (100 H'/F and 100 B/H'): fmb is the only other character showing a significant difference.

It might be thought that the evidence of Table II would be sufficient to justify the pooling of all the Australian skulls except those from the Northern Territory. It was felt safer, however, to re-group the material before making that assumption. Two roughly equal groups were made by clubbing the Western Australian, South Australian and Victorian skulls together to give one, and those from Queensland and New South Wales to give another. Both ♂ and ♀ means are in Table III and with them are the measurements of the short series of Australian skulls from unknown localities. The ♀ values for the Northern Territory specimens are in the same table, the ♂ having been given already in Table I. Coefficients for the two sexes between the three groups of skulls from known localities are in Table IV. The southern and eastern populations have the

TABLE III.

Mean Measurements of Groups of Australian Skulls.

Character	Male			Female			
	Western Australia, South Australia and Victoria	Queensland and New South Wales	Unknown localities	Western Australia, South Australia and Victoria	Queensland and New South Wales	Northern Territory	Unknown localities
C	1301·0 (77)	1287·7 (57)	1310·9 (30)	1149·9 (39)	1141·4 (27)	1142·5 (10)	1149·4 (9)
F	187·0 (74)	186·2 (39)	183·8 (22)	176·6 (31)	175·4 (22)	171·3 (10)	173·4 (5)
B	132·2 (87)	132·1 (62)	132·3 (39)	127·3 (43)	127·1 (30)	124·5 (10)	129·3 (12)
B'	96·4 (69)	96·2 (56)	96·1 (27)	91·5 (30)	91·7 (26)	91·2 (10)	91·8 (8)
LB	102·2 (79)	102·1 (46)	101·0 (30)	95·2 (36)	94·9 (28)	93·4 (10)	95·8 (9)
H'	132·3 (83)	134·9 (49)	135·1 (29)	124·9 (40)	126·5 (29)	128·1 (10)	127·7 (6)
S	372·0 (40)	373·5 (35)	373·8 (13)	352·6 (21)	353·9 (16)	345·2 (5)	358·5 (4)
U	516·3 (68)	515·1 (57)	510·7 (33)	489·2 (27)	488·3 (31)	477·5 (10)	492·6 (11)
fml	35·5 (73)	34·6 (45)	35·7 (34)	34·2 (35)	33·1 (28)	33·1 (10)	32·6 (9)
fmb	30·6 (62)	29·2 (37)	30·1 (34)	28·7 (29)	28·7 (23)	28·0 (10)	28·4 (9)
J	134·4 (72)	132·9 (55)	134·0 (37)	122·4 (36)	122·6 (28)	118·9 (9)	121·3 (9)
NH' ...	49·6 (68)	49·4 (38)	48·5 (28)	46·1 (27)	45·0 (21)	43·6 (10)	45·0 (8)
NB	27·0 (68)	26·4 (40)	27·4 (23)	25·1 (27)	25·9 (23)	24·7 (10)	27·6 (5)
O_2	33·4 (63)	33·8 (43)	33·4 (13)	33 1 (26)	32·7 (24)	31·4 (10)	30·7 (8)
100 B/F ...	70·9 (75)	70·7 (38)	71·9 (22)	72·6 (31)	72·1 (22)	72·8 (10)	73·8 (5)
100 H'/F ...	71·0 (72)	72·7 (38)	73·6 (22)	70·4 (30)	72·4 (14)	74·9 (10)	73·8 (5)
100 B/H' ...	100·0 (85)	98·0 (59)	98·0 (29)	102·3 (40)	99·8 (37)	97·2 (10)	100·0 (6)
100 fmb/fml	86·0 (62)	84·3 (38)	84·3 (34)	84·1 (29)	85·7 (24)	84·7 (10)	86·9 (8)
100 NB/NH'	54·3 (71)	54·5 (49)	56·5 (21)	54·4 (27)	56·7 (30)	56·8 (10)	60·9 (5)

♂ coefficient of $1·58 \pm ·21$ and the ♀ of $0·69 \pm ·21$*. The former differs quite significantly from zero, but it indicates a bond of affinity which is decidedly closer than one normally found between two distinct but neighbouring races†. The ♀ coefficient differs from zero by less than four times its probable error. Between the ♂ means the only significant differences ($\alpha > 6$ say) are for *fmb* ($\alpha = 9·82$), 100 H'/F (8·31), H' (8·23) and 100 B/H' (7·52): 100 B/H' ($\alpha = 7·25$) is the only ♀ character differentiated. From this evidence, it was supposed legitimate to conclude that the native population of the whole of Australia, with the exception of the Northern Territory, belongs to a single racial type which shows some slight local variation. That will be called, for convenience, the *A* type. The material considered is not sufficient to provide a reliable determination of the differences which may exist between sub-groups of the *A* type. The Northern Territory skulls

TABLE IV.

Coefficients of Racial Likeness between Groups of Australian Skulls[1].

	Male				Female			
		Western Australia, South Australia and Victoria (70·9)†	Queensland and New South Wales (46·4)	Northern Territory (17·1)		Western Australia, South Australia and Victoria (31·8)	Queensland and New South Wales (25·4)	Northern Territory (9·7)
Western Australia, South Australia and Victoria	(70·9)[2]	—	1·58 ± ·21	3·64 ± ·21	(31·8)[2]	—	0·69 ± ·21	3·36 ± ·21
Queensland and New South Wales	(46·4)	1·58 ± ·21	—	3·03 ± ·21	(25·4)	0·69 ± ·21	—	1·22 ± ·21
Northern Territory	(17·1)	3·64 ± ·21	3·03 ± ·21	—	(9·7)	3·36 ± ·21	1·22 ± ·21	—

[1] All the coefficients are based on the 19 characters given in Tables I and III.

[2] The numbers in brackets are the mean numbers of skulls available for the characters used in computing the coefficients.

* As in the writer's opinion, it is most reasonable to suppose that this ♂ coefficient is greater than the ♀ because the ♂ means are based on larger numbers of skulls, so it would be fallacious to assume that the ♂ types are less closely related than the ♀ ones. He believes that the coefficients ought not to be compared one with another unless allowance is made for the fact that the series concerned are made up by different numbers of skulls. Admitting that necessity, we have a reasonable explanation of the fact that all the ♂ coefficients in Table IV and in Table V below are greater than the corresponding ♀ coefficients. Again it is not disconcerting to find that the coefficients with the Northern Territory means become greater according as the means with which they are compared are based on larger numbers of skulls (Tables II, IV and V). The incidental evidence adduced in the present paper is quite insufficient to decide this question. It will make little difference to the conclusions arrived at here whether one view or the other be taken.

† Between two ♂ 17th-century London series, the Farringdon Street (91·9) and Whitechapel (90·3), Miss Hooke found a coefficient of $4·15 \pm ·18$ (*Biometrika*, Vol. XVIII. 1926, p. 24). Coefficients less than 5 are seldom found between European series of that length.

are sufficiently distinguished from all the others to make it necessary to assign them to a distinct, but closely allied, race. Their significant coefficients (Table IV) are occasioned almost entirely by their small calvarial direct measurements F, B, S and U, judging by both ♂ and ♀ means. The indices $100\ H'/F$ and $100\ B/H'$ also show some significant differences, but none are found between the basio-bregmatic heights (H'), the nasio-basion lengths (LB), the cephalic indices or the facial measurements. The pooled means of the skulls of the A type for all available characters are given in Table VI and the coefficients computed for the 19 characters hitherto considered (i.e. the ones in Tables I and III) are in Table V. The small groups of Australian skulls from unknown localities are compared with the others in the same table. Between the Northern Territory and A type Australians there is a ♂ coefficient of 3·69 ± ·21 and the significant differences are confined to six of the characters compared, viz. F ($\alpha = 17{\cdot}56$), U (14·17), B (10·85), $100\ H'/F$ (9·95), S (9·84) and C (6·17): the ♀ coefficient is 2·83 ± ·21 and the only α's greater than 6 are for $100\ H'/F$ (16·18), F (8·56), $100\ B/H'$ (8·56) and U (7·84). As it was reasonable to expect, the means of the skulls from unknown localities are intermediate between those of the two distinct racial types.

TABLE V.

Coefficients of Racial Likeness between Groups of Australian Skulls[1].

	Male				Female			
		Australian A[3] (129·6)[2]	Unknown localities (27·4)	Northern Territory (17·1)		Australian A[3] (58·9)	Unknown localities (7·4)	Northern Territory (9·7)
Australian A[3] ...	(129·6)[2]	—	0·93 ± ·21	3·69 ± ·21	(58·9)[2]	—	0·88 ± ·21	2·83 ± ·21
Unknown localities	(27·4)	0·93 ± ·21	—	1·27 ± ·21	(7·4)	0·88 ± ·21	—	0·74 ± ·21
Northern Territory	(17·1)	3·69 ± ·21	1·27 ± ·21	—	(9·7)	2·83 ± ·21	0·74 ± ·21	—

[1] All the coefficients are based on the 19 characters given in Tables I and III.
[2] The numbers in brackets are the mean numbers of skulls available for the characters used in computing the coefficients.
[3] The A group is made up by pooling the skulls from Queensland, New South Wales, Victoria, South Australia and Western Australia.

4. *Measurements of Tasmanian skulls.* Published measurements of native Tasmanian adult skulls were found in the following papers:

(a) A. de Quatrefages and E. T. Hamy: *Crania Ethnica. Les crânes des races humaines.* Paris, 1882. Measurements of 1 ♀ skull (*Tasmanien du nord*) are given on p. 228. The three series of means of small groups in the same table could not be used as the measurements are in whole mms. and there is no indication of the number of specimens on which each mean is based.

(b) W. R. Harper and A. H. Clarke: Notes on the measurements of the Tasmanian Crania in the Tasmanian Museum, Hobart. *Papers and Proceedings*

of the Royal Society of Tasmania for 1897. 1898, pp. 97—110. Of the 19 skulls in the Museum described as Tasmanians, the writers of this paper rejected three as being incorrectly classed. After comparing the measurements, they decided to reject three others (Nos. 1 *a*, 2 *a* and 3 *a*), supposing them to be half-castes. No. 1 *a* has a nasal index (100 NB/NH') of 42·8 and a gnathic index (100 GL/LB) of 93·4; No. 2 *a* has a nasal index of 44·2 and the gnathic index could not be found; No. 3 *a* has a nasal index of 53·2 and a gnathic index of 97·0. For 86 Tasmanian skulls of which the origin has not been questioned the lowest nasal index recorded is 49·0, and for 81 skulls the lowest gnathic index is 94·2. The exclusion of Nos. 1 *a* and 2 *a* would seem to be quite justified, but all the measurements of No. 3 *a* are in fair agreement with the Tasmanian means and there seems to be no sufficient reason for rejecting it*. Of the adult and sexed skulls measured by Harper and Clarke, Nos. 1—12 and 3 *a* were accepted.

(*c*) W. L. H. Duckworth: Craniological Notes on the Aborigines of Tasmania. *Journal of the Anthropological Institute.* Vol. XXXII. 1902, pp. 171—181. Individual measurements of three specimens in the Cambridge Anatomical Museum are given.

(*d*) H. Klaatsch: Bericht über einen anthropologischen Streifzug nach London und auf das Plateau von Süd-England. *Zeitschrift für Ethnologie.* Bd. XXXV. 1903, pp. 875—920. A few unusual measurements of 8 skulls in London and Paris museums are given in Tables I—V.

(*e*) Sir William Turner: The Craniology, Racial Affinities and Descent of the Aborigines of Tasmania. *Transactions of the Royal Society of Edinburgh.* Vol. XLVI. 1908, pp. 365—403. Individual measurements of 9 adult skulls in Edinburgh museums are given on p. 367; measurements of six other specimens in the Oxford University Museum are on p. 375†.

(*f*) Sir William Turner: The Aborigines of Tasmania. Part II.—The Skeleton. *Ibid.* Vol. XLVII. 1910, pp. 411—454. Measurements of one skull in the Royal Museum, Brussels, are on p. 414.

(*g*) H. Basedow: Der Tasmanierschädel, ein Insulartypus. *Zeitschrift für Ethnologie.* Bd. XLII. 1910, pp. 175—227. Individual measurements of 30 skulls in the museum of the Royal College of Surgeons are given on pp. 224—227. Most of them had been measured previously by Flower or Barnard Davis.

* R. J. A. Berry and A. W. D. Robertson in their Preliminary Communication on Fifty-three Tasmanian Crania, Forty-two of which are now recorded for the first time, *Proceedings of the Royal Society of Victoria*, Vol. XXII. (New Series), 1909, pp. 47—58, say of the specimens rejected by Harper and Clarke: "We have no hesitation whatsoever in stating that all six crania are the crania of pure-blood Tasmanians" (p. 49). And again: "Every one presents over 90 per cent. of the features so characteristically found in the skull of the Tasmanian aboriginal" (p. 50). The measurements of Nos. 1 *a* and 2 *a* do not support such emphatic statements.

† Measurements of the six Oxford specimens taken by Dr Gabriel Farmer had been published previously in: H. Ling Roth: *The Aborigines of Tasmania, with a Chapter on the Osteology by J. G. Garson.* 2nd edition. Halifax (England), 1899. The skulls were re-measured by F. H. S. Knowles and B. W. Freire-Marreco, and their results differed markedly from Farmer's. The latter supposed that 5 were ♂ and 1 ♀, but Turner gives 2 ♂ and 4 ♀. The measurements given in Turner's paper are the only ones of these skulls which have been used.

(*h*) R. J. A. Berry, A. W. D. Robertson and L. W. G. Büchner: The Craniometry of the Tasmanian Aboriginal. *Journal of the Royal Anthropological Institute.* Vol. XLIV. 1914, pp. 122—126. Individual measurements of 52 skulls are given in Table I. Sexes are not given, but the numbers correspond to those of the sexed series for which tracings have been provided by Berry and Robertson (Dioptographic Tracings in Four Normae of Fifty-two Tasmanian Crania. *Transactions of the Royal Society of Victoria.* Vol. V. 1909). Nos. 1—14 had been measured previously by Harper and Clarke (reference (*b*) above) and they are preserved in the Hobart Museum. The others are in the same Museum, in the National Museum, Melbourne, in the Launceston Museum and in private collections. No. 48 was omitted as it is young adult and un-sexed. Several of the measurements are insufficiently defined. The "orbital intervals" (Measurement No. 41 = *DC*?) of skull 24 and later numbers are obviously incorrect.

(*i*) R. Pöch: Ein Tasmanierschädel im k. k. naturhistorischen Hofmuseum. Die anthropologische und ethnographische Stellung der Tasmanier. *Mitteilungen der anthropologischen Gesellschaft in Wien.* Bd. XLVI. 1916, pp. 37—91. Detailed measurements are given on pp. 41—44.

5. *A Comparison of Australian and Tasmanian Skulls.* The ♂ and ♀ means of the pooled Tasmanian skulls are given in Table VI together with those of the Australian *A* series. All the available and sufficiently defined measurements are included in that table, and many of the means are based on such small numbers of individuals that they can be of no permanent value by themselves. They are recorded for future use when measurements of other skulls may be added to them. Enough reliable means are given, however, to establish all the more salient features of the two types: such characters are alone considered in the following comparison between them. The sexual differences between direct measurements are of the same order as those usually found, though particularly large ones are given for the Australian *A*, *L* and *U*, owing, no doubt, to the unusual growth of the superciliary ridge on the ♂ skull. There is a satisfactory agreement between the ♂ and ♀ indices and angles. From a casual inspection, it is clear that the two racial types differ quite markedly. Using 27 characters and the Australian *A* standard deviations (given in Table VII below) a ♂ coefficient of racial likeness is found of $13{\cdot}54 \pm {\cdot}18$ between the Australian *A* (113·1) and Tasmanian (42·6) means. For nine indices it is $18{\cdot}59 \pm {\cdot}30$. The most significant differences are for 100 B/F ($\alpha = 72{\cdot}33$), O_2 (55·13), 100 NB/NH' (30·18), 100 B/H' (29·36), Flower's orbital index (29·10), $G'H$ (25·84), B (25·79), LB (22·72), NH' (22·71), and F (21·38), while B', C, J, NB, Flower's O_1, fml, fmb, U, S, H', Broca's G_1 and G_2, 100 H'/F, 100 fmb/fml, 100 $G'H/J$, 100 GL/LB and Broca's palatal index are hardly, if at all, distinguished. As fewer ♀ Australian *A* standard deviations are available, the ♀ coefficient could only be computed for 16 characters. Between the *A* type Australians (61·6) and Tasmanians (29·3) it is $7{\cdot}18 \pm {\cdot}23$. The differentiated measurements are 100 B/F ($\alpha = 43{\cdot}66$), B (30·72), 100 B/H' (18·16) and 100 NB/NH' (9·28), while F, B', C, LB, J, NB, O_2, fml, fmb, U, H' and 100 fmb/fml show differences

which are only just, if at all, significant*. It is somewhat disconcerting to find a most marked difference between the ♂ orbital heights (O_2) and no appreciable difference between the ♀ values. Male mean orbital indices are usually smaller than the ♀ means for the same race, but those sexual differences are decidedly greater for the Australian and Tasmanian than for European series. It may be supposed that the greater growth of the superciliary ridges gives the ♂ skull a proportionately smaller orbital height. For the Tasmanians the ♂ O_2 is actually less than the ♀ value, which is quite unusual, so for that type the growth of the frontal bone seems to have had a greater effect on the orbits than is the case with Australians or more advanced types.

A curious fact may be noticed: the Tasmanian ♂ calvarial breadth (B) is significantly greater than the ♂ Australian value, but for all other calvarial breadths (viz. B', B'', bi-stephanic B, bi-asterionic B, external bi-orbital B and internal bi-orbital B) the Australian mean exceeds the Tasmanian and the same is found for the bi-zygomatic, orbital and palato-maxillary, but not for the nasal breadth. A similar excess in the Tasmanian parietal breadth and of defect in other breadth measurements are shown by most of the ♀ means. From the above comparison it is clear that there is a marked distinction between the Australian and Tasmanian racial types. They belong, doubtless, to the same family of races, but the breach between them is much wider than one normally found between two contiguous populations. The greatest differences found between Australian types are very small compared with their differences from the Tasmanian type.

6. *The Variability of the Australian and Tasmanian Skulls.* The only enquiries concerning the variability of sexed series of Australian and Tasmanian skulls which have hitherto been published appear to be those of Robertson†, and of Berry, Robertson and Cross‡. Although purporting to use biometric methods, these studies fail to fulfil several of the conditions which the biometrician would consider essential. The Australian material consisted of 78 ♂ skulls and 22 ♀, so the one group is just large enough to furnish constants of variability, but the other is quite inadequate. No attempt was made to test whether the sample was racially homogeneous or not. The 54 ♂ and 30 ♀ Tasmanian crania are again insufficient and it may be doubted whether the constants of an unsexed series are of any value§. The only characters dealt with were L, B and H', and the resulting

* The writer is again tempted to suggest that the ♀ coefficient, and the individual α's on which it is based, are smaller than the corresponding ♂ values owing to the fact that the ♀ means are based on smaller numbers of skulls. A direct comparison of the means would certainly not suggest that the ♀ types are more similar to one another than the ♂ types for characters other than the orbital height.

† A. W. D. Robertson: Craniological Observations on the Lengths, Breadths and Heights of a Hundred Australian Aboriginal Crania. *Proceedings of the Royal Society of Edinburgh*, Session 1910—1911, Vol. XXXI. Part I. No. 1.

‡ R. J. A. Berry, A. W. D. Robertson and K. Stuart Cross: A Biometrical Study of the Relative Degree of Purity of Race of the Tasmanian, Australian, and Papuan. *Ibid.* No. 2.

§ Standard deviations of 58 characters for an unsexed series of 52 Tasmanian skulls have also been given by R. J. A. Berry, A. W. D. Robertson and L. W. G. Büchner: The Craniometry of the Tasmanian Aboriginal. *Journal of the Royal Anthropological Institute*, Vol. XLIV. 1914, pp. 122—126.

Character*	Males: Australian A†	Males: Tasmanian	Females: Australian A†	Females: Tasmanian
C	1294·6 (146)	1264·3 (33)	1147·4 (67)	1153·8 (25)
L	187·8 (82)	182·2 (43)	175·9 (38)	174·6 (20)
L′	185·8 (26)	—	172·2 (6)	—
F	186·4 (126)	181·5 (49)	176·1 (55)	173·1 (26)
B	132·2 (162)	136·0 (60)	127·3 (75)	132·4 (36)
B′	96·1 (138)	94·0 (62)	91·7 (58)	90·1 (35)
B″	110·1 (11)	108·2 (24)	106·6 (7)	103·6 (10)
Bi-stephanic B	104·5 (62)	103·0 (22)	102·5 (23)	102·6 (14)
Bi-asterionic B	108·5 (62)	107·6 (40)	103·4 (24)	105·3 (21)
H′	133·1 (144)	130·9 (55)	125·4 (71)	125·3 (35)
H	138·1 (17)	—	130·2 (4)	—
OH	115·0 (13)	109·0 (1)	110·0 (7)	—
βOH	113·1 (10)	108·0 (1)	108·6 (5)	—
LB	102·1 (137)	98·8 (55)	95·1 (66)	92·7 (34)
Q′	300·8 (13)	300·0 (1)	290·0 (6)	—
βQ′	297·1 (27)	295·0 (1)	284·0 (12)	—
Broca's transverse arc	294·5 (49)	290·2 (40)	276·1 (17)	283·5 (17)
S	373·0 (93)	365·8 (36)	353·4 (39)	350·5 (15)
S_1	129·7 (77)	127·2 (44)	121·6 (25)	121·3 (23)
S_2	129·7 (75)	126·2 (42)	121·9 (24)	122·2 (23)
S_3	114·3 (72)	111·8 (37)	108·4 (22)	109·4 (16)
Arc lambda to inion	66·2 (40)	68·6 (9)	58·9 (10)	65·0 (7)
Arc inion to opisthion	49·1 (38)	43·5 (8)	47·9 (10)	43·3 (7)
S_1'	113·3 (11)	112·0 (1)	108·2 (5)	—
S_2'	116·6 (10)	108·0 (1)	108·0 (4)	—
S_3'	97·4 (11)	93·0 (1)	91·0 (6)	—
Chord lambda to inion	60·1 (9)	62·0 (1)	60·0 (4)	—
Chord inion to opisthion	50·3 (9)	47·0 (1)	44·6 (5)	—
U	515·6 (137)	511·3 (48)	488·8 (60)	489·5 (23)
Glabella U	530·2 (26)	502·1 (11)	495·7 (16)	489·0 (5)
PH	20·0 (1)	17·3 (4)	—	15·0 (4)
G′H	66·8 (79)	62·5 (36)	62·1 (32)	59·9 (16)
GB	93·9 (37)	88·3 (3)	89·9 (9)	91·0 (1)
J	133·6 (139)	131·0 (44)	122·5 (65)	122·0 (21)
Ext. bi-orbital B	107·9 (40)	105·4 (31)	105·2 (10)	101·5 (11)
Int. bi-orbital B	100·6 (7)	97·7 (6)	98·4 (5)	93·8 (4)
NH′	49·5 (118)	47·1 (58)	45·6 (49)	44·9 (30)
NH, R or L	52·3 (3)	—	44·0 (1)	—
NB	26·9 (120)	27·8 (57)	25·5 (51)	26·3 (29)
Ht. pyriform aperture	31·0 (56)	32·6 (10)	28·9 (27)	28·0 (13)
DC	23·5 (2)	22·4 (7)	21·0 (3)	23·6 (8)
Flower's inter-orb. B	26·2 (86)	25·3 (20)	24·7 (34)	23·8 (13)
SC	8·9 (8)	7·0 (1)	9·0 (5)	7·5 (1)

Character*	Males: Australian A†	Males: Tasmanian	Females: Australian A†	Females: Tasmanian
Flower's $\bar{O}_1$	41·3 (83)	41·1 (19)	38·7 (35)	38·8 (13)
O_1'	40·7 (33)	39·3 (40)	38·4 (17)	38·3 (18)
O_1	43·3 (11)	42·0 (1)	40·3 (4)	—
O_2	33·5 (118)	31·05 (60)	32·9 (51)	31·7 (31)
Palato-max. L	60·4 (53)	56·1 (37)	54·7 (18)	52·9 (14)
Broca's G_1	60·0 (54)	60·3 (15)	58·2 (26)	55·1 (11)
G_1'	50·9 (18)	—	50·1 (8)	—
Palato-max. B	65·8 (55)	64·8 (34)	61·7 (18)	61·7 (13)
Broca's G_2	38·2 (57)	37·8 (17)	36·2 (30)	35·4 (14)
G_2	41·1 (28)	—	39·1 (20)	—
GL	103·2 (106)	101·2 (50)	97·4 (44)	95·9 (29)
fml	35·2 (130)	35·7 (53)	33·6 (65)	34·2 (31)
fmb	30·0 (111)	29·6 (44)	28·7 (54)	28·4 (27)
Nasion-inion chord	169·3 (28)	175·0 (1)	158·6 (18)	—
Glabella-inion chord	177·0 (28)	177·7 (36)	164·9 (15)	166·3 (16)
Sub. to gb.-inion chord	—	99·7 (7)	—	100·0 (2)
100 B/L	70·8 (94)	74·2 (43)	72·2 (47)	75·1 (19)
100 B/F	71·0 (126)	75·2 (49)	72·4 (55)	76·5 (26)
100 H′/L	71·8 (91)	71·3 (37)	71·6 (46)	71·1 (19)
100 H′/F	71·6 (122)	72·2 (44)	70·9 (46)	72·3 (26)
100 H/L	72·1 (1)	—	69·8 (1)	—
100 B/H′	99·3 (156)	103·9 (55)	101·2 (79)	105·7 (34)
100 B/H	96·6 (17)	—	97·8 (4)	—
100 (B − H′)/L	− 1·0 (90)	+ 2·9 (37)	+ 0·6 (46)	+ 3·5 (19)
100 G′H/GB	72·2 (29)	67·5 (2)	69·0 (4)	—
100 NB/NH′	54·6 (132)	59·1 (57)	55·7 (58)	59·0 (29)
100 NB/NH, R or L	49·2 (3)	—	56·8 (1)	—
Flower's 100 O_2/O_1	82·0 (83)	74·3 (19)	84·1 (34)	81·3 (13)
100 O_2/O_1'	82·2 (45)	79·4 (40)	87·1 (25)	83·3 (17)
100 O_2/O_1	76·4 (11)	64·3 (1)	78·9 (5)	—
100 fmb/fml	85·0 (112)	82·1 (42)	85·0 (55)	83·3 (26)
Palato-max. index	110·0 (52)	114·8 (34)	112·9 (18)	114·7 (12)
Broca's palatal index	64·2 (54)	62·8 (15)	62·5 (26)	64·6 (11)
100 G_2/G_1'	81·8 (17)	—	77·9 (8)	—
Oc. I.	60·9 (9)	57·7 (1)	63·1 (4)	—
100 GL/LB	101·4 (101)	103·1 (48)	102·6 (47)	102·8 (28)
100 G′H/J	50·2 (75)	49·6 (23)	51·1 (30)	49·6 (7)
100 J/B	101·0 (139)	95·9 (44)	96·4 (65)	92·1 (21)
Schwalbe's index	—	55·1 (7)	—	59·4 (2)
P ∠	84°·2 (14)	—	79°·0 (2)	—
N ∠	72°·1 (44)	73°·5 (32)	71°·4 (13)	73°·2 (14)
A ∠	70°·6 (44)	69°·6 (32)	71°·7 (13)	70°·3 (14)
B ∠	37°·3 (44)	36°·9 (32)	36°·9 (13)	36°·5 (14)
Foraminal angle‡	− 4°·2 (9)	—	− 5°·0 (5)	—

* For definitions of the measurements see pp. 417—418 above.

† The numbers of skulls available for the A group means are rather larger than the sums of the corresponding numbers given for the component groups (i.e. all in Table I except the Northern Territory) because there were a few individuals who did not come from the Northern Territory, but who could not be assigned to any one of the other territorial divisions.

‡ The foraminal angle is negative if the opisthion is lower than the basion with regard to the Frankfort horizontal plane.

TABLE VII. *Constants of Variation for Australian and Tasmanian Skulls.*

Character	MALES						FEMALES		
	A Group Australians		Tasmanians		Egyptians E‡	English§	*A* Group Australians		Egyptians E‡
	Standard† Deviation	Coefficient† of Variation	Standard† Deviation	Coefficient† of Variation	Coefficient of Variation	Coefficient of Variation	Standard† Deviation	Coefficient† of Variation	Coefficient of Variation
C	120·0 ± 4·7	9·27 ± ·37	—	—	7·89 ± ·14	8·78 ± ·45	80·1 ± 4·7	6·98 ± ·41	7·08 ± ·16
F	6·29 ± ·27	3·37 ± ·14	—	—	3·14 ± ·05	3·43 ± ·12	4·79 ± ·31	2·72 ± ·17	2·71 ± ·05
L	6·67 ± ·35	3·55 ± ·19	—	—	3·09 ± ·05	3·42 ± ·12	—	—	2·66 ± ·05
B	4·95 ± ·19	3·74 ± ·14	5·32 ± ·33	3·91 ± ·24	3·43 ± ·05	4·14 ± ·17	4·54 ± ·25	3·57 ± ·20	3·34 ± ·07
B'	5·36 ± ·22	5·58 ± ·23	4·81 ± ·29	5·12 ± ·31	4·28 ± ·07	4·73 ± ·18	3·69 ± ·23	4·02 ± ·25	4·11 ± ·08
Bi-stephanic B	8·81 ± ·53	8·43 ± ·51	—	—	—	—	—	—	—
Bi-asterionic B	5·69 ± ·34	5·24 ± ·32	—	—	—	—	—	—	—
H'	4·94 ± ·20	3·71 ± ·15	4·76 ± ·31	3·64 ± ·23	—	3·90 ± ·17	5·30 ± ·30	4·23 ± ·24	—
H	—	—	—	—	3·75 ± ·06	4·09 ± ·23	—	—	3·39 ± ·07
LB	4·34 ± ·18	4·25 ± ·17	4·13 ± ·27	4·18 ± ·27	3·90 ± ·06	4·47 ± ·20	4·16 ± ·24	4·37 ± ·26	3·65 ± ·07
U	16·50 ± ·67	3·20 ± ·13	—	—	2·65 ± ·04	3·00 ± ·13	11·05 ± ·68	2·26 ± ·14	2·35 ± ·05
S	13·62 ± ·67	3·65 ± ·18	—	—	3·36 ± ·05	3·75 ± ·16	—	—	2·92 ± ·06
S_1	6·81 ± ·37	5·25 ± ·29	—	—	4·88 ± ·08	5·00 ± ·19	—	—	4·56 ± ·09
S_2	7·43 ± ·41	5·73 ± ·32	—	—	5·77 ± ·09	6·24 ± ·25	—	—	5·19 ± ·10
S_3	6·17 ± ·35	5·40 ± ·30	—	—	5·91 ± ·10	6·50 ± ·27	—	—	6·06 ± ·12
$G'H$	4·21 ± ·23	6·30 ± ·34	—	—	5·90 ± ·10	6·31 ± ·33	—	—	5·64 ± ·11
J	6·07 ± ·25	4·54 ± ·18	—	—	3·55 ± ·06	3·69 ± ·27	5·84 ± ·35	4·77 ± ·28	3·62 ± ·08
NH'	3·14 ± ·14	6·34 ± ·28	3·47 ± ·22	7·37 ± ·46	—	6·75 ± ·45	—	—	—
NH	—	—	—	—	5·65 ± ·09	6·03 ± ·31*	—	—	5·31 ± ·10
NB	2·00 ± ·07	7·43 ± ·33	1·69 ± ·11	6·06 ± ·38	7·27 ± ·12	8·17 ± ·43	1·89 ± ·13	7·41 ± ·50	6·98 ± ·14
Ht. pyriform aperture ...	2·91 ± ·19	9·39 ± ·60	—	—	—	—	—	—	—
Flower's O_1	2·56 ± ·13	6·20 ± ·33	—	—	—	—	—	—	—
O_1'	—	—	—	—	—	4·20 ± ·22	—	—	—
O_2	2·08 ± ·09	6·21 ± ·27	2·19 ± ·13	6·70 ± ·41	5·56 ± ·09*	6·70 ± ·34*	2·14 ± ·14	6·50 ± ·43	5·62 ± ·11
Flower's inter-orb. B ...	2·06 ± ·11	7·86 ± ·41	—	—	—	—	—	—	—
DC	—	—	—	—	—	9·73 ± ·51	—	—	—

Broca's G_1	4·54 ± ·29	7·57 ± ·49	—	—	—	—	—	—	—
G_1	—	—	—	—	6·70 ± ·11	5·59 ± ·33	—	—	6·26 ± ·13
Palato-max. L	3·93 ± ·26	6·51 ± ·43	—	—	—	—	—	—	—
Broca's G_2	2·93 ± ·19	7·67 ± ·49	—	—	—	—	—	—	—
G_2	—	—	—	—	6·78 ± ·12	7·76 ± ·51	—	—	6·83 ± ·15
Palato-max. B	3·31 ± ·21	5·03 ± ·32	—	—	—	—	—	—	—
GL	5·88 ± ·27	5·70 ± ·37	5·40 ± ·36	5·34 ± ·36	5·10 ± ·08	5·66 ± ·34	—	—	4·49 ± ·09
fml	2·52 ± ·11	7·16 ± ·30	2·82 ± ·18	7·90 ± ·52	6·95 ± ·11	8·15 ± ·36	2·28 ± ·13	6·79 ± ·40	6·37 ± ·12
fmb	2·17 ± ·10	7·23 ± ·33	—	—	7·18 ± ·11	7·19 ± ·33	2·25 ± ·15	7·84 ± ·51	6·96 ± ·14
100 B/L	3·88 ± ·19	5·48 ± ·27	—	—	3·57 ± ·06	4·62 ± ·19	—	—	3·34 ± ·07
100 B/F	2·89 ± ·12	4·07 ± ·17	—	—	—	—	2·48 ± ·16	3·43 ± ·22	—
100 H'/L	3·70 ± ·19	5·15 ± ·26	—	—	—	4·72 ± ·21	—	—	—
100 H'/F	3·09 ± ·13	4·32 ± ·19	—	—	—	—	—	—	—
100 H/L	—	—	—	—	4·06 ± ·07	4·65 ± ·26	—	—	3·90 ± ·08
100 B/H'	5·41 ± ·21	5·45 ± ·21	4·05 ± ·26	3·90 ± ·25	—	4·80 ± ·21	5·15 ± ·28	5·09 ± ·27	—
100 B/H	—	—	—	—	4·14 ± ·07	4·35 ± ·24	—	—	3·88 ± ·08
100 NB/NH'	5·17 ± ·21	9·47 ± ·40	4·75 ± ·30	8·04 ± ·51	—	—	4·76 ± ·30	8·55 ± ·54	—
100 NB/NH	—	—	—	—	8·08 ± ·13	9·22 ± ·49*	—	—	7·90 ± ·16
Flower's orb. index ...	5·61 ± ·29	6·84 ± ·36	—	—	—	—	—	—	—
100 O_2/O_1'	—	—	—	—	—	7·45 ± ·40	—	—	—
Broca's palatal index ...	6·98 ± ·45	10·87 ± ·71	—	—	—	—	—	—	—
100 G_2/G_1	—	—	—	—	8·67 ± ·16	7·04 ± ·50	—	—	8·40 ± ·19
Palato-max. index ...	7·74 ± ·51	7·04 ± ·47	—	—	—	—	—	—	—
100 fmb/fml	5·83 ± ·26	6·86 ± ·31	—	—	6·86 ± ·11	7·07 ± ·33	6·50 ± ·42	7·65 ± ·49	6·69 ± ·13
100 J/B	4·59 ± ·19	4·54 ± ·18	—	—	—	—	4·32 ± ·26	4·48 ± ·27	—
100 $G'H/J$	3·45 ± ·19	6·87 ± ·38	—	—	—	—	—	—	—
100 GL/LB	4·12 ± ·20	4·06 ± ·19	—	—	—	—	—	—	—

* These measurements were taken on the left side.

† The numbers of skulls on which the Australian A and Tasmanian constants of variation were calculated are precisely the same as the corresponding ones for the means given in Table VI.

‡ Quoted from: Karl Pearson and Adelaide G. Davin: On the Biometric Constants of the Human Skull. *Biometrika*, Vol. XVI. 1924, pp. 338—9.

§ Quoted from: Beatrix G. E. Hooke: A Third Study of the English Skull with special reference to the Farringdon Street Crania. *Ibid.* Vol. XVIII. 1926, p. 40.

indices 100 B/L, 100 H'/L and 100 H'/B, and it is not to be expected that any reliable determination of the variability of the skull as a whole will be given by considering only six correlated measurements. In dealing with the Tasmanian skulls, Flower's ophryo-occipital lengths (F) were pooled with others measured from the glabella.

Standard deviations and coefficients of variation of the Australian A ♂ and ♀ and the Tasmanian ♂ series dealt with in the present paper are in Table VII. Constants are given for all characters for which 50 or more skulls had been measured, and none quoted are based on fewer than that number of individuals. The Australian skulls of the A type were drawn from such an enormous area that they may be expected to show a high variability. An unusually great variation might be an inherent characteristic of the race, or it might be occasioned by the conditions to which it had been subjected, or again, it might be due to racial heterogeneity. The personal equations of the different workers who measured the series may also have influenced the observed variation to some extent. By dividing the population into groups (Section 3 above) the conclusion was reached that the skulls from all parts of the continent, except the Northern Territory, belonged to a single race. That assumption has been made in spite of the fact that a significant, though slight, difference was found between a group comprising crania from Western Australia, South Australia and Victoria, and another representing Queensland and New South Wales. A further test of the validity of that assumption will be given

TABLE VIII.

Standard Deviations of Groups of Australian Male Skulls.

		C	F	B	B'	H'	LB	U	J
W. Australia, S. Australia and Victoria	σ	118·1 ± 6·4	6·36 ± ·35	4·75 ± ·19	5·30 ± ·30	4·90 ± ·26	4·32 ± ·23	17·42 ± 1·01	4·69 ± ·26
	No. of Skulls	77	74	87	69	83	79	68	72
Australian A	σ	120·0 ± 4·7	6·29 ± ·27	4·95 ± ·19	5·36 ± ·22	4·94 ± ·20	4·34 ± ·18	16·50 ± ·67	6·07 ± ·25
	No. of Skulls	146	126	162	138	144	137	137	139

		NH'	NB	O_2	fml	$100\,B/F$	$100\,H'/F$	$100\,B/H'$	$100\,NB/NH'$
W. Australia, S. Australia and Victoria	σ	2·83 ± ·16	2·07 ± ·12	2·07 ± ·12	2·57 ± ·14	3·06 ± ·17	3·57 ± ·20	5·27 ± ·27	5·00 ± ·28
	No. of Skulls	68	68	63	73	75	72	85	71
Australian A	σ	3·14 ± ·14	2·00 ± ·07	2·08 ± ·09	2·52 ± ·11	2·89 ± ·12	3·09 ± ·13	5·41 ± ·21	5·17 ± ·21
	No. of Skulls	118	120	118	130	126	122	156	132

by comparing the variability of the former group with the variability of the *A* group as a whole. Male standard deviations for 18 characters are given in Table VIII.

It will be seen that the variabilities of the combined Western Australia, South Australia and Victoria series are almost identically the same as those of the larger *A* group, of which they form approximately one-half. Of the 18 characters compared the standard deviations of 12 are increased by the addition of the Queensland and New South Wales skulls, the other six are reduced. When the σ's of the two independent sub-samples are compared directly, it is found that the Western Australia, South Australia and Victoria series is more variable than that from Queensland and New South Wales for 11 characters and less variable for 7. Only 3 of the 18 differences are greater than 2·5 times their probable errors. For J the ratio is 3·6, for $100\,H'/F$, 3·1, and for NH', 2·57. Judging from all the characters, it is reasonable to conclude that the Western Australia, South Australia and Victoria series is not appreciably less variable than the total population of *A* type skulls.

From the constants given in Table VII it should be possible to obtain an accurate idea of the variabilities of the Australian *A* and Tasmanian types relative to one another and to other racial populations. The table includes the coefficients of variation of the long Egyptian *E* series of 26th—30th Dynasty skulls, provided by Pearson and Davin, and of a London series from a 17th century cemetery in Farringdon Street, measured by Hooke. The former is remarkable on account of its small variation, and the constants of the latter may be considered typical for a racially homogeneous European population. The *A* group Australians have 7 ♂ coefficients of variation greater than the corresponding Tasmanian ones and 4 less, so that the population which is scattered over the larger area appears to be rather more variable than the restricted island one. Both are decidedly more variable than the Egyptian series. The Australians only have a lesser variation than the Egyptians for the characters S_2 and S_3 and the Tasmanians only for NB. The English coefficients of variation exceed the Australian for 14 out of 24 characters and the Tasmanian for 7 out of 10 characters. Finally, it may be noticed that the English coefficients exceed the Egyptian for 26 out of 28 characters. Arranging the types in order of their variability, as judged by all the cranial measurements available, we have the increasing sequence: Egyptian—Tasmanian—Australian *A*—English. The population which extends over the greater part of a continent is less variable than a collection of individuals from a single London cemetery. From the coefficients of variation, the relative variation of the ♂ Australian skull is seen to be greater than that of the ♀ skull, but the excess of one over the other is not so evident as is that of the ♂ over the ♀ variation in the case of the Egyptians. The ♀ Australian constants exceed the corresponding Egyptian ones in 9 cases out of 12.

When computing coefficients of racial likeness between groups of Australian skulls (Section 3 above) the Egyptian *E* standard deviations were used. If they were replaced by the more appropriate Australian *A* values which are now available, all the coefficients would be slightly reduced. Only one was re-calculated.

Between the Western Australia, South Australia and Victoria ♂ group and the Queensland and New South Wales ♂ group a coefficient was found of $1{\cdot}58 \pm {\cdot}21$, using Egyptian standard deviations (Table IV). For the same 19 pairs of means and Australian A standard deviations the coefficient is $1{\cdot}25 \pm {\cdot}21$.

7. *Additional Measurements of Australian and Tasmanian Skulls.* It might be thought that the foregoing analysis of the measurements of Australian and Tasmanian skulls provided by a considerable number of different workers would be sufficient, in itself, to establish the main characters of the craniological types. The evidence appeared to be perfectly consistent. After excluding a considerable part of the material, there was good reason to believe that the remainder had been dealt with by strictly comparable methods of technique. We are able to test the reliability of the results by comparing the pooled means with constants given for series not hitherto considered. The majority of these are unsexed series.

Of the Australian skulls in Paris museums, 10 only have been hitherto used. They are represented by a few means of 7 ♂ and 3 ♀ specimens given by Pruner-Bey in his paper of 1865. As the localities from which they had been obtained were not stated, the measurements could only be used in compiling the groups from unknown localities given in Table III above. In 1879 Broca provided the means of a series of 27 unsexed Australian skulls to which he had access in Paris*. In the same paper there are means of 35 unsexed Tasmanian skulls. Of that series, Klaatsch's unusual measurements of four specimens (see p. 426 (d)) and others of 1 ♀ given in the *Crania Ethnica* have alone been used in the present paper. Comparison is made with the pooled data in Table IX. The Australian skulls in Paris may have included a few from the Northern Territory, but there can be little doubt that the majority belong to the A group. Cauvin's last paper on Australian skulls was published in 1883†. It provides the individual cephalic, nasal and orbital indices of 50 unsexed skulls, one having come from the Northern Territory and the remainder from other known parts of the continent. Of that number, two were in the *Musée Broca*, and it is probable that Broca himself had measured them before 1879, but the others appear to have been in museums or private collections in Australia. Individual indices of 23 of the series had been given by Cauvin at an earlier date (see p. 419 above) and as they were sexed it was possible to use them when compiling the Australian A means. As they could not be separated from the others, all 49 were pooled to give the values in Table IX. There is a most satisfactory agreement between the groups represented in that table. All the absolute measurements of Broca's unsexed series are intermediate between the corresponding ♂ and ♀ means (with the unimportant exceptions of the Tasmanian B', bi-stephanic B and O_1') and they incline more towards the ♂ than the ♀ means.

* Paul Broca: Méthode des moyennes. Etude des variations craniométriques et de leur influence sur les moyennes; détermination de la série suffisante. *Bulletins de la Société d'Anthropologie de Paris*, Tome II, 3ième série, 1879, pp. 756—820.

† Cauvin: Sur les races de l'Océanie (Analyse). *Bulletins de la Société d'Anthropologie de Paris*, Tome VI, 3ième série, 1883, pp. 245—256. The reference to Cauvin's earlier paper is given on p. 419 above.

TABLE IX.

Means of Sexed and Unsexed Series of Australian and Tasmanian Skulls.

Character	Australians				Tasmanians		
	Pooled Australian *A*. Male	Broca's unsexed series	Cauvin's unsexed series	Pooled Australian *A*. Female	Pooled Tasmanians. Male	Broca's unsexed series	Pooled Tasmanians. Female
L	187·9 (82)	182·4 (27)	—	175·9 (38)	182·2 (43)	180·4 (35)	174·6 (20)
B	132·2 (162)	130·4 (27)	—	127·3 (75)	136·0 (60)	135·5 (35)	132·4 (36)
B′	96·1 (138)	93·7 (27)	—	91·7 (58)	94·0 (62)	94·0 (35)	90·1 (35)
Bi-stephanic *B* ...	104·5 (62)	102·7 (27)		102·5 (23)	103·0 (22)	104·8 (35)	102·6 (14)
H′	133·1 (144)	131·8 (27)	—	125·4 (71)	130·9 (55)	128·5 (35)	125·3 (35)
LB	102·1 (137)	98·0 (27)	—	95·1 (66)	98·8 (55)	96·1 (35)	92·7 (34)
Ext. bi-orbital *B*	107·9 (40)	106·8 (27)	—	105·2 (10)	105·4 (31)	104·5 (35)	101·5 (11)
J	133·6 (139)	127·3 (27)	—	122·5 (65)	131·0 (44)	125·7 (35)	122·0 (21)
NH′	49·5 (118)	47·0 (27)	—	45·6 (49)	47·1 (58)	45·6 (35)	44·9 (30)
NB	26·9 (120)	26·1 (27)	—	25·5 (51)	27·8 (57)	26·6 (35)	26·3 (29)
O_1'	40·7 (33)	40·4 (27)	—	38·4 (17)	39·3 (40)	37·7 (35)	38·3 (18)
100 *B*/*L*	70·8 (94)	{71·5 (27)}[1]	72·2 (48)	72·2 (47)	74·2 (43)	{75·1 (35)}	75·1 (19)
100 *H′*/*L*	71·8 (91)	{72·2 (27)}	—	71·6 (46)	71·3 (37)	{71·2 (35)}	71·1 (19)
100 *B*/*H′*	99·3 (156)	{98·9 (27)}	—	101·2 (79)	103·9 (55)	{105·5 (35)}	105·7 (34)
100 *NB*/*NH′* ...	54·6 (132)	{55·6 (27)}	56·0 (49)	55·7 (58)	59·1 (57)	{58·4 (35)}	59·0 (29)
100 O_2/O_1' ...	82·2 (45)	—	83·0 (49)	87·1 (25)	79·4 (40)	—	83·3 (17)

[1] The indices in curled brackets were calculated from the means of the component lengths instead of from individual indices.

The differences between the indices are statistically insignificant. This welcome evidence encourages us to place greater confidence in the Australian *A* and Tasmanian constants for the two sexes which have already been determined.

In 1897 Krause published individual measurements of 170 native skulls which were preserved at that time in various museums and private collections in Australia*. The district from which each had come was known in the case of all except four of them and of the remainder 21 were from the Northern Territory. The sex of 21 ♂ and 15 ♀ individuals was known with certainty, but unfortunately no attempt was made to sex the others†. Measurements were taken in accordance with the instructions of the *Frankfurter Verständigung*. Excluding the juvenile specimens and all from the Northern Territory and unknown localities, we find the unsexed means given in Table X. In 1910 Robertson dealt with 100 Australian skulls in the Museum of the Anatomical Department of the University of Melbourne‡; 96 were from States other than the Northern Territory and the provenance of the other four was not known. The number of measurements taken

* Wilhelm Krause: Australische Schädel. *Zeitschrift für Ethnologie. Verhandlungen der Berliner Gesellschaft für Anthropologie, Ethnologie und Urgeschichte*, Jahrgang 1897, pp. 508—558.

† On p. 512 Krause gives a list of 16 of the unsexed skulls which were ♀ in his opinion, but he made no attempt to distinguish the specimens which were ♀?.

‡ A. W. D. Robertson: Craniological Observations on the Lengths, Breadths and Heights of a Hundred Australian Aboriginal Crania. *Proceedings of the Royal Society of Edinburgh*, Session 1910—1911, Vol. XXXI. Part I. No. 1.

(viz. L, B and H' and three resulting indices) was too small to justify any of the wider generalisations which are broached in the paper. The lengths were taken in accordance with the instructions of the Monaco International Conference of 1906. Of the 100 skulls, 78 were supposed ♂ and 22 ♀, so there is an unusual preponderance of the one sex over the other. That might be explained by the fact that they were selected from a larger number of specimens, but it is not stated whether that was so or not. It is probable that all, or nearly all, of the 23 skulls described by Cauvin, which we have used in computing the Australian A pooled means, were re-measured by Krause and some of them may have been included in Robertson's material. Some others of Robertson's series may also have been measured previously by Krause. It is not possible to decide such points from the published writings of these authors. Turning to Table X, it will be seen that there is an excellent agreement between the indices of Robertson's series and the pooled Australian A means, but the absolute measurements show differences that are clearly significant. The ♂ L, B and H' of the Melbourne series are all decidedly smaller than the pooled ♂ means and the sexual differences for those characters are also smaller and most markedly so in the case of B. It is only possible to reconcile this evidence by supposing that part of the material compared had been incorrectly sexed. The assumption that some of the skulls in the Melbourne Museum which were judged ♂ are in reality ♀ would lead to a satisfactory agreement. That seems to be the most reasonable hypothesis: a transformation of that kind would give more usual sexual differences and the sex ratio would be more in accordance with that which might be expected for a museum collection.

The measurements of Krause's unsexed series are only given (Table X) for the characters for which reliable pooled means are available. Of the 15 absolute measurements, 9 are intermediate between the ♂ and ♀ values of the A group while they are nearer to the ♂ than to the ♀ mean. One other (J) is also intermediate, but it is rather nearer to the ♀ mean. Four other characters (NB, O_2, fml and fmb) have unsexed means 0·1 mm. greater than the corresponding ♂ values. But the calvarial breadth of Krause's series is decidedly smaller than the ♀ value for the pooled group. All the indices and angles agree well except the three involving B (viz. $100\ B/L$, $100\ B/H'$ and $100\ J/B$) and the differences for the latter are perfectly significant in virtue of the considerable numbers on which the means are based. The peculiarly small value of B is alone responsible for the discordance of Krause's series and it is not associated with small values of such related characters as B', U and J. This comparison suggests that the calvarial breadths used in computing the pooled mean and Krause's calvarial breadth had not been determined in the same way. If that could be demonstrated, then it might be possible to reconcile the two series, but no difference can be found between the methods by which the calvarial breadths were said to have been taken. Before attempting to find a way out of this *impasse* some further evidence may be considered. In compiling the pooled Australian means given in Table VI a few sources were overlooked. The following papers contain further measurements of ♂ adult skulls from localities other than the Northern Territory:

TABLE X.

Further Mean Measurements of Series of Australian Skulls.

Character	♂			♂ + ♀		♀	
	Pooled A Group: 1st Sample	Pooled A Group: 2nd Sample	Robertson's Series	Krause's Series	Robertson's Series	Pooled A Group	Robertson's Series
L ...	187·8 (82)	185·8 (28)	183·6 (78)	184·6 (135)	181·8 (100)	175·9 (38)	175·6 (22)
B ...	132·2 (162)	133·7 (14)	130·6 (78)	124 8 (135)	130·2 (100)	127·3 (75)	128·7 (22)
B' ...	96·1 (138)	96·2 (10)	—	95·8 (134)	—	91·7 (58)	—
H' ...	133·1 (144)	137·8 (6)	131·1 (78)	132·1 (95)	129·7 (100)	125·4 (71)	124·5 (22)
LB ...	102·1 (137)	101·4 (28)	—	99·6 (128)	—	95·1 (66)	—
S ...	373·0 (93)	376·5 (13)	—	368·6 (128)	—	353·4 (39)	—
U ...	515·6 (1[illegible])	514·5 (13)	—	502·6 (135)	—	488·8 (60)	—
J ...	133·6 (1[illegible])	134·6 (14)	—	127·4 (100)	—	122·5 (65)	—
$G'H$ ...	66·8 (79)	68·7 (13)	—	65·4 (122)	—	62·1 (32)	—
NB ...	26·9 (120)	27·9 (23)	—	27·0 (128)	—	25·5 (51)	—
GL ...	103·2 (106)	101·1 (22)	—	100·4 (97)	—	97·4 (44)	
fml ...	35·2 (132)	35·5 (24)		35·3 (125)	—	33·6 (65)	—
fmb ...	30·0 (111)	30·0 (23)	—	30·1 (124)	—	28·7 (54)	—
O_2 ...	33·5 (118)	33·7 (23)	—	33·6 (105)	—	32·9 (51)	—
C ...	1294·6 (146)	1282·1 (14)	—	1242·8 (47)	—	1147·4 (67)	—
$100\,B/L$ ...	70·8 (94)	71·0 (28)	71·3 (38)	{67·6 (135)}	71·8 (100)	72·2 (47)	73·4 (22)
$100\,H'/L$ ...	71·8 (91)	72·1 (5)	71·5 (78)	{71·6 (95)}	71·4 (100)	71·6 (46)	71·0 (22)
$100\,B/H'$ ...	99·3 (156)	97·8 (6)	{99·6 (78)}[1]	{94·5 (95)}	{100·4 (100)}	101·2 (79)	{103·4 (22)}
$100\,J/B$ ...	101·0 (139)	100·6 (14)	—	{102·0 (100)}	—	96·4 (65)	—
$100\,fmb/fml$	85·0 (112)	84·2 (23)	—	{85·3 (124)}	—	85·0 (55)	—
$100\,G'H/J$ ...	50·2 (75)	51·1 (13)	—	{51·4 (100)}	—	51·1 (30)	—
$100\,GL/LB$	101·4 (101)	100·0 (22)	—	{100·8 (97)}		102·6 (47)	—
$N\angle$ ...	72°·1 (44)	70°·2 (21)	—	{71°·4 (97)}	—	71°·4 (13)	—
$A\angle$ ...	70°·6 (44)	71°·0 (21)	—	{70°·4 (97)}	—	71°·7 (13)	
$B\angle$ ...	37°·3 (44)	38°·8 (21)	—	{38°·2 (97)}	—	36°·9 (13)	—

[1] The indices in curled brackets were calculated from the means of the component lengths.

(*a*) K. Brackebusch: *Die Australierschädel der Sammlung des anatomischen Instituts.* Inaugural-Dissertation. Göttingen, 1905. This paper gives a large number of measurements of 19 ♂ skulls. Of that number 16 figure in Spengel's catalogue of 1874 (reference (*b*), p. 419), but additional measurements given by Brackebusch can now be used. All of No. XVI may be included as the measurements of this skull were only used previously in the group from unknown localities.

(*b*) J. T. Wilson: Craniometry. Abstract Report on the Craniology of Australian Aborigines, pp. 96—100 in John Fraser: *The Aborigines of New South Wales.* Published by authority of the New South Wales Commissioners for the World's Columbian Exposition, Chicago, 1893. Measurements of 4 ♂ skulls in the Australian Museum, Sydney, are given. They may have been measured previously by Cauvin and subsequently by Krause.

(*c*) A. Ecker: *Die anthropologischen Sammlungen Deutschlands.* Freiburg section, 1878. Nos. 2, 3 and 5 (p. 53) had been omitted in error when compiling

the first series of pooled means. No. 11 (p. 54) was only included then in the group from unknown localities, so it may be re-used now.

(*d*) H. Basedow: Der Tasmanierschädel, ein Insulartypus. *Zeitschrift für Ethnologie.* 42 Jahrgang, 1910, pp. 176—227. No. 29 (pp. 212—215) was only included previously in the group from unknown localities.

(*e*) *Craniological Data from the Indian Museum, Calcutta.* Ethnographic Survey of India. Calcutta, 1909. Measurements of one skull are given on p. 52.

This small group of additional material leads to the ♂ means (called the Pooled *A* Group: 2nd Sample) given in column 3 of Table X. It will be seen that they are in very close agreement with those previously found. Using 15 characters for which Australian *A* standard deviations are available (viz. all given in Table X except H', S, GL, 100 H'/L, 100 B/H', 100 J/B, 100 $G'H/J$, $N\angle$, $A\angle$ and $B\angle$) the coefficient of racial likeness between the 1st (120·4) and 2nd (20·0) samples is $-0{\cdot}05 \pm {\cdot}24$. The mean B of the 2nd sample actually exceeds that of the 1st, though by an insignificant amount.

Professor Wilhelm Krause was one of the German anthropologists who ratified the craniometric scheme known as the *Frankfurter Verständigung**. In measuring the Australian skulls he followed that system and no mention whatever is made of a deviation from it†. The only calvarial breadth defined is the "Grösste Breite: senkrecht zur Sagittalebene, gemessen mit dem Schiebezirkel, wo sie sich findet, nur nicht am Zitzenfortsatz, Processus mastoides, oder an der hinteren Temporalleiste; die Messpunkte müssen in derselben Horizontalebene liegen" (*loc. cit.* p. 2, No. 4). That greatest calvarial breadth measured on either the parietal bones or the temporal squamae is the one defined by all the above cited craniometricians with two exceptions. Flower determined the greatest parietal breadth only and he was followed by Basedow. Their mean B for 60 ♂ skulls of the Australian *A* type is 132·4 and when the parieto-squamous breadths of 102 other ♂ Australian *A* skulls is added to that number the mean is 132·2. It is clear that, for the statistically small numbers dealt with, the pooling of the breadths determined by the two methods will be quite a safe procedure and that was done in compiling the means given in Table VI. It may be inferred that the greatest calvarial breadth of the Australian skull is found almost invariably on the parietal bones, as normally for modern racial types‡. Strangely enough Krause failed to notice the discordance between his measurements for the race and others previously given. He gives a mean cephalic index of 69·7 for 155 unsexed skulls, but that is calculated for the glabella horizontal length (L') which is, for ♂ mean values, several millimetres less than the greatest glabella-occipital length (L) according to his own measurements. Using the latter, and excluding the Northern Territory skulls

* Verständigung über ein gemeinsames craniometrisches Verfahren. *Archiv für Anthropologie,* Band xv. 1884, pp. 1—8.

† Krause also followed the *Frankfurter Verständigung* without modification in the third part of the Berlin section of *Die anthropologischen Sammlungen Deutschlands,* 1898.

‡ Pöch (*loc. cit.* p. 72) found for adult ♂ Australian skulls that the greatest breadth fell on the parietal bones in 19 cases out of 21.

and a few immature specimens, the mean of the usual cephalic index (100 B/L) is 67·6 for 135 unsexed skulls. Brackebusch (*loc. cit.* p. 14) and others have wrongly compared Krause's value of 69·7 with the mean cephalic indices given for other series of Australian skulls and have hence failed to notice the wide discrepancy between the records. On the one hand we have 19 anthropologists providing measurements of over 300 skulls which are in close agreement when comparison is restricted to those which are said to have been taken by precisely similar methods of technique; on the other hand a single anthropologist has given measurements which agree perfectly well with those of the other workers except in the case of one chord. It is extremely probable that the discordance does not indicate a racial difference between the series, but a difference between the ways in which that chord was measured. The most reasonable explanation seems to be that Krause did not measure the parieto-squamous but some other calvarial breadth*.

8. *Conclusions.* The main purpose of this paper has been to place on record some reliable statistical constants for the Australian and Tasmanian skulls. It appears that only two racial types can be found among the natives of the whole continent of Australia. The first is probably confined to the Northern Territory and it differs distinctly from the second—the A type, which is spread over all the remaining parts—by having a smaller brain-box and slightly differing proportions of height to length and breadth. No differences can be found between the facial skeletons of the two. The Australian races are as similar to one another as contiguous races are usually found to be. The Northern Territory type is not represented by a large enough sample to furnish statistical constants of any permanent value. The A type is to-day represented by a small population which is spread homogeneously, as far as we can tell, over an enormous area. A similar wide extension of a single racial variety is found in China for a large and dense population†, and the condition is extremely dissimilar to that which is found in Europe. The Australian A type is represented by an adequately large sample. The variability of the population is distinctly greater than that of the standard series Egyptian of 26th—30th Dynasty skulls, but rather less than that of a 17th century London series from a single cemetery. In general the more primitive existing races of man show smaller variability than advanced races, but it must be remembered that in the recent history of the world the former have been more restricted by inhabiting islands and more inaccessible regions and that they have had fewer opportunities of establishing contact with neighbouring races. The

* If Krause did not measure the parieto-squamous breadth, then his must have been less than the bi-parietal and greater than the maximum frontal and bi-asterionic breadths. I have only been able to find one measurement which would satisfy these conditions and which was fairly commonly used before 1897. Hermann Welcker (*Untersuchungen über Wachsthum und Bau des menschlichen Schädels. Erster Theil.* Leipzig, 1862, p. 24) defines a breadth taken between the points of intersection of a horizontal and a transverse circumference. Rudolf Virchow (Die altnordischen Schädel zu Kopenhagen. *Archiv für Anthropologie*, Bd. IV. 1870, p. 60) mentions a breadth taken between the *tubera parietalia*, but he uses the maximum bi-parietal breadth.

† See *Biometrika*, Vol. XVIII. 1926, pp. 415—416.

evidence is not inconsistent with the hypothesis that all races have potentially the same inherent faculty of variation, while the actual variability of each is determined by the conditions, whether ethnic or environmental, to which it has been subjected. The measurements of Australian skulls which have been published by 20 different anthropologists are in very satisfactory agreement except in one particular. An unsexed series described by Krause has means which are consistent with the others except for one dimension—the calvarial breadth. The present writer has been unable to escape from the conclusion that Professor Krause had measured that diameter in some most unusual way and not by the method which he professes to have used, but any more satisfactory explanation of the discordance would be greatly welcomed.

It is not possible to test the homogeneity of the native population of Tasmania by dividing into groups the sample of skulls of which measurements have been provided. The variability indicated is rather less than that of the Australian *A* type, but still distinctly greater than that of the Egyptian population. The Tasmanian and Australian skulls are quite widely differentiated and perfectly significant differences are found between the cephalic indices and several other calvarial and facial measurements. It cannot be said that the Tasmanian is nearer to the one Australian type than to the other. All three doubtless belong to the same family of prognathous Oceanic races. A comparison with the mean measurements of other types* shows that they have one or two extreme measurements, but the greater number of their characters fall well within the inter-racial distributions. The Australian *A* skull has a peculiarly small index 100 L/S dependent on its low vault and retreating frontal bone, but it is not quite extreme and its height-length index (100 H'/L) is less remarkable. But for the same type the proportion of the bi-asterionic to the calvarial breadth is apparently greater than for any other modern race. The Tasmanian nasal index (100 NB/NH') also seems to be the greatest recorded. Both Australians and Tasmanians have peculiarly small foraminal lengths (*fml*). The view advocated by some writers that the two races are distinguished from all other modern races on account of their ultra-primitive characters can receive no support whatever from a comparison of these measurements.

* The material for other races used in this comparison was that collected by the present writer in: Studies of Palaeolithic Man. II. *Annals of Eugenics*, Vol. II. 1927, pp. 318—380.

MISCELLANEA.

I.

Further Contributions to the Theory of Small Samples.

(Vol. XVII. pp. 176 *et seq.*)

I regret to say that an error has crept into the above paper on p. 179. In my Lecture Notes from which this paper was printed the equation line 3 is correctly given as

$$F\left(\tfrac{1}{2}n, \tfrac{1}{2}n, \tfrac{1}{2}(n-1), \rho^2\right) = (1-\rho^2)^{-\frac{1}{2}(n+1)}\, F\left(-\tfrac{1}{2}, -\tfrac{1}{2}, \tfrac{1}{2}(n-1), \rho^2\right),$$

and not equal to $(1-\rho^2)^{-\frac{1}{2}(n+1)}\, F\left(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}(n-1), \rho^2\right)$ as printed on p. 179.

Further the general term in (ix) is correctly given as it is below:

$$H(\rho^2) = \left(1 + \frac{\rho^2}{1\,!}\,\frac{1}{2n-2} + \frac{\rho^4}{2\,!}\,\frac{1}{(2n-2)(2n+2)} + \dots \right.$$
$$\left. + \frac{\rho^{2p}}{p\,!}\,\frac{1^2 . 3^2 . 5^2 \dots (2p-3)^2}{(2n-2)(2n+2)\dots(2n+4p-6)} + \dots \right).$$

Very unfortunately the general term was overlooked in the next step in the work, and the

TABLE I (*Biometrika*, Vol. XVII. p. 182, revised).

Values of the Correlation of Standard Deviations in Samples for Different Values of Variate Correlation in a Sampled Normal Population.

Values of ρ.

n	·0	·1	·2	·3	·4	·5	·6	·7	·8	·9	1·0
2	·0000	·0088	·0352	·0796	·1421	·2239	·3261	·4501	·5988	·7772	1·0000
3	·0000	·0092	·0367	·0828	·1479	·2326	·3376	·4642	·6141	·7905−	1·0000
4	·0000	·0094	·0375+	·0846	·1510	·2371	·3435−	·4712	·6216	·7965+	1·0000
5	·0000	·0095−	·0380	·0857	·1528	·2398	·3470	·4754	·6258	·7999	1·0000
6	·0000	·0096	·0383	·0864	·1540	·2415+	·3493	·4783	·6285−	·8019	1·0000
7	·0000	·0096	·0386	·0870	·1549	·2428	·3510	·4799	·6304	·8032	1·0000
8	·0000	·0097	·0388	·0873	·1556	·2438	·3522	·4813	·6317	·8042	1·0000
9	·0000	·0097	·0389	·0876	·1561	·2445−	·3531	·4823	·6327	·8050−	1·0000
10	·0000	·0097	·0390	·0879	·1565−	·2451	·3538	·4832	·6335+	·8055+	1·0000
11	·0000	·0098	·0391	·0881	·1568	·2455+	·3544	·4838	·6342	·8060	1·0000
12	·0000	·0098	·0392	·0882	·1571	·2459	·3549	·4844	·6347	·8064	1·0000
13	·0000	·0098	·0392	·0884	·1573	·2462	·3553	·4848	·6351	·8067	1·0000
14	·0000	·0098	·0393	·0885+	·1575+	·2465−	·3557	·4852	·6355+	·8069	1·0000
15	·0000	·0098	·0393	·0886	·1577	·2467	·3560	·4856	·6358	·8072	1·0000
16	·0000	·0098	·0394	·0887	·1578	·2469	·3562	·4859	·6361	·8074	1·0000
17	·0000	·0099	·0394	·0888	·1580	·2471	·3564	·4861	·6364	·8075+	1·0000
18	·0000	·0099	·0395−	·0888	·1581	·2473	·3566	·4863	·6366	·8077	1·0000
19	·0000	·0099	·0395−	·0889	·1582	·2474	·3568	·4865+	·6368	·8078	1·0000
20	·0000	·0099	·0395+	·0890	·1583	·2476	·3570	·4867	·6369	·8079	1·0000
21	·0000	·0099	·0395+	·0890	·1584	·2477	·3571	·4869	·6371	·8080	1·0000
22	·0000	·0099	·0396	·0891	·1584	·2478	·3573	·4870	·6372	·8081	1·0000
23	·0000	·0099	·0396	·0891	·1585+	·2479	·3574	·4871	·6374	·8082	1·0000
24	·0000	·0099	·0396	·0891	·1586	·2480	·3575−	·4872	·6375−	·8083	1·0000
25	·0000	·0099	·0396	·0892	·1586	·2481	·3576	·4873	·6376	·8084	1·0000
50	·0000	·0100	·0398	·0896	·1593	·2490	·3588	·4887	·6388	·8092	1·0000
100	·0000	·0100	·0399	·0898	·1597	·2495+	·3594	·4894	·6394	·8096	1·0000
400	·0000	·0100	·0400	·0900	·1599	·2499	·3599	·4899	·6399	·8099	1·0000
ρ^2	·0000	·0100	·0400	·0900	·1600	·2500	·3600	·4900	·6400	·8100	1·0000

first three terms only retained, from which a new general term appears to have been written down and given to the computer of Table I on p. 182 of the paper. The last column of that Table had previously troubled me, but I had not perceived the source of the error until Mr J. Wishart kindly drew my attention to the source of the trouble in the general term of equation (ix). I am very grateful to Miss M. Moul for a recomputation of the Table, which has been considerably modified in the right-hand upper region.

Mr Wishart among other elegant results of a memoir to be published in the next number of *Biometrika* has shown that the correlation of the variances, $r_{\sigma_x^2\sigma_y^2}$, where selection is made from an infinite normal population, is given by

$$r_{\sigma_x^2\sigma_y^2}=\rho^2_{xy},$$

which had hitherto only been obtained by approximate methods. I had suggested to him that this could be confirmed by my equation (v) of p. 177 of the paper cited, which gives the correlation surface of σ_1, σ_2 (and therefore of course of σ_1^2, σ_2^2) in samples of any size. This he found followed easily. I have since succeeded in showing from the same equation (v) that the regression lines of the variances are straight.

In fact one finds, if $\tilde{\sigma}_2^2$ be the probable value of σ_2^2 for a given σ_1^2, that:

$$\tilde{\sigma}_2^2=\left(1-\frac{1}{n}\right)\Sigma_2^2(1-\rho^2)+\frac{\Sigma_2^2}{\Sigma_1^2}\rho^2\sigma_1^2,$$

where Σ_1, Σ_2 are the standard deviations and ρ the correlation in the population sampled.

If on the other hand we take the correlation of σ_1 and σ_2, or $r_{\sigma_1\sigma_2}$, we find that the value of $r_{\sigma_1\sigma_2}$ changes with the size of the sample n as well as with the correlation ρ of the variates in the sampled population, as the Table indicates. Further, the regression is not linear, but of rather a complicated nature. It might therefore be thought advisable to use the variances rather than the standard deviations. Unfortunately it is the latter rather than the former which more frequently occur in the problems to which such results must be applied. K. P.

II.

In discussing in the July issue of this Journal* the work of A. J. Lotka and Vito Volterra regarding the interdependence of species living together, I stated that while the latter obtained the fundamental differential equation, (viii) of my article, as a reasonable form of relation to be expected, based on the hypothesis of encounters, the former reached it only as an approximation derived from a Taylor expansion of the more general theoretical equation. I have to thank Dr Lotka for since pointing out to me that I had overlooked the fact that he has himself considered this aspect of the problem in a later section of his book (Chapter XXV, p. 359), where he shows how the interpretation of encounters or of collisions and capture may be given to the term KN_1N_2, reached in the earlier part of the book without a precise analysis of its physical significance. In this later section Dr Lotka also discusses the effect of velocity of motion upon frequency of encounter. E. S. P.

Erratum.

Mr J. Wishart wishes to draw attention to a slip in his memoir in Vol. XIX. p. 27, two lines below equation (26), where the sign of the radical has been omitted in the value of u_s, which should read

$$u_s=\frac{mx}{\sqrt{l+s+1}}.$$

This correct value is applied in the examples on p. 28.

Subscribers to Vol. XIX are requested to make this change in the text.

* *Biometrika*, Vol. XIX. pp. 216—222.

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CONTENTS

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Date of issue.	Date of issue.	Date of issue.
26.10.44		
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